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# Strategic Decision Making in Service Systems

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Submitted in total fulfilment of the requirements of the degree of

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# Abstract

Decision-making has been studied by economists, mathematicians, and engineers for centuries. The optimal decision can depend on the system state or others' decisions. The former case is usually formulated as a Markov decision process, while the latter case can be analyzed as a non-cooperative game among the individuals. In this dissertation, we illustrate these two cases through decision-making problems in different service systems. We also analyze a problem of parameter estimation for the latter case.

We begin by considering a collection of statistically identical two-state continuous time Markov chains (channels). A controller continuously selects a channel with the view of maximizing infinite horizon average reward. A switching cost is paid upon channel changes. We consider two cases: full observation where all channels are observed simultaneously, and partial observation where only the current channel is observed. We analyze the difference in performance between these cases for various policies. For the partial observation case with two channels, or an infinite number of channels, we explicitly characterize an optimal threshold for two sensible policies which we name "call-gapping"and "cool-off". Our results present a qualitative view on the interaction of the number of channels, the available information, and the switching costs.

In the channel selection problem above, there is a central controller who can make decisions based on the physical conditions. Next, we turn our attention to the case where everyone can make her decision independently but her optimal choice depends on others' decisions and everyone is aware of this interaction. The first model of this type we consider is a particular type of service system with two strategic servers. The model was discussed by Guglielmi and Badia [33] in 2015, where each server can either choose to be active or inactive and an active server is requested to transmit a packet. The servers have

varying probabilities of successfully transmitting when they are active, and both servers receive a unit reward if the packet is transmitted successfully. Guglielmi and Badia [33] provided an analysis of optimal strategies in four scenarios: where each server does not know the other's successful transmission probability; one of the two servers is always inactive; each server knows the other's successful transmission probability; each server knows the other's successful transmission probability and they are willing to cooperate.

Unfortunately the analysis in Guglielmi and Badia [33] contained errors. We correct these errors and discuss three cases where both servers (I) communicate and cooperate; (II) neither communicate nor cooperate; (III) communicate but do not cooperate. In particular, we obtain the unique Nash equilibrium strategy in Case II through a Bayesian game formulation, and demonstrate that there is a region in the parameter space where there are multiple Nash equilibria in Case III. We also quantify the value of communication and cooperation by comparing the social welfare in the three cases, and propose possible regulations to make the Nash equilibrium strategy socially optimal for both Cases II and III.

Then we consider an $M/M/1$ feedback queue where price and time sensitive customers are homogeneous with respect to service valuation and cost per unit time of waiting, and investigate the behavior of equilibria. Upon arrival, customers can observe the number of customers in the system and then decide to join or to balk. Customers are served in order of arrival. After being served, each customer either successfully completes the service and departs the system with probability $q$, or the service fails and the customer immediately joins the end of the queue to wait to be served again until she successfully completes it.

We analyse this decision problem as a non-cooperative game among the customers. For two payoff settings, we show that there exists a unique symmetric Nash equilibrium threshold strategy. Moreover, if we relax the strategy restrictions by allowing customers to renege, in the new Nash equilibrium, customers have a greater incentive to join. However, this does not necessarily increase the equilibrium expected payoff, and for some parameter values, it decreases it.

Finally we consider a problem of estimating customer delay and tardiness sensitivity

from periodic queue length observations. Every day, a single server queueing system commences its service at time zero. A random number of customers decide when to arrive at the system so as to minimize the sum of the waiting and the tardiness costs, where the tardiness is quantified by the interval between time zero and a customer's service completion time. The waiting and the tardiness costs are proportional to the waiting time and the tardiness with rates $\alpha$ and $\beta$, respectively. Clearly, each customer's best response depends on the others' decisions, thus the resulting strategy is a Nash equilibrium. In this work, we consider the estimation of the ratio $\theta \equiv \beta/(\alpha+\beta)$ from queue length data observed at discrete time points each day on several days, given that every customer uses a Nash equilibrium arrival strategy. The sampling instants are not necessarily equally spaced. Our method does not require estimation of the Nash equilibrium arrival strategy or an accurate estimation of its support. The estimator is strongly consistent and the estimation error is asymptotically normal. Moreover, the asymptote of the estimation error as a function of the queue length covariance matrix (at samplings times) is derived. The estimator performance is demonstrated through simulations, and we observed that it was robust to the number of sampling instants each day.

# Declaration

This is to certify that

1. the thesis comprises only my original work towards the PhD,
2. due acknowledgement has been made in the text to all other material used,
3. the thesis is less than 100,000 words in length, exclusive of tables, maps, bibliographies and appendices.

Jiesen Wang

ORCID Identifier: 0000-0003-2709-6586, 2021

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# Preface

This dissertation contains original research in Chapters 3, 4, 5, 6, 7. Chapter 3 is based on

- [86]: Wang, J., Nazarathy, Y., Taimre, T. (2020). The Value of Information and Efficient Switching in Channel Selection. In revision.

Chapter 4 is based on the published work

- [27]: Fackrell, M., Li, C., Taylor, P., Wang, J. (2019). The Value of Communication and Cooperation in a Two-Server Service System. The ANZIAM Journal **2019**,*61(4)*, 349-367.

Chapter 5 is based on the published work

- [28]: Fackrell, M., Taylor, P., Wang, J. (2021). Strategic customer behavior in an $M/M/1$ feedback queue. Queueing Systems **2021**, *97(3)*, 223-259.

We added Section 5.3.2 and Section 5.7 to Chapter 5 in the dissertation. Chapter 6 extends the work in Chapter 5. We plan to write up the results in Chapter 6 in preparation for a publication. Chapter 7 is based on

- [75]: Ravner, L., Wang, J. (2021). Estimating customer delay and tardiness sensitivity from periodic queue length observations. Submitted.

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# Chapter 1

# Introduction

Decision making problems are concerned with finding the best option from a set of available alternatives, given one's belief about the environment. The environment cannot be manipulated by the decision maker, but interacts with her. Decision making problems can be classified into two types, depending on whether the environment is a physical condition or others' behaviors. The former case is frequently modeled as a Markov decision process (MDP), and the latter can be formulated as a game theory problem.

In the MDP, a decision maker is sometimes referred to as a controller. At each decision epoch, the controller observes the state of the system, and makes the decision which optimizes her expected payoff. The effect of a decision depends on the current state and is twofold. The controller will obtain an immediate reward, and the decision will also define a transition probability distribution for the future state. In the above formalization, a single controller interacts with an environment evolving according to a probabilistic transition function affected by the controller's decisions. However, when multiple controllers are included in a shared environment, and each controller's expected payoff is affected by others' actions, then this is beyond the scope of MDP models. Game theory which we introduce in the next section is suitable for analyzing this scenario.

## 1.1 Game theory and Nash equilibria

Game theory (Osborne and Rubinstein [72]) provides a formal way to describe the interactions among individuals, thus it allows us to model "rules of the game" and widen our view to include multiple controllers with interacting goals. Specifically, it assumes that each individual can independently make her own decision to maximize her expected payoff which depends on others' decisions, so she needs to anticipate others' actions. The resulting solution is not an optimization resolution since each individual's objective function is different.

In 1950, John Nash proposed a notion of equilibrium in [67] (see Nash [68] for a more complete description), which is now called *Nash equilibrium*, as a solution for $n-$person games. Nash equilibrium, which is a centerpiece in game theory, describes a strategy profile including one strategy for each player, such that no one has an incentive to deviate, given the strategies of the other players. This was stated originally in Nash [67] as

> *Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the n − 1 strategies of the other players in the countered n-tuple. A self-countering n-tuple is called an equilibrium point.*

That is, each individual cannot improve her payoff by unilaterally changing her behavior.

Applying game theory to analyze a system with individuals who interact to obtain services and who can make their decisions independently, can help understand complex behavioral relationships such as competition and cooperation between them.

Another important quantity in this field is social welfare, which is the aggregate benefit derived by all the potential individuals. A Nash equilibrium solution does not necessarily lead to the maximal social welfare. Naor [66] is the pioneering work recognizing that the Nash equilibrium in an $M/M/1$ queue is not socially optimal, and proposing a regulation that leads to the attainment of social optimality.

As it was written in Hassin [35], the advantage of a queueing model is that its structure enables a more definite answer. My research mainly focuses on finding Nash equilibrium strategies of individuals in queueing systems. The individuals can be customers, servers, or both, which depends on the structure of the model. My study also calculates the optimal social welfare, compares it with that resulting from the Nash equilibrium, and regulates the Nash equilibrium to be socially optimal.

## 1.2 Organization of the thesis

This dissertation is organized as follows.

- Chapter 2 introduces preliminaries about queueing networks and strategic decision making.

- In Chapter 3, we consider a channel selection problem, and analyse it in two cases: the full observation case where states of all channels are observed, and the partial observation case where only the current channel is observed. The optimal policy depends on beliefs concerning the system state. We propose two policy classes which we name "call-gapping" and "cool-off", and explicitly characterize an optimal threshold for the two policies.

- In Chapter 4, we consider a two-server system with strategic servers. Each server can independently choose to transmit a packet, and has a varying probability of successfully transmitting. The server that takes the transmitting task needs to pay a one-off cost, and both servers receive a unit reward if the sequence of packets is transmitted successfully. We work out the Nash equilibria for two levels of communication, and propose possible regulations to make the Nash equilibrium strategy socially optimal.

- In Chapter 5 and 6, we consider customers' strategic behavior in an $M/M/1$ feedback queue, where arriving customers decide independently to join or not join after observing the queue size. Feedback queues are used to model the situation where customers need to retry for service after an unsuccessful one. We assume two settings for a customer's payoff. In the first setting, the payoff is the difference between the reward the customer obtains when she successfully finishes her service and her waiting cost, which is linear in her sojourn time. In the other setting, the payoff is exponentially discounted by the customer's sojourn time. We use a game theory framework to analyse the interaction between people's decisions, and derive a Nash equilibrium strategy for the two settings. Every time a service fails, the individual may have an incentive to give up. If customers are allowed to leave after joining, rather than remain in the system until their service is successful, then more people will choose to join. However, our analysis shows that this leads to a more crowded system which does not make anyone better off, and in some cases, makes everyone worse off.

- In Chapter 7, we consider a single server system where customer can choose when to arrive. While the literature on strategic arrivals to queueing systems provides a number of ways for describing and obtaining the Nash equilibrium, the problem of estimation has received little attention. In this chapter, we take advantage of Nash equilibrium properties to estimate the customer delay and tardiness sensitivity from periodic queue length observations.
- The thesis concludes with Chapter 8, where we propose some directions for future research.

## 1.3 Mathematical conventions

In this thesis we take the following conventions.

- Let $\mathbb{1}_{\{.\}}$ denote the indicator function.
- Let $\boldsymbol{e}_i$ denote the vector with a 1 in the $i$th coordinate and 0's elsewhere.
- We use capital alphabet and lowercase bold alphabet to denote matrices and vectors, respectively.
- let $|\mathcal{S}|$ where $\mathcal{S}$ is a set, be the cardinal number of $\mathcal{S}$.

# Chapter 2

# Preliminaries

## 2.1 Queueing networks

### 2.1.1 Markov processes

Let $\{X(t), t \in \mathcal{T}\}$ be a Markov process taking values in a countable state space $\mathcal{S}$. For a continuous-time Markov process, $\mathcal{T}$ is the set of real numbers $\mathbb{R}$; for a discrete-time Markov process, $\mathcal{T}$ is the set of integers $\mathbb{Z}$ (see Kelly [54, Section1.1]). In Walrand [84], if $X(t)$ is a continuous-time Markov process, denote the rate matrix of $X(t)$ by $Q = [q(i,j), i,j \in \mathcal{S}]$ such that, for $0 < \delta \ll 1$,

$$q(i,j)\delta \approx \mathbb{P}\{X(t+\delta) = j \mid X(t) = i\} \qquad j \neq i \tag{2.1}$$

$$q(i,i)\delta = -\sum_{j \neq i} q(i,j)\delta \approx -\mathbb{P}\{X(t+\delta) \neq i \mid X(t) = i\}. \tag{2.2}$$

If $X(t)$ is a discrete-time Markov process, denote the transition probability matrix of $X(t)$ by $P = [p(i,j), i,j \in \mathcal{S}]$ such that

$$p(i,j) = \mathbb{P}\{X(t+1) = j \mid X(t) = i\} \qquad j \neq i \tag{2.3}$$

$$p(i,i) = 1 - \sum_{j \neq i} p(i,j) = \mathbb{P}\{X(t+1) = i \mid X(t) = i\}. \tag{2.4}$$

A discrete-time Markov process is called a Markov chain in Kelly [54]. This terminology is used for the rest of this dissertation so that when we refer to a Markov process (chain) we mean a continuous (discrete) time process.

The Markov process (chain) $X(t)$ may possess an equilibrium distribution. From Kelly [53], for a non-explosive Markov process, if we can find a collection of positive numbers $\{\pi(i), i \in \mathcal{S}\}$ summing to unity that satisfy the balance equations

$$\pi(i)\,q(i) = \sum_{j \neq i} \pi(j) q(j,i) \qquad i \in \mathcal{S}\,, \tag{2.5}$$

for a Markov process, or

$$\pi(i) = \sum_{j \in \mathcal{S}} \pi(j) p(j,i) \qquad i \in \mathcal{S}\,, \tag{2.6}$$

for a Markov chain, then $\{\pi(i), i \in \mathcal{S}\}$ is the equilibrium distribution of $X(t)$. As it was written in Kelly [53, p2], if we can find a collection of positive numbers satisfying equations (2.5) for a Markov process (or (2.6) for a Markov chain) whose sum is finite, then the collection can be normalized to produce an equilibrium distribution. When an equilibrium distribution exists it is unique and

$$\lim_{t \to \infty} P(X(t) = i \mid X(0) = j) = \pi(i)\,. \tag{2.7}$$

Note that this does not hold for explosive Markov processes. The following example, which is from Miller, R. G. [65], shows that it is possible to find a summable solution to Equation (2.5) that is not the stationary distribution. Let $\{X(t)\}$ be a birth and death process taking values in a state space $\mathcal{S} = \{0, 1, \ldots\}$ with transition rates

$$q(i, i+1) = 4^i \qquad i = 0, 1, \ldots \tag{2.8}$$

$$q(i, i-1) = 4^i/2 \qquad i = 1, 2, \ldots \tag{2.9}$$

This is an explosive process with $+\infty$ capable of being reached in finite time. It is readily verified that $\{\pi(i) = 2^{-i-1}, i \in \mathcal{S}\}$ satisfies the detailed balance condition

$$\pi(i)\,q(i, i+1) = \pi(i+1)\,q(i+1, i) \qquad \forall i \in \mathcal{S}\,, \tag{2.10}$$

so it is a unique solution to Equation (2.5). However, the process cannot possibly be recurrent since the probability of a birth is twice that of a death.

### 2.1.2 Jump process

For $i, j \in \mathcal{S}$, let $q(i) = \sum_{j \neq i} q(i,j)$ where $q(i,j)$is the transition rate from $i$ to $j$, then a Markov process remains in state $i$ for an exponentially distributed time with rate $q(i)$. For $i \neq j$, when it departs from state $i$, it transits to state $j$ with probability

$$p(i,j) = \frac{q(i,j)}{q(i)} .$$

Following Kelly [54], if we associate a Markov chain $X^J(t)$ with a Markov process $X(t)$ by letting $X^J(0) = X(0)$, letting $X^J(1)$ be the next state the Markov process $X(t)$ transits to after time 0, letting $X^J(2)$ be the next state after that, then the Markov chain $X^J(t)$ is referred to as *jump chain* of the Markov process $X(t)$.

It is possible for a Markov process to possess an equilibrium distribution and for its jump process not to, and vice versa. The following example from Kelly [54, Exercise 1.1.5] explains the condition in which the jump chain has an equilibrium distribution if the process does.

**Example 1.** *If a Markov process has an equilibrium distribution $\pi(i), i \in \mathcal{S}$, then its jump process has an equilibrium distribution if and only if $\sum_{i \in \mathcal{S}} \pi(i) q(i)$ is finite.*

*Proof.* To see this, we need to first work out the equilibrium distribution for the jump process if it exists. Observe that

$$\begin{aligned} \sum_{j \neq i} A\pi(j)\, q(j)\, p(j,i) &= A \sum_{j \neq i} \pi(j)\, q(j)\, \frac{q(j,i)}{q(j)} \qquad i \in \mathcal{S} \qquad (2.11) \\ &= A \sum_{j \neq i} \pi(j)\, q(j,i) \\ &= A\pi(i)\, q(i) , \end{aligned}$$

so

$$\pi^J(i) = A\, \pi(i)\, q(i) \qquad (2.12)$$

where $A = \frac{1}{\sum_{i \in \mathcal{S}} \pi(i) q(i)}$.

Hence the jump process has an equilibrium distribution if and only if $\sum_{i \in \mathcal{S}} \pi(i) q(i)$ is finite. $\square$

Note that if $q(i)$ does not depend on $i$, then it is a Poisson process.

### 2.1.3 Reversibility

Let $\{X(t), t \in \mathcal{T}\}$ be a stochastic process taking values in a countable space $\mathcal{S}$. For $t_1 < t_2 < \cdots < \tau$, $X(t)$ is reversible if $(X(t_1), X(t_2), \ldots, X(t_n))$ has the same distribution as $(X(\tau - t_1), X(\tau - t_2), \ldots, X(\tau - t_n))$ (see Kelly [54, Section 1.2]). That is, when the direction of the time is reversed, the resulting process and the original process are statistically the same.

A stationary Markov process (or chain) is reversible if and only if there exists a collection of positive numbers $\{\pi(i), i \in \mathcal{S}\}$ summing to unity, that satisfy the detailed balance condition

$$\pi(i) q(i,j) = \pi(j) q(j,i) \quad (or \quad \pi(i) p(i,j) = \pi(j) p(j,i)) \qquad i, j \in \mathcal{S}\,.$$

See Kelly [54, Themorem 1.2, 1.3] for a detailed proof. Time reversibility is a powerful tool for analysing the networks. We illustrate its applications by the following theorem and a corollary about an $M/M/1$ queue.

**Theorem 1.** *Stationary birth-and-death processes are reversible.*

*Proof.* Consider a general birth-and-death process $\{X(t), t \geq 0\}$ on $\mathcal{S} = \{0, 1, 2, \ldots\}$ with birth and death rates

$$q(i, i+1) = \lambda_i \quad i \in \mathcal{S}\,, \qquad q(i, i-1) = \mu_i \quad i \in \mathcal{S}/\{0\}\,.$$

See Figure 2.1 for an illustration of the process. Suppose that $X(t)$ has a stationary distri-

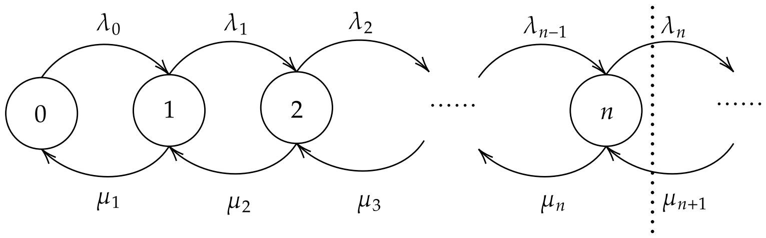



Figure 2.1: Transition rate diagram of a birth and death process.

bution $\{\pi(i), i \in \mathcal{S}\}$, then it follows from Equation (2.5) that $\{\pi(i), i \in \mathcal{S}\}$ satisfies

$$\pi(0)\lambda_0 = \pi(1)\mu_1 \tag{2.13}$$

$$\pi(i)(\lambda_i + \mu_i) = \pi(i+1)\mu_{i+1} + \pi(i-1)\lambda_{i-1} \qquad i \in \{1, 2, \ldots\}\,. \tag{2.14}$$

Let

$$M_i \equiv \lambda_i \pi(i) - \mu_{i+1}\pi(i+1) \qquad i \in \mathcal{S}\,.$$

After rearranging terms in Equation (2.13) and (2.14), we have

$$M_0 = \pi(0)\lambda_0 - \pi(1)\mu_1 = 0 \tag{2.15}$$

$$M_i = \pi(i)\lambda_i - \mu_{i+1}\pi(i+1) = \pi(i-1)\lambda_{i-1} - \mu_i\pi(i) = M_{i-1} \qquad i \in \mathcal{S} \tag{2.16}$$

Thus $M_i = 0$ for $i \in \mathcal{S}$. That is, a stationary birth-and-death process satisfies the detailed balance condition

$$\lambda_i\, \pi(i) = \mu_{i+1}\, \pi(i+1) \qquad i \in \mathcal{S}\,.$$

It follows from Kelly [54, Themorem 1.2] that it is reversible. □

If we associate a graph with a Markov process by letting $(i, j)$ be an edge if $q(i, j) > 0$ or $q(j, i) > 0$. The graph is said to have a *tree shape* if the graph does not have cycles. If the graph associated with a stationary Markov process has a tree shape, then the process is reversible. Note that the tree shape is a sufficient condition for reversibility, but not a necessary condition. Since every birth-and-death process has a tree shape, every stationary birth-and-death process is reversible.

**Corollary 1.** *The output process of an M/M/1 queue is a Poisson process with the same rate as*

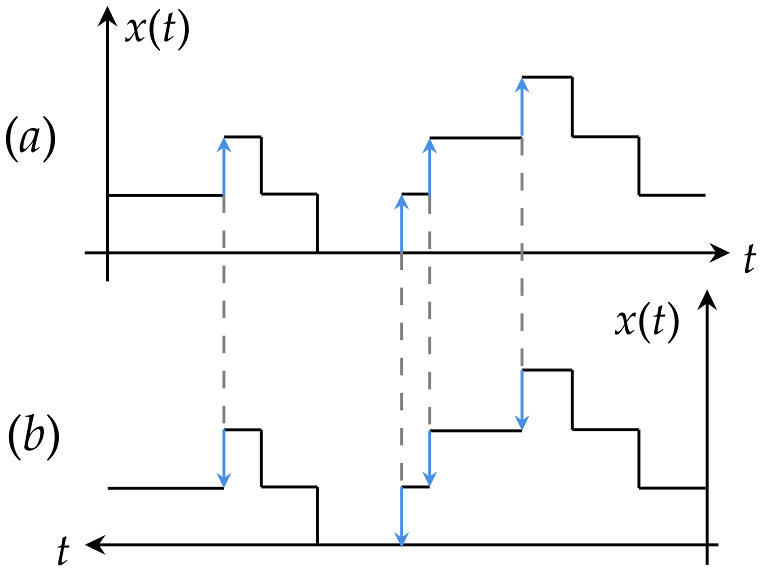



Figure 2.2: An illustration of Reversibility in an $M/M/1$ queue.

*the input process.*

*Proof.* Let $X(t)$ be the queue size of an $M/M/1$ queue at time $t$, then $X(t)$ is a birth-and-death process and a sample path of $X(t)$ is illustrated in Figure 2.2 (a). If we invert the time like in Figure 2.2 (b), it is clear that the output of the reversed process is the input of the original process. Moreover, from Theorem 1, since $\{X(t)\}$ is reversible, the output of the original process is stochastically same as the output of the reversed process. Hence, the output of an $M/M/1$ queue is a Poisson process with the same rate as the input process. □

Corollary 1 is part of the results in Burke's theorem, which is stated below.

**Theorem 2.** ***Burke's theorem*** *[17]. For $M/M/1$ queues, $M/M/c$ queues or $M/M/\infty$ queues in the steady state with arrivals a Poisson process with rate $\lambda$:*

- *The departure process is a Poisson process with rate $\lambda$.*
- *At time $t$ the number of customers in the queue is independent of the departure process prior to time $t$.*

### 2.1.4 Embedded Markov chain

In many cases, the continuous-time Markov process which models a system is insufficient to answer the question of interest. For example, suppose we have an $M/M/1$ queue. To obtain the dynamics of the queue size, we construct a continuous-time Markov process on the state space $\mathcal{S} = \{0, 1, 2, \ldots\}$. One question of interest is what is the probability of observing $i \in \mathcal{S}$ customers when a random customer arrives to a system? To answer this question, we need to construct the embedded discrete-time Markov chain by looking at the continuous-time Markov chain only at arrival instances. Note that the jump process defined earlier is an embedded Markov chain where the continuous-time Markov process is observed at every transition point. We analyze the embedded discrete-time Markov chain observed by arriving customers in detail in the following.

First observe that the probability that there is an arrival in the interval $(t, t+\delta)$ is approximately

$$\sum_{i=0}^{\infty} \pi(i)\, q(i, i+1)\delta + o(\delta)\,,$$

The probability that a customer arrives and finds $i$ customers present in the queue is approximately

$$\pi(i)\, q(i, i+1)\delta\,.$$

Thus, conditioning on that there is an arrival in the interval $(t, t+\delta)$, the probability that this arriving customer finds $i$ customers in the queue is

$$\frac{\pi(i)\, q(i, i+1)\delta}{\sum_{i=0}^{\infty} \pi(i)\, q(i, i+1)\delta} + o(\delta)\,. \tag{2.17}$$

If $q(i, i+1) = \lambda$ for any $i$, then the probability that an arriving customer observes $i$ customers is $\pi(i)$. This shows that the configuration of customers observed by an arriving customer has the same invariant distribution as the stationary distribution. This is known as the Poisson Arrivals See Time Averages (or PASTA) phenomenon.

### Waiting times

Another question of interest is the distribution of an arriving customer's waiting time. For $i \in \{1, 2, \dots\}$, let $S_i$ be the service time of the $i$th customer in the system. Assume that $S_i$'s are independent and identically distributed with probability density function $f(x) = \mu e^{-\mu x} \mathbb{1}_{\{x \geq 0\}}$. Let $W$ be the waiting time of an arriving customer to an $M/M/1$ queue. By making use of the stationary distribution of the embedded Markov chain in Equation (2.17), for any $w > 0$, we have

$$\mathbb{P}\left(W \leq w\right) = \sum_{k=1}^{\infty} \pi(k) \mathbb{P}\left(\sum_{i=1}^{k} S_i \leq w\right).$$

If the question is only concerned with the expected waiting time, not the distribution, then a basic but very useful result is stated in the following theorem.

**Theorem 3.** ***Little's law*** *[49]. In a queuing process, let $1/\lambda$ be the mean time between the arrivals of two consecutive units, $\ell$ be the mean number of units in the system, and $w$ be the mean time spent by a unit in the system. If both $\lambda$ and $w$ exist and are finite, then $\ell = \lambda w$.*

Little's Law is a sample path property that holds whenever the quantities involved are finite.

### Jumps in Jackson networks

The following description of Jackson networks (see Jackson [47]) can be found in Walrand [84, p41, p49]. A Jackson network is a queueing system with $J$ queues. Let $\mathcal{N} = \{1, 2, \dots, J\}$, then for $i \in \mathcal{N}$, external arrivals to queue $i$ form a Poisson process with rate $\gamma_i$, and the service time of queue $i$ is exponentially distributed with rate $\mu_i$. Upon finishing service at queue $i$, each customer is either sent to queue $k$ with probability $r_{ik}$, or leaves the network with probability $r_{i0} = 1 - \sum_{i \in \mathcal{S}} r_{ij}$. All random variables defined here are independent. Figure 2.3 gives an example of a Jackson network.

For $i \in \mathcal{N}, t \geq 0$, let $N_i(t)$ be the queue length of queue $i$ at time $t$. Let $\mathbf{N}(t) \equiv (N_1(t), N_2(t), \dots, N_J(t))$ represent the state of a Jackson network at time $t$. Since individuals are allowed to enter, leave, or move between the queues in the system, there are

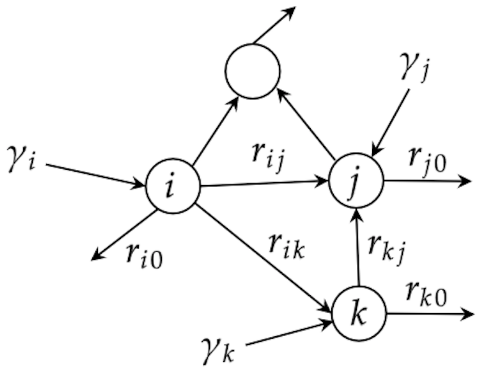



Figure 2.3: An example of a Jackson network.

three possible events. For $i, j \in \mathcal{N}$, let

$$T_{i\cdot}\,\boldsymbol{n} = \boldsymbol{n} - \boldsymbol{e}_i \tag{2.18}$$

$$T_{\cdot j}\,\boldsymbol{n} = \boldsymbol{n} + \boldsymbol{e}_j \tag{2.19}$$

$$T_{ij}\,\boldsymbol{n} = \boldsymbol{n} - \boldsymbol{e}_i + \boldsymbol{e}_j\,, \tag{2.20}$$

then $\boldsymbol{N}(t)$ forms a Markov process on the state space $\mathcal{S} = \{0, 1, 2, \ldots\}^J$, whose rate matrix is $Q = [q(\boldsymbol{n}\,,\,\boldsymbol{n}'),\ \boldsymbol{n}, \boldsymbol{n}' \in \mathcal{S}]$, where

$$q(\boldsymbol{n}\,,\,T_{i\cdot}\,\boldsymbol{n}) = \mu_i\, r_{i0}\, \mathbb{1}_{\{n_i>0\}} \tag{2.21}$$

$$q(\boldsymbol{n}\,,\,T_{ij}\,\boldsymbol{n}) = \mu_i\, r_{ij}\, \mathbb{1}_{\{n_i>0\}} \tag{2.22}$$

$$q(\boldsymbol{n}\,,\,T_{\cdot j}\,\boldsymbol{n}) = \gamma_i \tag{2.23}$$

The Markov process has a unique invariant distribution $\{\pi(\boldsymbol{n}), \boldsymbol{n} \in \mathcal{S}\}$ if and only if there exists a nonnegative solution $\{\lambda_i, i \in \mathcal{N}\}$ that satisfies, for any $i \in \mathcal{N}$, $\lambda_i < \mu_i$ and

$$\lambda_i = \gamma_i + \sum_{j=1}^{J} \lambda_j r_{ji}, \qquad 1 \le i \le J\,. \tag{2.24}$$

Equation (2.24) are called the flow conservation equations in Walrand [84]. In equilibrium, $n_i's$, where $i \in \mathcal{N}$ are independent and

$$\pi_i(n_i) = (1-\rho_i)\,\rho_i^{n_i}\,,$$

where $\rho_i := \lambda_i/\mu_i$. The invariant distribution has a product form, and is given by

$$\pi(\boldsymbol{n}) = \pi_1(n_1)\ldots\pi_J(n_J)\,. \tag{2.25}$$

To see why this is the case, we need to check whether $\{\pi(\boldsymbol{n}), \boldsymbol{n} \in \mathcal{S}\}$ satisfies the equilibrium equations

$$\begin{aligned} \pi(\boldsymbol{n}) & \left[\sum_i q(\boldsymbol{n}, T_{i\cdot}\,\boldsymbol{n}) + \sum_i\sum_j q(\boldsymbol{n}, T_{ij}\,\boldsymbol{n}) + \sum_j q(\boldsymbol{n}, T_{\cdot j}\,\boldsymbol{n})\right] \qquad i,j \in \mathcal{N} \\ &= \sum_i \pi(T_{i\cdot}\,\boldsymbol{n})q(T_{i\cdot}\,\boldsymbol{n},\boldsymbol{n}) + \sum_i\sum_j \pi(T_{ij}\,\boldsymbol{n})q(T_{ij}\,\boldsymbol{n},\boldsymbol{n}) + \sum_j \pi(T_{\cdot j}\,\boldsymbol{n})q(T_{\cdot j}\,\boldsymbol{n},\boldsymbol{n}) \end{aligned} \tag{2.26}$$

which will be satisfied if we can find a distribution $\pi(\boldsymbol{n})$ which satisfies the partial balance equations

$$\pi(\boldsymbol{n})\left[q(\boldsymbol{n}, T_{i\cdot}\,\boldsymbol{n}) + \sum_{j\neq 0} q(\boldsymbol{n}, T_{ij}\,\boldsymbol{n})\right] = \pi(T_{i\cdot}\,\boldsymbol{n})q(T_{i\cdot}\,\boldsymbol{n},\boldsymbol{n}) + \sum_{j\neq 0}\pi(T_{ij}\,\boldsymbol{n})q(T_{ij}\,\boldsymbol{n},\boldsymbol{n}) \qquad i,j \in \mathcal{N} \tag{2.27}$$

and

$$\pi(\boldsymbol{n})\sum_j q(\boldsymbol{n}, T_{\cdot j}\,\boldsymbol{n}) = \sum_j q(T_{\cdot j}\,\boldsymbol{n},\boldsymbol{n}) \qquad j \in \mathcal{N}\,. \tag{2.28}$$

After substituting $\pi(\boldsymbol{n})$, the left hand side of Equation (2.27) becomes $\pi(\boldsymbol{n})\mu_i$, and the right hand side of Equation (2.27) becomes

$$\frac{\pi(\boldsymbol{n})}{\rho_i}\left[\gamma_i + \sum_j \rho_j\mu_j r_{ji}\right].$$

To verify (2.27) is identical to verify $\mu_i\rho_i = \gamma_i + \sum_j \rho_j\mu_j r_{ji}$, that is $\lambda_i = \gamma_i + \sum_j \lambda_j r_{ji}$ for $i,j \in \mathcal{N}$, which is the flow conservation equations (2.24). Equation (2.28) reduces to, after substitution,

$$\sum_j \gamma_j = \sum_j \rho_j \mu_j r_{j0} \qquad j \in \mathcal{N}\,. \tag{2.29}$$

By summing equations (2.24), we have

$$\sum_j \lambda_j = \sum_j \gamma_j + \sum_j \sum_i \lambda_i r_{ij} \Longleftrightarrow \sum_j \lambda_j \left( \sum_i r_{ji} + r_{j0} \right) = \sum_j \gamma_j + \sum_j \sum_i \lambda_i r_{ij} \tag{2.30}$$

$$\Longleftrightarrow \sum_j \lambda_j r_{j0} = \sum_j \gamma_j \qquad i, j \in \mathcal{N}\,, \tag{2.31}$$

which verifies Equation (2.29).

Next we look at an example of jumps in a Jackson network.

**Example 2.** *This example is from Walrand [84, Section 2.10]. Let $S_m$ be the mth jump time from queue i to queue j, then the probability of the system state after the customer leaves queue i but before she reaches queue j*

$$\mathbb{P}\left( \boldsymbol{N}(S_m^-) = \boldsymbol{n} \right) = \pi(\boldsymbol{n} - \boldsymbol{e}_i)\, \mathbb{1}_{\{n_i > 0\}}\,. \tag{2.32}$$

*Walrand [84, p72] describes this result as the customer who jumps sees the others with their invariant distribution in the network.*

*Proof.* We first calculate the quantity $\sum_{\boldsymbol{n}} \pi(\boldsymbol{n})\, \mu_i\, r_{ij}\, \mathbb{1}\{n_i > 0\}$.

$$\begin{aligned}
\sum_{\boldsymbol{n}} \pi(\boldsymbol{n})\, \mu_i\, r_{ij}\, \mathbb{1}\{n_i > 0\} &= \sum_{\boldsymbol{n}:n_i>0} \pi(\boldsymbol{n})\, \mu_i\, r_{ij} \\
&= \sum_{\boldsymbol{n}:n_i>0} \pi_1(n_1) \ldots \pi_i(n_i) \ldots \pi_J(n_J)\, \mu_i\, r_{ij} \\
&= \sum_{\boldsymbol{n}:n_i>0} \pi_1(n_1) \ldots (1 - \frac{\lambda_i}{\mu_i}) \left( \frac{\lambda_i}{\mu_i} \right)^{n_i} \ldots \pi_J(n_J)\, \mu_i\, r_{ij} \\
&= \lambda_i\, r_{ij} \sum_{\boldsymbol{n}:n_i>0} \pi_1(n_1) \ldots (1 - \frac{\lambda_i}{\mu_i}) \left( \frac{\lambda_i}{\mu_i} \right)^{n_i - 1} \ldots \pi_J(n_J) \\
&= \lambda_i\, r_{ij} \sum_{\boldsymbol{n}} \pi_1(n_1) \ldots (1 - \frac{\lambda_i}{\mu_i}) \left( \frac{\lambda_i}{\mu_i} \right)^{n_i} \ldots \pi_J(n_J) \\
&= \lambda_i\, r_{ij} \sum_{\boldsymbol{n}} \pi(\boldsymbol{n}) = \lambda_i\, r_{ij}\,.
\end{aligned} \tag{2.33}$$

Next noticing that the probability flux of the event we are interested in is $\pi(\boldsymbol{n})\,\mu_i\,r_{ij}\,\mathbb{1}_{\{n_i>0\}}$, we have

$$\begin{aligned}\mathbb{P}(\boldsymbol{N}(S_n^-)=\boldsymbol{n}) &= \frac{\pi(\boldsymbol{n})\,\mu_i\,r_{ij}\,\mathbb{1}_{\{n_i>0\}}}{\sum_n \pi(\boldsymbol{n})\,\mu_i\,r_{ij}\,\mathbb{1}_{\{n_i>0\}}} = \frac{\pi(\boldsymbol{n})\,\mu_i\,r_{ij}\,\mathbb{1}_{\{n_i>0\}}}{\lambda_i\,r_{ij}} = \frac{\pi(\boldsymbol{n})\,\mathbb{1}_{\{n_i>0\}}}{\lambda_i/\mu_i} \\ &= \pi_1(n_1)\,\ldots\,(1-\frac{\lambda_i}{\mu_i})\left(\frac{\lambda_i}{\mu_i}\right)^{n_i-1}\mathbb{1}_{\{n_i>0\}}\,\ldots\,\pi_J(n_J) \\ &= \pi(\boldsymbol{n}-\boldsymbol{e}_i)\,\mathbb{1}_{\{n_i>0\}}\,,\end{aligned} \tag{2.34}$$

where the second step follows from (2.33).

Equation (2.34) shows that the configuration of customers who remain unchanged at time $S_m$ has the invariant distribution in the network with the customer that jumps at $S_m$ removed. This result is called the *arrival theorem* for Jackson networks. □

### 2.1.5 Quasi-birth-and-death processes

A quasi-birth-and-death process (QBD) is a Markov process $X(t)$ on a two-dimensional state space $\{(i,j)\ :\ j\geq 0, 1\leq i\leq M_j\}$, where $M_j$ is a positive integer, $i$ is referred to as the phase, and $j$ is referred to as the level (see Neuts [70]). The transition rate matrix

$$Q=\begin{bmatrix} Q_0^{(0)} & Q_1^{(0)} & 0 & 0 & \ldots \\ Q_{-1}^{(1)} & Q_0^{(1)} & Q_1^{(1)} & 0 & \ldots \\ 0 & Q_{-1}^{(2)} & Q_0^{(2)} & Q_1^{(2)} & \ldots \\ 0 & 0 & Q_{-1}^{(3)} & Q_0^{(3)} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{2.35}$$

has the block partitioned form. Matrices $Q_{-1}^{(j)}$, $Q_0^{(j)}$ and $Q_1^{(j)}$ are of size $M_{j-1}\times M_j$, $M_j\times M_j$ and $M_j\times M_{j+1}$, respectively. If $Q_1^{(j)}$ is identical for all $j\geq 0$, and $Q_0^{(j)}$, $Q_{-1}^{(j)}$ are identical for all $j\geq 1$, then $X(t)$ is a homogeneous quasi-birth-and-death process, which is also referred to as level independent quasi-birth-and-death process.

In a quasi-birth-and-death process, a state cannot transit up or down by more than one level. For $j\geq 0$, $0\leq i\leq M_j$, a state $(i,j)$ transits to state $(k,j+1), 1\leq k\leq M_{j+1}$

with probability $(Q_1^{(j)})_{ik}$, to state $(k,j), k \neq i, 1 \leq k \leq M_j$ with probability $(Q_0^{(j)})_{ik}$, to state $(k, j-1), 1 \leq k \leq M_{j-1}$ with probability $(Q_{-1}^{(j)})_{ik}$ (if $j > 0$). The diagonal entries of $Q_0^{(j)}$ are strictly negative, with the row sums of $Q$ equal to 0:

$$(Q_0^{(j)})_{ii} = -\left(\left(\sum_{1 \leq k \leq M_{j-1}} (Q_{-1}^{(j)})_{ik}\right) \mathbb{1}_{\{j>0\}} + \sum_{\substack{1 \leq k \leq M_j \\ k \neq i}} (Q_0^{(j)})_{ik} + \sum_{1 \leq k \leq M_{j+1}} (Q_1^{(j)})_{ik}\right),$$

for $j \geq 0, 1 \leq i \leq M_j$.

### 2.1.6 Matrix analytic methods

Marcel Neuts pioneered matrix-analytic methods in the study of applied probability models. As he wrote in Neuts [69],

> *The numerical investigation of probability models is an essential, but undeveloped part of their solution. For complex problems, the difficulties of the numerical analysis are comparable to those of an analytic discussion. A thorough understanding of the structural properties is essential to a well-planned algorithm.*

See Neuts [71] and Latouche and Ramaswami [56] for more applications of the matrix-analytic methods.

In this dissertation, we will use the knowledge of matrix analytic methods of level dependent QBDs in Chapter 5, but the model is a discrete-time level dependent QBD, which can be characterized by the transition matrix

$$P = \begin{bmatrix} A_0^{(0)} & A_1^{(0)} & 0 & 0 & \dots \\ A_{-1}^{(1)} & A_0^{(1)} & A_1^{(1)} & 0 & \dots \\ 0 & A_{-1}^{(2)} & A_0^{(2)} & A_1^{(2)} & \dots \\ 0 & 0 & A_{-1}^{(3)} & A_0^{(3)} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{2.36}$$

If $X(t)$ is a discrete time level dependent QBD, the following theorem from Latouche and Ramaswami [56, Theorem 12.1.1] shows that the stationary distribution of $X(t)$ has a matrix-product form.

**Theorem 4.** *Assume that the level dependent QBD is irreducible, aperiodic, and positive recurrent. Its limiting probability vector* $\boldsymbol{\pi} \equiv [\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \ldots]$ *with* $\boldsymbol{\pi}_j$ *the sub-vector of stationary probability for level j, satisfies the relation*

$$\boldsymbol{\pi}_j = \boldsymbol{\pi}_{j-1} R^{(j)} \qquad \text{for} \quad j \geq 1 \tag{2.37}$$

*where the matrix* $R^{(j)}$ *records the expected number of visits to the states in level j between two visits to level* $j-1$.

We introduce another two matrix sequences $\{G^{(j)} : j \geq 1\}$ and $\{U^{(j)} : j \geq 1\}$ where $G^{(j)}$ records the first passage probabilities from level $j$ to level $j-1$, and $U^{(j)}$ records the first return probabilities to level $j$ under taboo of levels $0, 1, \ldots, j-1$. The following theorem is from Latouche and Ramaswami [56, Theorem 12.1.1], which states the relationship between $\{G^{(j)}\}$, $\{U^{(j)}\}$, and $\{R^{(j)}\}$.

**Theorem 5.** *Any one of the sequences* $\{G^{(j)}\}$, $\{U^{(j)}\}$, *and* $\{R^{(j)}\}$ *determines the other two through the set of relations below.*

$$G^{(j)} = (I - U^{(j)})^{-1} A_{-1}^{(j)}, \tag{2.38}$$

$$R^{(j)} = A_1^{(j-1)} (I - U^{(j)})^{-1}, \tag{2.39}$$

$$U^{(j)} = A_0^{(j)} + A_1^{(j)} G^{(j+1)}, \tag{2.40}$$

$$U^{(j)} = A_0^{(j)} + R^{(j+1)} A_{-1}^{(j)}. \tag{2.41}$$

### 2.1.7 Poisson's equation

Let $\{X(t), t \geq 0\}$ be an irreducible, aperiodic and positive recurrent Markov chain with transition probability matrix $P$, state space $\mathcal{S}$, and stationary distribution $\boldsymbol{\pi}$. Let $\boldsymbol{g}$ be a given column vector, then Poisson's equation is defined in Makowski and Shwartz [60] as

$$(I - P)\boldsymbol{h} = \boldsymbol{g} - \omega \mathbf{1}, \tag{2.42}$$

with $\boldsymbol{h}$ and $\omega$ the solution. If the state space is finite, then the solution is

$$\boldsymbol{h} = (I - P)^{\#} + c\mathbf{1} \qquad \text{and} \qquad \omega = \boldsymbol{\pi g}\,, \tag{2.43}$$

where $(I - P)^{\#}$ is the group inverse of $I - P$ (see Meyer [63]), and $c$ is an arbitrary constant. There is a simple interpretation for the solution: assume that $\boldsymbol{g}$ is a state-dependent reward vector. That is, any visit of state $i$ brings a reward of $g_i$. Then the stationary expected reward per unit time is $\boldsymbol{\pi g}$, and the $i$th component of $(I - P)^{\#}$ is the total expected difference between the actual reward and its stationary mean, given that $X(0) = i$ (see Makowski and Shwartz [60, Chapter 9], Dendievel, Latouche, and Liu [20]). The arbitrary constant $c$ is needed to specify the solution of Poisson's equation when $I - P$ is singular. When $P$ is irreducible and substochastic but not stochastic, we have (also see [55, Theorem 3.2.4])

$$\begin{aligned}(I-P)^{-1}_{ij} &= \left(\sum_{n=0}^{\infty} P^n\right)_{ij} \\ &= \sum_{n=0}^{\infty} P\left[X(n) = j \mid X(0) = i\right] \\ &= \sum_{n=0}^{\infty} \mathbb{E}\left[\mathbb{1}_{\{X(n)=j\}} \mid X(0) = i\right] \\ &= \mathbb{E}\left[\sum_{n=0}^{\infty}\left[\mathbb{1}_{\{X(n)=j\}} \mid X(0) = i\right]\right] \\ &= \mathbb{E}\left[\text{the number of visits to state } j \mid X(0) = i\right]\,.\end{aligned} \tag{2.44}$$

Hence, $\left((1-P)^{-1}\right)_{ij}, i, j \in \mathcal{S}$ is the total number of times that the process is in $j$, given that it starts at $i$. Moreover, the Markov chain with transition probability matrix $P$ is absorbing, so $\boldsymbol{\pi} = 0$. Thus, the stationary expected reward per unit of time is zero, that is, the solution for (2.42) is $\omega = 0$, and $\boldsymbol{h} = (I - P)^{-1}\boldsymbol{g}$ which is the vector of the total expected reward accumulated during the whole history.

**Remark:** The irreducible property for matrix $P$ is necessary. For example, matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0.5 & 0 \end{bmatrix} \tag{2.45}$$

is strictly substochatic, but is not irreducible since it has two classes. For such a $P$, $(I - P)^{-1}$ does not exist.

Dendievel, Latouche, and Liu [20] studied Poisson's equation for discrete time QBDs, and exploited the special transition structure of QBDs to obtain its solutions in two forms. One is based on a decomposition through first passage times to lower levels, and the other is based on a recursive expression for the deviation matrix.

### 2.1.8 Non-Markov process

In Markov processes, the future is independent of the past given the present. There are other random processes where the evolution of the process depends on more information than is recorded in the current state. In such a case, we need to modify the current state to include all the information that determines the future state. We use the following model to illustrate this kind of process and how we derive the stationary distribution for it.

#### Reneging from a single-buffered single server queue

Consider a single server system with a single buffer in front of the server as that shown in Figure 2.4. We assume that customers arrive according to a Poisson process with rate $\lambda$ and the service discipline is first-come first-served. The service time is exponentially distributed with rate $\mu$. Upon her arrival, each customer is aware of whether the server is idle or busy. If the server is busy but the buffer is available, she will wait at the buffer. If the buffer is occupied, she has to leave the system. While a customer is waiting at the buffer, if she cannot enter the server in $T$ time, she will renege from the system. Time $T$ is a random variable with its distribution function denoted by $G$, and the probability density function denoted by $g$. Let the hazard function of $T$ be $h \equiv g/(1 - G)$. We are interested in the stationary distribution of the number of customers in the system.

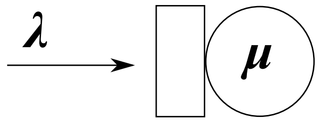



Figure 2.4: The single-buffered single server queue.

In this example, since a customer's reneging depends on the time she has already spent in the system, we need to define a finer classification of states recording the spent sojourn time of the current waiting customer if the buffer is not empty (see Taylor [82]). On the other hand, we only need to record the server state, since the service time is assumed to have an exponential distribution which is memoryless, and we assume that customers are not allowed to renege once they start their service. As a result, we have a state space $\mathcal{S} = \{(0,0), (0,1), (x,1)\}$, where $x$ is the spent sojourn time of the current waiting customer; the first coordinate is 0 if the buffer is empty; the second coordinate represents the server state with 0 being idle and 1 being busy.

**Theorem 6.** *For the single-buffered single server queueing system described above, the limiting distribution is*

$$\pi(x,1) = ae^{-\int_0^x(\mu+h(u))du} = a\, e^{-\mu x}\,(1-G(x)), \tag{2.46}$$

$$\pi(0,1) = \frac{\pi(0^+,1)}{\lambda} = \frac{a}{\lambda}, \tag{2.47}$$

$$\pi(0,0) = \frac{\mu}{\lambda}\,\pi(0,1) = \frac{a\,\mu}{\lambda^2}, \tag{2.48}$$

*where*

$$a = \frac{\lambda^2\mu}{(\mu^2+\lambda\mu+\lambda^2\,(1-\mathbb{E}(e^{-\mu x})))}. \tag{2.49}$$

Proof. Let $X(t), t \in \mathbb{R}$ be the system state at time $t$, and $P_t(s) \equiv P(X(t) = s)$. To find its limiting distribution which herein is assumed to exist, we start with the case of $(x,1)$,

$$P_{t+\delta}(x+\delta,1) = P_t(x,1)\left(1-\mu\delta-\int_0^\delta h(x+u)du\right)+o(\delta) \tag{2.50}$$

The value of $h(t)$ is the reneging rate of the waiting customer given that she has not reneged at time $t$. Thus $\int_{u=0}^{\delta} h(x+u)du$ is the probability that the waiting customer does not renege during time $(x, x+\delta)$. The term on the right hand side reflects the situation in which no departure or reneging occurs in the time $(x, x+\delta)$ and all that happens is spent sojourn time of the customer in the waiting room who is present at time $t$ age by an amount $\delta$. For state $(0,0)$, we have

$$P_{t+\delta}(0,0) = P_t(0,0)(1-\lambda\delta) + P_t(0,1)\mu\delta + o(\delta) \tag{2.51}$$

The first term on the right hand side represents the situation in which the system was empty at time $t$ and there are no new arrivals in the time $(t, t+\delta)$, while the second term reflects the situation in which there is one customer in her service at time $t$ and the service finishes by time $t+\delta$. Next, we deal with the boundary condition by observing that

$$\int_0^{\delta} P_t(u,1)\, du = \lambda\delta P_t(0,1) + o(\delta)\,. \tag{2.52}$$

Both sides of this equation describe the event that the server was busy at time $t$, a further customer arrived, and was allocated to the waiting room, before time $t+\delta$.

Rearranging and dividing equation (2.50), (2.51) and (2.52) by $\delta$, and letting $\delta \to 0$, we get

$$\frac{\partial P_t(x,1)}{\partial t} + \frac{\partial P_t(x,1)}{\partial x} = -P_t(x,1)(\mu + h(x))\,, \tag{2.53}$$

$$\frac{\partial P_t(x,1)}{\partial t} = \mu P(0,1) - \lambda P(0,0)\,, \tag{2.54}$$

$$\lambda P_t(0,1) = P_t(0^+,1)\,. \tag{2.55}$$

Next, we put the time derivative to zero to get equations for the stationary densities

$$\frac{d\,\pi(x,1)}{dx} + \pi(x,1)(\mu + h(x)) = 0\,, \tag{2.56}$$

$$\mu\pi(0,1) = \lambda\pi(0,0)\,, \tag{2.57}$$

subject to $\lambda\,\pi(0,1) = \pi(0^+,1)$. Hence

$$\pi(x,1) = ae^{-\int_{u=0}^{x}(\mu+h(u))du} = a\,e^{-\mu x}\,(1-G(x))\,, \tag{2.58}$$

$$\pi(0,1) = \frac{\pi(0^+,1)}{\lambda} = \frac{a}{\lambda}\,, \tag{2.59}$$

$$\pi(0,0) = \frac{\mu}{\lambda}\,\pi(0,1) = \frac{a\,\mu}{\lambda^2}\,, \tag{2.60}$$

where $a$ is the normalization factor. Equation (2.58) arises from the facts that, for any absolutely continuous $G$,

$$\frac{d\,log(1-G(x))}{dx} = -\frac{g(x)}{1-G(x)} = -h(x)\,. \tag{2.61}$$

To find out the constant $a$, we normalize the probabilities

$$\pi(0,0) + \pi(0,1) + \int_{x=0}^{\infty} \pi(x,1)\,dx = 1\,, \tag{2.62}$$

which leads to

$$a\,\left(\frac{1}{\lambda} + \frac{\mu}{\lambda^2} + \frac{1}{\mu}\,(1-\mathbb{E}(e^{-\mu x}))\right) = 1\,, \tag{2.63}$$

where $\mathbb{E}(e^{-\mu x}) = \int_{x=0}^{\infty} e^{-\mu x}\,g(x)\,dx$.

Example 3 and 4 illustrate how Theorem 6 can be used for some specific distributions of the reneging time $T$.

**Example 3.** *The limiting distribution, when the reneging time is $d$, is*

$$\pi(0,0) = \frac{a_d\,\mu}{\lambda^2}\,, \tag{2.64}$$

$$\pi(0,1) = \frac{a_d}{\lambda}\,, \tag{2.65}$$

$$\pi(x,1) = a_d\,e^{-\mu x}\,, 0 < x < d\,, \tag{2.66}$$

*where*

$$a_d = \frac{\lambda^2\,\mu}{\mu^2 + \lambda\,\mu + \lambda^2\,(1-e^{-\mu d})} \tag{2.67}$$

*is the normalization factor .*

*Proof.* This can be shown by considering the deterministic distribution as a special case of the general distribution and using the results obtained earlier. When the reneging time is $d$, then $g(x)$ is a delta function at $d$ and

$$G(x) = \begin{cases} 0 & \text{if } x < d \\ 1 & \text{if } x \geq d\,, \end{cases} \tag{2.68}$$

which leads to $\mathbb{E}(e^{-\mu x}) = e^{-\mu\, d}$ and $\pi(x,1) = a_d\, e^{-\mu x}\,, 0 < x < d$, respectively. After this, the conclusion in Example 3 can be obtained directly.

An alternative method to obtain the results in Example 3 can be stated as follows. For state $(x,1)$ and $(0,1)$, observe that

$$P_{t+\delta}(x+\delta,1) = P_t(x,1)\,(1-\mu\delta) \qquad\qquad x \in (0,d)\,, \tag{2.69}$$

$$P_{t+\delta}(0,1) = P_t(0,1)\,(1-\lambda\,\delta-\mu\,\delta) + \int_{u=0}^{\delta} P_t(d-u,1)\,du + P_t(0,0)\,\lambda\,\delta + \int_{u=0}^{d-\delta} P_t(u,1)\,du\,\mu\delta\,. \tag{2.70}$$

The second term on the right hand side of Equation (2.70) reflects the situation in which the spent service time of the waiting customer is greater than or equal to $d-\delta$ at time $t$, thus she will renege by time $t+\delta$. The fourth term reflects the situation in which the spent time of the waiting customer is less than $d-\delta$ and there is one customer in her service at time $t$, at time $t+\delta$, the service finishes while the waiting customer remains in the system. Similarly, we rearrange, divide Equations (2.69) and (2.70) by $\delta$ and let $\delta \to 0$, to derive

$$\frac{\partial P_t(x,1)}{\partial t} + \frac{\partial P_t(x,1)}{\partial x} = -\mu P_t(x,1)\,, \tag{2.71}$$

$$\frac{dP_t(0,1)}{dt} = P_t(d,1) - P_t(0,1)\,(\lambda+\mu) + P_t(0,0)\lambda + \int_{u=0}^{d} P_t(u,1)du\,\mu\,. \tag{2.72}$$

After putting the time derivative to zero, and using the fact that $\lambda\,\pi(0,0) = \mu\,\pi(0,1)$, we get

$$\pi(x,1) = a_d\, e^{-\mu x} \qquad 0 < x < d \tag{2.73}$$

$$\lambda\, \pi(0,1) = ae^{-\mu d} + a(1-e^{-\mu d}) = a = \pi(0^+,1)\,. \tag{2.74}$$

Note that the outcome of Equation (2.70) corresponds with the boundary condition in Lemma 6 □

**Example 4.** *The limiting distribution of $\pi(0,0)$ and $\pi(0,1)$ when the reneging time is exponentially distributed with rate $\gamma$, are the same as that of $\pi(0,0)$ and $\pi(0,1)$ in the case where the reneging time is d, if and only if*

$$\gamma = \frac{\mu e^{-\mu d}}{1-e^{-\mu d}}\,. \tag{2.75}$$

*Proof.* From Theorem 6, we have

$$\pi(0,0) = \frac{a_e\,\mu}{\lambda^2} \tag{2.76}$$

$$\pi(0,1) = \frac{a_e}{\lambda} \tag{2.77}$$

$$\pi(x,1) = a_e\, e^{-(\gamma+\mu)x} \qquad x > 0\,. \tag{2.78}$$

Then

$$\gamma = \frac{\mu e^{-\mu d}}{1-e^{-\mu d}} \Leftrightarrow \int_{x=0}^{\infty} e^{-(\gamma+\mu)x}\, dx = \frac{1}{\gamma+\mu} = \frac{1-e^{-\mu d}}{\mu} = \int_{x=0}^{d} e^{-\mu x} dx \Leftrightarrow a_e = a_d\,. \tag{2.79}$$

□

The conclusion in Example 4 is the same as the result in Barrer [10].

## 2.2 Strategic Decision Making

We have explained in Section 1.1 that in the situation where the best response of each individual depends on others' decisions, the resulting solution is a Nash equilibrium and this kind of problem is called strategic decision making in this dissertation. The decision can have different forms. For example, it can be whether to join the queue or not, what time to give up the service after joining due to deteriorate condition, what time to arrive, or whether to purchase priority or not. See Hassin and Haviv [40] for a comprehensive survey of strategic behavior in queueing systems. In this thesis, we mainly focus on symmetric Nash equilibria, which means individuals are homogeneous and everyone makes the same decision.

In this section, we first introduce the pioneering work Naor [66], which first recognized the difference between the Nash equilibrium solution and the socially optimal one, and proposed a regulation to eliminate this difference. Then we use one simple example to show how to find a Nash equilibrium if customers cannot observe the queue size before making their decisions.

### 2.2.1 Naor's result

Naor [66] considered a single server first-come-first-served system where arriving customers can decide to join or not to join after observing the queue length. Assume that customers can receive a reward $R$ after completing their service, and their waiting cost is linear in their waiting time with rate $C$. Customers choose to join when their expected payoff defined as the difference between $R$ and the expected waiting cost, is not negative. Let $\xi$ be the queue size observed by an arriving customer, then customers choose to join if

$$\frac{C\xi}{\mu} \leq R$$

and balk if

$$\frac{C\xi}{\mu} > R\,.$$

Thus the Nash equilibrium is a threshold strategy with threshold value

$$n_e = \left\lfloor \frac{R\,\mu}{C} \right\rfloor\,. \tag{2.80}$$

Let $\{\pi_i^{(n)}, 0 \leq i \leq n\}$ be the stationary distribution of the queue size $i$ when customers use threshold $n$. The social welfare $S(n)$, which is defined as the total expected payoff per unit time when others use threshold $n$ is

$$S(n) = \lambda R(1 - \pi_n^{(n)}) - C\mathbb{E}[\xi] \tag{2.81}$$

$$= \lambda R \frac{1-\rho^n}{1-\rho^{n+1}} - C\left[\frac{\rho}{1-\rho} - \frac{(n+1)\rho^{n+1}}{1-\rho^{n+1}}\right]. \tag{2.82}$$

From Naor [66], the socially optimal threshold $n^*$ which maximizes $S(n)$ satisfies

$$n^* = \lfloor x^* \rfloor, \tag{2.83}$$

where $x^*$ satisfies

$$\frac{x^*(1-\rho) - \rho(1-\rho^{x^*})}{(1-\rho)^2} = R\mu/C. \tag{2.84}$$

The value of $S(n)$ is increasing in $n$ when $n \le n^*$, and decreasing in $n$ when $n > n^*$. Moreover, $n_e > n^*$, and $S(n_e) < S(n^*)$.

In Naor's model, customers can observe the queue size before deciding whether to join or not. In some situations, the system state is unobservable to customers when they make their decisions. In that case, the Nash equilibrium can be characterized by a joining probability $p_e$ such that an arriving customer is indifferent between joining and balking, when others use probability $p_e$ to join. See Hassin and Haviv [40, Chapter 3] for details. In the next section, we use an example to illustrate how to calculate $p_e$ for a queueing system with increasing service rate.

### 2.2.2 Nash equilibrium for an invisible system with increasing service rate

Consider a two-server system with exponentially distributed service time with rate $\mu$ for each server. When the number of customers in the system does not exceed $n$, only one server is working. When the number is greater than $n$, the spare server is turned on, and it won't be turned off until the system is empty again. Assume that customers cannot observe the number of customers in the system, but are aware of whether one or two servers are working. Each customer obtains a reward of $R$ after the service but she needs to pay a cost which is linear in her waiting time with rate 1. Since the service rate can be either $\mu$ or $2\mu$ when there are less than $n+1$ customers in the system, we also need to record the servers' state. We use $i$ and $k$, where $i \in \{0,1,2,\ldots\}$ and $k \in \{0,1\}$ to denote the number of customers in the system and whether the spare server is working or not (1 is working, 0 is not working), respectively.

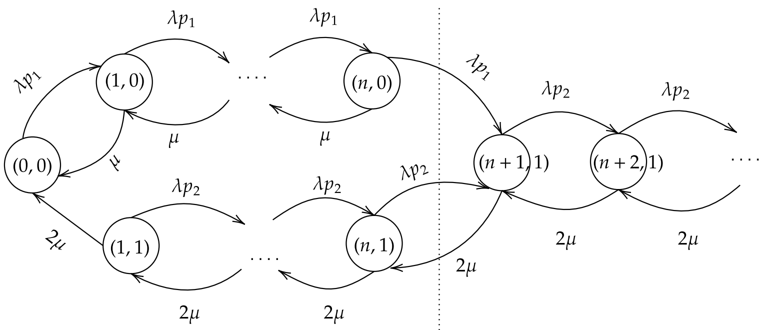



Figure 2.5: State transition diagram for the system with increasing service rate.

Let $\mathcal{S}_{\mathcal{W}} \equiv \{(i,0) : i \in \{0,1,\ldots,n\}\}$, $\mathcal{S}_{\mathcal{V}} \equiv \{(i,1) : i \in \{1,2\ldots\}\}$, and $\mathcal{S} \equiv \mathcal{S}_{\mathcal{W}} \cup \mathcal{S}_{\mathcal{V}}$. If customers join with probability $p_1$ when one server is working, and $p_2$ when both servers are working, then for $(i,k) \in \mathcal{S}$, the transition rate diagram is depicted in Figure 2.5, and the stationary distribution $\pi(i,k)$ of customers in the system satisfies

$$\lambda\, p_1\pi(0,0) = \mu\, \pi(1,0) + 2\mu\pi(1,1) \tag{2.85}$$

$$(\lambda\, p_1 + \mu)\pi(i,0) = \lambda\, p_1\pi(i-1,0) + \mu\, \pi(i+1,0) \qquad 1 \le i \le n-1 \tag{2.86}$$

$$(\lambda\, p_2 + 2\mu)\pi(i,1) = \lambda\, p_2\pi(i-1,1) + 2\mu\, \pi(i+1) \qquad 2 \le i \le n \tag{2.87}$$

$$(\lambda\, p_2 + 2\mu)\pi(1,1) = 2\mu\, \pi(2,1) \tag{2.88}$$

$$(\lambda\, p_1 + \mu)\pi(n,0) = \lambda p_1\pi(n-1,0) \tag{2.89}$$

$$(\lambda\, p_2 + 2\mu)\pi(n+1,1) = \lambda p_1\pi(n,0) + \lambda p_2\pi(n,1) + 2\mu\pi(n+2,1) \tag{2.90}$$

$$(\lambda\, p_2 + 2\mu)\pi(i,1) = \lambda p_2\pi(i-1,1) + 2\mu\pi(i+1,1)\,. \tag{2.91}$$

Let

$$\rho_1 = \frac{\lambda p_1}{\mu} \qquad \rho_2 = \frac{\lambda p_2}{2\mu}\,. \tag{2.92}$$

When $\rho_2 < 1$, the stationary distribution exists and the normalization yields that

$$\pi(i,0) = \left( \frac{\rho_1^i - \rho_1^{n+1}}{\frac{1-\rho_1^{n+1}}{1-\rho_1} + \left(\frac{1-\rho_1}{2(1-\rho_2)} - 1\right)(n+1)\rho_1^{n+1}} \right) \mathbb{1}_{\{\rho_1 \neq 1\}} + \left( \frac{(1-\rho_2)(n-i+1)}{(1-\rho_2)(n+1)(n+2)+(n+1)} \right) \mathbb{1}_{\{\rho_1 = 1\}} \quad i = 0,\dots,n \tag{2.93}$$

$$\pi(i,1) = \left( \frac{\rho_2^{(i-(n+1))^+}(1-\rho_2^{\min\{i,n+1\}})}{\frac{2(1-\rho_2)}{1-\rho_1}\left(\frac{1-\rho_1^{n+1}}{(1-\rho_1)\rho_1^{n+1}} - (n+1)\right) + (n+1)} \right) \mathbb{1}_{\{\rho_1 \neq 1\}} + \left( \frac{\rho_2^{(i-(n+1))^+}(1-\rho_2^{\min\{i,n+1\}})}{(1-\rho_2)(n+1)(n+2)+(n+1)} \right) \mathbb{1}_{\{\rho_1 = 1\}} \quad i = 1,2,\dots, \tag{2.94}$$

where $i^+ \equiv \max\{i,0\}$. Moreover, we have the conditional probabilities

$$\pi_{\mathcal{S}_\mathcal{W}}(i,0) \equiv \frac{\pi(i,0)}{\pi_{\mathcal{S}_\mathcal{W}}} = \left( \frac{\rho_1^i - \rho_1^{n+1}}{\frac{1-\rho_1^{n+1}}{1-\rho_1} - (n+1)\rho_1^{n+1}} \right) \mathbb{1}_{\{\rho_1 \neq 1\}} + \left( \frac{n-i+1}{(n+2)(n+1)/2} \right) \mathbb{1}_{\{\rho_1 = 1\}} \quad i = 0,\dots,n \tag{2.95}$$

$$\pi_{\mathcal{S}_\mathcal{V}}(i,1) \equiv \frac{\pi(i,1)}{\pi_{\mathcal{S}_\mathcal{V}}} = \frac{\rho_2^{(i-(n+1))^+}(1-\rho_2^{\min\{i,n+1\}})}{n+1} \qquad i = 1,2,\dots, \tag{2.96}$$

where

$$\pi_{\mathcal{S}_\mathcal{W}} \equiv \sum_{i=0}^{n} \pi(i,0) \tag{2.97}$$

$$= \left( \frac{2}{1-\rho_1}\left( \frac{1-\rho_1^{n+1}}{(1-\rho_1)\rho_1^{n+1}} - (n+1) \right) \mathbb{1}_{\{\rho_1 \neq 1\}} + (n+1)(n+2)\mathbb{1}_{\{\rho_1 = 1\}} \right) \pi(1,1)$$

$$\pi_{\mathcal{S}_\mathcal{V}} \equiv \sum_{i=1}^{\infty} \pi(i,1) = \frac{n+1}{1-\rho_2}\,\pi(1,1) \tag{2.98}$$

$$\tag{2.99}$$

**Remark**: In Figure 2.5, it is clear that the rate from state $(n+1,1)$ to state $(n,0)$ is 0. Hence the process is not reversible, and we cannot use the detailed balance equations to compute the stationary distribution. Figure 2.5 does not have a tree shape, but if we cut the graph by the dotted line, we can show that

$$\lambda p_1 \pi(n') + \lambda p_2 \pi(n) - 2\mu\pi(n+1) = 2\mu \left(1 + \rho_2 \frac{1-\rho_2^n}{1-\rho_2} - \frac{1-\rho_2^{n+1}}{1-\rho_2}\right) \pi(1) = 0\,. \quad (2.100)$$

### Nash equilibrium

To work out the Nash equilibrium $(p_1^e, p_2^e)$, besides the stationary distribution, we need to obtain the expected waiting time when a customer joins at different positions. Note that when there is only server working, it is possible that the queue size exceeds $n$ during a customer's waiting, thus we need to record the total number. However, once the second server is working, the expected waiting time depends only on the position the customer stands at.

For $0 \le i \le j \le n$, given $p_1$ and $p_2$, let $w_{i,j}$ be a customer's expected waiting time if she joins at position $i$, there are $j$ customers in the system, and there is only one server working when she joins. Then

$$w_{i,j} = \frac{1}{\lambda\, p_1 + \mu} + \frac{\lambda\, p_1}{\lambda\, p_1 + \mu} \left(\frac{i}{2\mu}\, \mathbb{1}_{\{j=n\}} + w_{i,j+1}\, \mathbb{1}_{\{j<n\}}\right) + \frac{\mu}{\lambda\, p_1 + \mu}\, w_{i-1,j-1}\, \mathbb{1}_{\{i>1\}}$$

which can be solved recursively using the following Algorithm 1. Then the Nash equi-

**Algorithm 1**

```
Set w_{1,n} = 1/(λ p1+μ) + (λ p1)/(λ p1+μ) · 1/(2μ)
for j = n − 1 : −1 : 1 do
    w_{1,j} = 1/(λ p1+μ) + (λ p1)/(λ p1+μ) · w_{1,j+1}
for i = 2 : n do
    w_{i,n} = 1/(λ p1+μ) + (λ p1)/(λ p1+μ) · i/(2μ) + μ/(λ p1+μ) · w_{i−1,n−1}
    for j = n − 1 : −1 : 1 do
        w_{i,j} = 1/(λ p1+μ) + (λ p1)/(λ p1+μ) · w_{i,j+1} + μ/(λ p1+μ) · w_{i−1,j−1}
```

librium $(p_1^e, p_2^e)$ can be obtained by solving (see Hassin and Haviv [40, Chapter 3])

$$\sum_{i=1}^{n} \pi_{\mathcal{S}_W}(i,0)\, w_{i,i} = R \tag{2.101}$$

$$\sum_{i=1}^{\infty} \pi_{\mathcal{S}_V}(i,1)\, \frac{i}{2\mu} = \frac{\frac{n}{2} + \frac{1}{1-\rho_2}}{2\mu} = R\,. \tag{2.102}$$

In Figure 2.6, the Nash equilibrium probabilities $(p_1^e, p_2^e)$ are depicted for two different values of $R$. In Figure 2.6 (a), customers have a higher incentive to join when there is only one server working as it is a signal of a less crowded system. As $R$ increases a bit, $p_2^e > p_1^e$, as depicted in Figure 2.6 (b). A possible explanation is that when $R$ is larger, customers are willing to wait longer, which will allow the system more time to open the second server that in turn will help reduce the waiting time of the joining customers. But when $R$ is large enough, all arriving customers join the system if there is only one server is working.

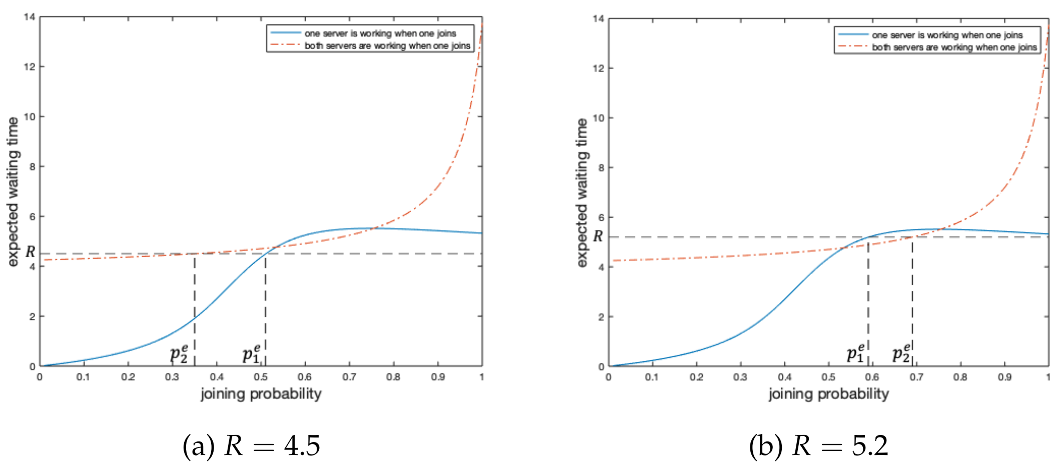



Figure 2.6: Nash equilibrium $(p_1^e, p_2^e)$ when $\lambda = 1.9, \mu = 1$.

This result can lead to other interesting questions regarding information disclosure, and operating strategies. For example, is it better to let arriving customers know the number of working servers, or the number of waiting customers? From the system manager's perspective, what is the optimal $n$?

# Chapter 3

# The Value of Information and Efficient Switching in Channel Selection

## 3.1 Introduction

Many scenarios in mining, finance, telecommunications, medical research, and other fields, involve a situation where the reward collected from a resource fluctuates over time in a stochastic manner. For example the yield obtained from mineral resource extraction varies as digging persists; the returns from financial investments vary based on many random market effects; the communication bit rate of communication channels varies based on physical conditions; and the findings in medical trials vary over time. Such a theme of randomly varying rewards, or a randomly modulated reward rate, reappears in multiple application areas. Hence designing, managing, and controlling such uncertain situations has been a focal point of stochastic operations research in a variety of contexts. The theory of restless bandits presents one general paradigm for dealing with such problems. See for example Gittins, Glazebrook, and Weber [31], and Weber and Weiss [88] for the general restless bandits problem. The main theme in such research is the efficient selection of channels/projects/arms over time.

In this chapter we add to the body of literature by considering problems with switching costs. Switching costs have been considered in Agrawal, Hegde, and Teneketzis [3], Banks and Sundaram [9] and Dusonchet and Hongler [25] in the context of multi-armed bandit processes, but to the best of our knowledge have not been studied as we do here. The general problem which we discuss is one in which a resource yields random rewards over time, with some known average reward rate and a maximal reward rate. One way to improve performance is to introduce additional independent instances of this resource and to consider a situation where at any time we use an instance of our (dynamic) choice. By doing so, we are potentially able to increase the obtained reward rate from the average reward rate towards the maximal reward rate.

As an example consider a communication link that yields an average bit rate of 4 Mbit/sec and a maximal bit rate of 10 Mbit/sec where the actual instantaneous bit rate varies stochastically over time and achieves the maximal bit rate for random finite durations. By introducing multiple instances of such a link and allowing the system to switch between the instances without constraint, we are able to get an effective average bit rate that exceeds 4 Mbit/sec and potentially nears the maximal bit rate of 10 Mbit/sec. The key is clearly to use the right instance of the channel at "the right time", so as to on-average use channels that do better than the average bit rate. At the extreme positive case, by introducing an increasing number of instances and assuming their stochastic behavior is independent between instances, we can get arbitrarily close to the 10 Mbit/sec bit rate. The other extreme is not switching between channels at all and settling for the average bit rate of 4 Mbit/sec, obtained from a single channel.

While the usage of such redundant channels can be of benefit, it is clearly not without cost. First there are the structural costs of setting up additional channels. The exact nature of these costs depends on the application and is not our focus. Then there are costs associated with setting up systems for gaining information about the instantaneous state of all channels. Finally there are dynamic instantaneous switching costs involved when switching between channels. In this chapter we study the tradeoffs and costs associated with this problem using the simplest example model that we could consider. The elegance of our simple model is that it captures the value of real-time information and at the same time allows us to compare how the addition of more channels to the system increases rewards. Real systems are bound to be more complex than the model which we present, however the results from our model capture the essential tradeoffs that one can expect.

After reparameterization, our model is that each of $n$ channels yield instantaneous rewards of either 0 or 1 where switching between the reward states is according to a

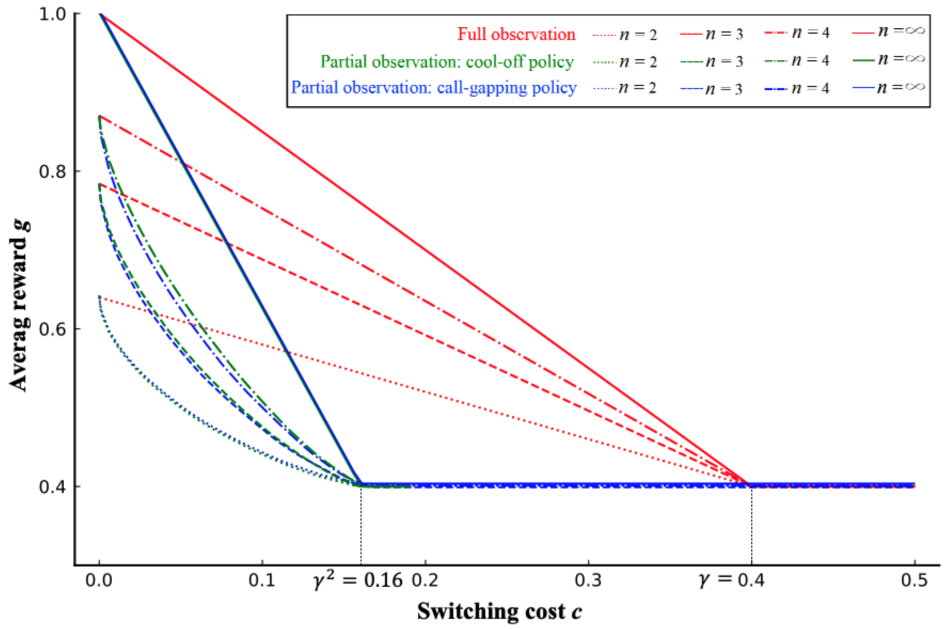



Figure 3.1: The optimal average reward for a $\gamma = 0.4$ system with a varying number of channels $n$. The red curves are for a case of full observation. The green curves are for a case of partial observation with a cool-off policy. The blue curves are for the case of partial observation with a call-gapping policy. Note that for $n = 2$ and $n = \infty$ both of the latter policies are identical.

two state continuous time Markov chain. The long term average reward obtained by a channel is $\gamma \in (0, 1)$ and the instantaneous cost of switching between channels is $c$. We distinguish between a case of *full observation* where the state of all channels is observable, and a case of *partial observation* where only the state of the current channel is available. For each of these cases, we suggest parameterized channel selection policies, and we are able to analyze and optimize the parameters of these policies in certain situations.

An illustration of the performance of such a model for the case of $\gamma = 0.4$ is in Figure 3.1. There are two threshold values at $\gamma = 0.4$ and at $\gamma^2 = 0.16$. We see that in the full observation case, when $c > \gamma$ it is not worthwhile to use additional channels and otherwise the total reward increases in an affine manner as $c \to 0$. Further we see that in the partial observation case, usage of redundant channels is only worthwhile if $c < \gamma^2$. In this case, the total reward rate increases in a non linear manner as $c \to 0$. In this case, we compare two policies that are described in the sequel, namely *call-gapping* and *cool-off*. While we don't have optimality proofs for these policies within the class of all policies, we show that these policies agree in performance for $n = 2$ and $n \to \infty$. We further obtain explicit results for the case of $n = 2$ and at $n = \infty$. Note that in all cases, as the number of channels increases, the achievable reward rate increases towards the maximal reward for small switching costs $c$. Our results in this chapter help understand the tradeoffs between the number of channels, switching costs, information, and policy qualitatively.

The remainder of the chapter is structured as follows: In Section 3.2 we present the model, summarize our main results, and derive the simple result for the case of full observation. In Section 3.3 we derive results associated with the case of partial observations. We then conclude in Section 3.5.

## 3.2 Model and Summary of Results

We consider $n$ statistically identical channels $i = 1, 2, \dots, n$, with state $X_i(t) \in \{0, 1\}$, evolving independently according to a continuous-time, time-homogenous, two-state Markov chain model with generator

$$Q = \begin{bmatrix} -\lambda & \lambda \\ \mu & -\mu \end{bmatrix},$$

where $\lambda, \mu > 0$.

At any given time, the controller may only use one of the channels. The controller's choice is indicated through $U(t) \in \{1, \dots, n\}$, where $U(t) = i$ means that channel $i$ is being used at time $t$. There is a fixed switching cost of $\kappa \geq 0$ for every instant at which $U(t)$ is changed; that is, every time $t$ where $U(t^-) \neq U(t)$, an additional cost of $\kappa$ is incurred.

When $X_i(t) = x \in \{0, 1\}$, the reward rate of that channel is denoted by $r(x)$, a function assumed identical for all channels. The goal of the controller is then to maximize the average reward,

$$g = \liminf_{t \to \infty} \frac{\mathbb{E}\left[\int_0^t r\big(X_{U(s)}(s)\big)\, ds\right] - \kappa\, N(t)}{t}, \tag{3.1}$$

where $N(t)$ denotes the number of channel changes during the time interval $[0, t]$.

For $s \in [0,t]$, $\int_0^t X_{U(s)}(s)\,ds$ and $t - \int_0^t X_{U(s)}(s)\,ds$ are the total time that $X_{U(s)}(s) = 1$ and $X_{U(s)}(s) = 0$, respectively. Thus, we have

$$\int_0^t r\big(X_{U(t)}(t)\big)\,dt = \left(\int_0^t X_{U(s)}(s)ds\right) r(1) + \left(t - \int_0^t X_{U(s)}(s)\,ds\right) r(0)$$
$$= t\,r(0) + \big(r(1) - r(0)\big) \int_0^t X_{U(s)}(s)\,ds\,.$$

Without loss of generality, we are able to set $r(0) = 0$ and $r(1) = \zeta > 0$. Further, by rescaling time by a factor of $\eta = \lambda^{-1}$ and setting $\zeta/\eta$ to be 1, we parameterize the channel model only by $\gamma = \lambda/(\lambda+\mu)$ (the stationary probability of a channel being in state 1) and $c = \kappa/\eta$.

In what follows, we write $I(t) = X_{U(t)}(t)$, which denotes the state of the selected channel at time $t$. Then, the average reward $g$ defined in (3.1) corresponding to the parametrization just outlined is expressed as:

$$g = \liminf_{t\to\infty} \frac{(\zeta/\eta)\int_0^t \tilde{I}(u)\,du - (\kappa/\eta)\tilde{N}(t)}{t} = \liminf_{t\to\infty} \frac{\int_0^t \tilde{I}(u)\,du - c\tilde{N}(t)}{t}\,,$$

where $\tilde{I}(t) = I(\eta t)$ and $\tilde{N}(t) = N(\eta t)$.

Hence our model parameters are $\gamma \in (0,1)$ and $c \geq 0$ with a rescaled generator

$$Q = \begin{bmatrix} -1 & 1 \\ 1/\gamma - 1 & -(1/\gamma - 1) \end{bmatrix}.$$

We distinguish two situations:

**Full Observation (Case I)**: The controller fully observes $\{X_1(t),\ldots,X_n(t)\}$.

**Partial Observations (Case II)**: When $U(t) = i$ the controller only observes $X_i(t)$.

In considering this partial observation case, it is useful to consider belief states for $i = 1,\ldots,n$, via

$$\omega_i(t) = \mathbb{P}\big(X_i(t) = 1 \,|\, \text{All information observed up to time } t\big),$$

and in constructing the belief states, it is useful to denote the *last time in channel i* via,

$$\mathcal{T}_i(t) = \sup\{t' \leq t \,:\, U(t') = i\}\,.$$

This allows the construction of the belief state of channel $i$ via

$$\omega_i(t) = p\big(t - \mathcal{T}_i(t)\,;\, X_i(\mathcal{T}_i(t))\big),$$

where $p(t\,;\,x) := \mathbb{P}\big(X_i(t) = 1 \,|\, X_i(0) = x\big)$ which can be represented explicitly as

$$p(t\,;\,x) = \begin{cases} \gamma(1 - e^{-\frac{t}{\gamma}}), & x = 0\,, \\ \gamma + (1-\gamma)e^{-\frac{t}{\gamma}}, & x = 1\,. \end{cases} \tag{3.2}$$

Observe that if $U(t) = i$ then $\mathcal{T}_i(t) = t$ and then $\omega_i(t) = X_i(t)$ which is either 0 or 1.

**Control policies:** We consider several policies all of which limit channel switching to times $t$ when $X_i(t) = 0$. This means that in the partial observation case (II), the belief state at any time $t$ for $i \neq U(t)$ is strictly monotonically increasing in time according to the first expression in (3.2). In both case I and case II, one trivial policy denoted $\pi^{(s)}$ is the policy of never switching channel. Further, in case I, there is the policy $\pi^{(a)}$ which switches to an arbitrary channel $j \neq U(t)$ if $X_j(t) = 1$ and $X_{U(t)} = 0$. That is, this policy ensures that the current channel is in state 1 if such a channel exists, and further this policy does not switch excessively if not needed.

In case II, in addition to the policy $\pi^{(s)}$, we consider a *call-gapping policy*, $\pi^{(\tau)}$, and a *cool-off* policy $\pi^{(\sigma)}$. In brief, call-gapping does not switch out of a channel prior to a time $\tau > 0$. Such a policy was analyzed in a different context in Lin and Ross [58] where it was termed call-gapping. Cool-off does not switch into a channel prior to a time $\sigma > 0$.

Like all our policies, call-gapping does not switch out of a good channel, however if the current channel $U(t)$ is in bad state, i.e. $X_{U(t)} = 0$ then this policy switches to the channel with highest belief state as long as the time since the last switch is not less than $\tau$. That is, at time $t$ denote the *last switching time* via,

$$\mathcal{T}^{\ell}(t) = \sup\{t' \leq t \,:\, U(t'^{-}) \neq U(t')\}\,,$$

then the call-gapping policy switches only if $X_{U(t)} = 0$ and $t - \mathcal{T}^{\ell}(t) \geq \tau$. Note that in our context, since the channels are homogenous and $\omega_i(t)$ is monotonically increasing, when the call-gapping policy yields a switch then it is to a channel that has not been used for the longest duration. This naturally yields *round-robin* behavior as we switch from channel $i$ to channel $i+1$ (modulo $n$).

As with the call-gapping policy, the cool-off policy does not switch out of a good channel and once switching occurs we switch with a round-robin manner, to the channel that has been visited longest ago. However, in contrast to the call-gapping policy, we choose to switch into a channel only if it hasn't been visited for a time that is at least the cool-off parameter $\sigma$. This essentially means, that the cool-off policy considers the belief states of all channels and switches into a channel $i$ only if $U(t) = 0$ and $\omega_i(t)$ exceeds the threshold $p(\sigma\,;\,0)$. Note that unlike the call-gapping policy, the cool-off policy does not require a minimum staying time at each channel, thus we can switch out of a channel immediately after switching to it if it is in bad state. That is, the cool-off policy may potentially incur multiple instantaneous switches. Also note that when $n = 2$, the call-gapping and cool-off policies are identical. Both the call-gapping and cool-off policies ensure a strict upper limit on the switching costs, $c\tilde{N}(t)/t$, as with these policies, almost surely,

$$\limsup_{t\to\infty}\, c\frac{\tilde{N}(t)}{t} \leq \frac{c}{\tau}, \qquad \text{and} \qquad \limsup_{t\to\infty}\, c\frac{\tilde{N}(t)}{t} \leq \frac{nc}{\sigma}.$$

Our results for these policies are as follows. First for case I, the performance of the system under $\pi^{(s)}$ and $\pi^{(a)}$ is tractable, for any value of $n$. This also allows us to characterize when each of these policies is preferable as a function of the switching cost $c$ and the parameter $\gamma$.

In contrast, for case II, analysis is more difficult. When $n = 2$, we are able to characterize the optimal call-gapping parameter $\tau^*$. For finite $n > 2$, we have not found such an analytic characterization. However, we can consider systems with large $n$. For this we construct an approximate model, that we label with $n = \infty$. Such a system has only two channels, say $i = 1$ and $i = 2$. When $U(t) = 1$, then $X_2(t)$ is assumed to be in steady state. Similarly when $U(t) = 2$ then $X_1(t)$ is assumed to be in steady state. That is, the current channel $X_{U(t)}$ behaves normally, however at the moment of switching to the other channel, the state of that channel is drawn from the steady state distribution $[(1-\gamma) \quad \gamma]$, and the evolution continuous. Under a round-robin based policy such as our $\pi^{(\tau)}$ and $\pi^{(\sigma)}$, with large $n$, the $n = \infty$ model approximates the system because we can expect the next channel in the round-robin to be at approximate steady state.

Our analytic results compare the different policies (and their parameters) for Case I, and Case II, and for different values of $n$. That is, for Case I, we determine when $\pi^{(s)}$ is preferable to $\pi^{(a)}$ and vice versa. We also determine the value of the long term reward $g$. Similarly for Case II (under $n = 2$ and $n = \infty$), we compare $\pi^{(\tau)}$, $\pi^{(\sigma)}$, and $\pi^{(s)}$, determine the optimal parameters for $\tau$ (or $\sigma$), and obtain expressions for $g$.

**Theorem 7.** *In Case I, the optimal choice between $\pi^{(a)}$ and $\pi^{(s)}$ is given by:*

$$\pi^* = \begin{cases} \pi^{(a)}, & c < \gamma\,, \\ \pi^{(s)}, & c \geq \gamma\,. \end{cases}$$

*Further, under $\pi^*$ the optimal expected average reward is given by:*

$$g^* = \begin{cases} 1 - (1-\gamma)^n - c\frac{1-\gamma-(1-\gamma)^n}{\gamma} & c < \gamma\,, \\ \gamma & c \geq \gamma\,. \end{cases}$$

□

For Case II, we consider two scenarios. Firstly, restrict the policy space to be the set of call-gapping policies and denote the optimal policy therein by $\pi^*_C$. Secondly, restrict the policy space to be the set of all cool-off policies and denote the optimal policy therein by $\pi^*_D$. We now have,

**Theorem 8.** *For Case II, we give a similar partial result: for any $n$, if $c > \gamma^2$ then the optimal policy is $\pi^{(s)}$ when compared against $\pi^{(\tau)}$ and $\pi^{(\sigma)}$. For $n \geq 2$, we have:*

$$\pi^*_C = \begin{cases} \pi^{(\tau^*)}, & c < \gamma^2, \\ \pi^{(s)}, & c \geq \gamma^2, \end{cases} \qquad \pi^*_D = \begin{cases} \pi^{(\sigma^*)}, & c < \gamma^2, \\ \pi^{(s)}, & c \geq \gamma^2. \end{cases}$$

*When* $c < \gamma^2$ *and* $n = 2$, *the* optimal call-gapping time, $\tau^*$ *is the unique non-negative solution of the equation*

$$e^{\frac{2\tau^*}{\gamma}}(\gamma^2 - c)(\gamma - 2) + 2e^{\frac{\tau^*}{\gamma}}\gamma(\gamma - \tau^*(\gamma - 1)) - \gamma(\gamma^2 - c) = 0. \tag{3.3}$$

*The optimal cool-off level is given by* $\sigma^* = \tau^*$. *When* $c < \gamma^2$, $n = \infty$,

$$\sigma^* = \tau^* = 0\,.$$

*Further with* $\pi_C^*$ *or* $\pi_D^*$, *the optimal expected average reward is given by:*

$$g_C^* = g_D^* = \begin{cases} \frac{A_1(\gamma,\tau^*) - c\, A_2(\gamma,\tau^*)}{A_3(\gamma,\tau^*)} & c < \gamma^2, \quad n = 2\,, \\ 1 - c\,(1-\gamma)/\gamma^2 & c < \gamma^2, \quad n = \infty\,, \\ \gamma & c \geq \gamma^2\,, \end{cases} \tag{3.4}$$

*where,*

$$A_1(\gamma,\tau) = e^{\frac{2\tau}{\gamma}}((\tau - 1)\gamma^3 - (3\tau - 2)\gamma^2 + 2\tau\gamma) - 2e^{\frac{\tau}{\gamma}}\gamma^2(1-\gamma)^2 + 2\gamma^4 + (\tau - 3)\gamma^3 - \tau\gamma^2,$$
$$A_2(\gamma,\tau) = (\gamma - 1)(e^{\frac{2\tau}{\gamma}}(\gamma - 2) + \gamma),$$
$$A_3(\gamma,\tau) = \gamma^3 + (\tau - 2)\gamma^2 - \tau\gamma - e^{\frac{2\tau}{\gamma}}(\gamma^3 - (\tau + 2)\gamma^2 + 3\tau\gamma - 2\tau).$$

□

Note that the expression for $g_C^*$ or $g_D^*$ as in Eqaution (3.4) in Theorem 8 can be used to evaluate the performance for $n = 2$ for any value of $\tau$ (alt. $\sigma$), by replacing $\tau^*$ in the expression with $\tau$ (alt. $\sigma$). Also note that empirical evidence, see Figure 3.1, suggests that, when $2 < n < \infty$ that the optimal cool-off policy has better expected average reward than the optimal call-gapping policy.

**Proof of Theorem 7:** We consider the process $\{M(t), t \geq 0\}$ on the state space $\{0, 1, \ldots, n\}$ where,

$$M(t) = \sum_{i=1}^{n} X_i(t).$$

The fact that the channels are independent and satisfy the same probability law, implies that $M(t)$ is a birth and death continuous time Markov chain, with birth and death rates

$$\lambda_i = n - i, \qquad \mu_i = i\Big(\frac{1}{\gamma} - 1\Big), \tag{3.5}$$

for $i \in \{0, \ldots, n-1\}$ and $i \in \{1, \ldots, n\}$ respectively. Such a birth and death process is known as an Ehrenfest model with binomial stationary distribution,

$$\pi_i = \binom{n}{i} \gamma^i \, (1-\gamma)^{n-i} \qquad i = 0, \ldots, n.$$

Under the policy $\pi^{(a)}$ a reward rate of 1 is accrued at times $t$ during which $M(t) > 0$. Further, switching costs are incurred at certain transition times of $M(t)$. Specifically at times $t$ when $M(t^-) = 0$ and $M(t) = 1$ a switching cost $c$ is incurred with probability $(n-1)/n$. This is the probability that the channel that changes to 'good' is not the currently selected channel and therefore an immediate channel change takes place. Similarly, at times $t$ when $M(t^-) = k \geq 2$ and $M(t) = k-1$, a cost of $c$ is incurred with probability $1/k$. This is the probability of the current channel turning 'bad' and hence requiring a switch. Now considering the reward as a Markov reward process, we have that average under policy $\pi^{(a)}$ is,

$$\begin{aligned} g &= (1-\pi_0) - c\frac{n-1}{n}\lambda_0\pi_0 - c\sum_{k=2}^{n}\frac{1}{k}\mu_k\pi_k \\ &= 1 - (1-\gamma)^n - c\frac{1-\gamma-(1-\gamma)^n}{\gamma}, \end{aligned}$$

where the second line follows from the structure of the birth and death rates (3.5). Further with $\pi^{(s)}$ we have that $g$ is trivially $\gamma$. Hence $\pi^{(a)}$ is preferable only when $c \leq \gamma$. $\square$

## 3.3 Renewal Reward Analysis

In this section we prove Theorem 8 via a regenerative analysis for $n = 2$ and later for $n = \infty$. We consider the regenerative structure based on regeneration points where switching has just occurred into a bad channel ($= 0$). That is, take the time-axis and consider time points at which the controller switched (from a bad channel) into a bad channel. These are regeneration points which we denote by $T_0 = 0, T_1, T_2, \ldots$. We can assume that at time $t = 0$ the system starts at such a point because we are looking at the infinite horizon average cost. Note that between $T_{n-1}$ and $T_n$ it is possible that there were other switches — namely switches into a channel that is in good state (= 1).

The net reward obtained during the interval $(T_{n-1}, T_n]$, is given by

$$V_n = \int_{T_{n-1}}^{T_n} \tilde{I}(u)\, du - c\big(\tilde{N}(T_n) - \tilde{N}(T_{n-1})\big).$$

That is the total time during the interval when the channel was in good state less the switching costs.

From the regenerative property of the process, we can conclude that the sequence $\big(T_n - T_{n-1}, V_n\big)$ is an i.i.d. sequence. We denote by $W$ a generic random variable distributed as $T_n - T_{n-1}$ and by $V$ a generic random variable distributed as $V_n$. It then follows from the "Renewal Reward Theorem", see for example Theorem 1.2, Chap VI [8], that the average reward is given by

$$g = \frac{\mathbb{E}[V]}{\mathbb{E}[W]}\,. \tag{3.6}$$

Note that by construction $W$ is non-lattice and both $V$ and $W$ have a finite mean. Hence our goal is now to compute the expectation of $V$ and $W$ under a call-gapping policy with parameter $\tau$. For $n = 2$ or $n = \infty$ this is equivalent to the cool-off policy with $\sigma = \tau$.

We now construct two generic random variables via their probability distributions. We denote these as $W_0$ and $W_1$ and their CDFs by $F_{W_i}(t)$ for $i = 0, 1$. We have,

$$F_{W_i}(t) = \mathbb{P}(W_i \le t) = \begin{cases} 0, & t < \tau, \\ (1 - p(\tau\,;\, i)) + p(\tau\,;\, i)(1 - e^{-\frac{1-\gamma}{\gamma}(t-\tau)}), & \tau \le t. \end{cases} \tag{3.7}$$

These random variables are mixtures of a mass at $\tau$ and a shifted exponential. The random variable $W_i$ denotes the time of switching under a policy with call-gapping parameter $\tau$ when at time 0 a switch occurred into a channel in state $i$. It is constructed by observing that if at time $t = \tau$ the state is 0 we switch with probability $1 - p(\tau\,;\,i)$; otherwise we wait an exponentially distributed duration until switching. Note that,

$$\mathbb{E}[W_i] = \tau + p(\tau\,;\,i)\frac{\gamma}{1-\gamma}. \tag{3.8}$$

With the generic random variables $W_0$ and $W_1$ at hand, we can analyse a complete regenerative cycle of duration $W$. For this denote,

$$W = \widetilde{W}_0 + \widetilde{W}_1 + \widetilde{W}_2 + \ldots + \widetilde{W}_M,$$

with $\widetilde{W}_i$ denoting the inter-switching times within the cycle. Here $M$ is a random variable with support $0, 1, 2, \ldots$, denoting the number of times we switch into a good channel within such a cycle. By construction that,

$$\widetilde{W}_0 \;=^d\; W_0, \qquad\qquad \widetilde{W}_i \;=^d\; W_1, \;\; i = 1, 2, \ldots.$$

This follows since our regeneration points are the time points at which the controller switches from a bad channel into a bad channel. Then each cycle starts with a duration distributed as $W_0$. It is then followed by $M$ additional durations which are switches from bad states to good states, each distributed as $W_1$.

The following two lemmas yield explicit expressions for the denominator and numerator of (3.6).

**Lemma 1.** *The denominator of* (3.6) *can be represented as,*

$$\begin{aligned}\mathbb{E}[W] &= \mathbb{E}[W_0] + \frac{\mathbb{E}[p(W_0\,;\,0)]}{1 - \mathbb{E}[p(W_1\,;\,0)]}\mathbb{E}[W_1] \\ &= \frac{e^{\frac{2\tau}{\gamma}}(\gamma^3 - (\tau+2)\gamma^2 + 3\tau\gamma - 2\tau) - \gamma^3 - (\tau-2)\gamma^2 + \tau\gamma}{(\gamma-1)^2(e^{\frac{2\tau}{\gamma}}(\gamma-2) - 2e^{\frac{\tau}{\gamma}}\gamma + \gamma)}.\end{aligned}$$

□

**Proof:**

Observe that,

$$W =^d \widetilde{W}_0 + \widetilde{I}_0\widetilde{W}_1 + \widetilde{I}_0\widetilde{I}_1\widetilde{W}_2 + \cdots + \widetilde{I}_0\widetilde{I}_1 \cdots \widetilde{I}_{n-1}\widetilde{W}_n + \cdots ,$$

where the indicator, $\widetilde{I}_i$, for $i = 0, 1, 2, \ldots$ is equal to 1 if the jump at the end of the interval associated with $\widetilde{W}_i$ was to a good channel. Note that the random variables $\widetilde{I}_i$ for $i = 1, 2, \ldots$ are identically distributed and $\widetilde{I}_0$ follows a different distribution. We can now denote generic random variables, $I_0$ and $I_1$ satisfying

$$\widetilde{I}_0 =^d I_0 , \qquad \widetilde{I}_i =^d I_1, \ i = 1, 2, \ldots .$$

For these two random variables, by conditioning on $W_i$ (for $i = 0, 1$), we have

$$\begin{aligned}
\mathbb{E}[I_i] &= \mathbb{E}[\mathbb{E}[I_i \mid W_i]] \\
&= \mathbb{E}[p(W_i\,;0)] \\
&= (1 - p(\tau\,;i))p(\tau\,;0) + p(\tau\,;i) \int_\tau^\infty p(t\,;0) \frac{1-\gamma}{\gamma} e^{-\frac{1-\gamma}{\gamma}(t-\tau)}\, dt \\
&= \gamma - \frac{e^{-\frac{\tau}{\gamma}}\gamma(\gamma + p(\tau\,;i) - 2)}{\gamma - 2},
\end{aligned} \tag{3.9}$$

where the second step follows because every switch is out of a bad channel and the third step follows from (3.7).

Observe that the sequence $\{\widetilde{I}_i\}$ is a mutually independent sequence of random variables and for each $i$, $\widetilde{I}_i$ is independent of $\widetilde{W}_j$ for all $j \neq i$. Based on the fact that $\mathbb{E}[\widetilde{W}_i]$ is same as $\mathbb{E}[W_1]$ for $i = 1, 2, \ldots$, we have

$$\begin{aligned}
\mathbb{E}[W] &= \mathbb{E}[W_0] + \mathbb{E}[I_0]\mathbb{E}[W_1] + \mathbb{E}[I_0]\mathbb{E}[I_1]\mathbb{E}[W_1] + \ldots + \mathbb{E}[I_0](\mathbb{E}[I_1])^{n-1}\mathbb{E}[W_1] + \cdots \\
&= \mathbb{E}[W_0] + \mathbb{E}[I_0]\frac{1}{1 - \mathbb{E}[I_1]}\mathbb{E}[W_1] .
\end{aligned} \tag{3.10}$$

After manipulation using (3.2), (3.8), and (3.9), the result follows. □

**Lemma 2.** *The numerator,* $\mathbb{E}[V]$ *of* (3.6) *can be represented as*

$$\mathbb{E}[V] = \mathbb{E}[R] - c\,(\mathbb{E}[M] + 1),$$

*where,*

$$\begin{aligned}\mathbb{E}[R] &= \mathbb{E}[R_0] + \frac{\mathbb{E}[p(W_0\,;0)]}{1-\mathbb{E}[p(W_1\,;0)]}\mathbb{E}[R_1] \\ &= \frac{e^{\frac{2\tau}{\gamma}}((1-\tau)\gamma^3 + (3\tau-2)\gamma^2 - 2\tau\gamma) + 2e^{\frac{\tau}{\gamma}}\gamma^2(\gamma-1)^2 - 2\gamma^4 + (3-\tau)\gamma^3 + \tau\gamma^2}{(\gamma-1)^2(e^{\frac{2\tau}{\gamma}}(\gamma-2) - 2e^{\frac{\tau}{\gamma}}\gamma + \gamma)},\end{aligned}$$

*and,*

$$\begin{aligned}\mathbb{E}[M] &= \frac{\mathbb{E}[p(W_0\,;0)]}{1-\mathbb{E}[p(W_1\,;0)]}, \\ &= \frac{e^{\frac{2\tau}{\gamma}}(2-\gamma) - \gamma}{(e^{\frac{2\tau}{\gamma}}(\gamma-2) - 2e^{\frac{\tau}{\gamma}}\gamma + \gamma)(\gamma-1)} - 1.\end{aligned}$$

□

**Proof:** The proof follows similar lines to the previous proof. We have that $V = R - c\,(M+1)$ with $M$ as defined above and the reward $R$ being the net time during $[0, W]$ in which a good channel is selected. As with the analysis of the cycle duration, $W$, we denote,

$$R = \widetilde{R}_0 + \widetilde{R}_1 + \widetilde{R}_2 + \ldots + \widetilde{R}_M,$$

where $\tilde{R}_i$ is the reward accumulated over the period corresponding to $\tilde{W}_i$. Similar to the previous analysis, we construct generic random variables $R_0$ and $R_1$ with

$$\widetilde{R}_0 \stackrel{d}{=} R_0, \qquad \widetilde{R}_i \stackrel{d}{=} R_1, i = 1, 2, \ldots.$$

The expectations of these can be computed to be

$$\begin{aligned}\mathbb{E}[R_0] &= \int_0^\tau p(t\,;0)dt + p(\tau\,;0)\frac{1-\gamma}{\gamma} = \gamma(\tau - p(\tau\,;0)) + \frac{p(\tau\,;0)(1-\gamma)}{\gamma}, \\ \mathbb{E}[R_1] &= \int_0^\tau p(t\,;1)dt + p(\tau\,;1)\frac{1-\gamma}{\gamma} = \gamma(\tau - p(\tau\,;1) + 1) + \frac{p(\tau\,;1)(1-\gamma)}{\gamma}.\end{aligned}$$

Using $\tilde{I}_i$ as in the previous proof, we observe,

$$R =^d \widetilde{R}_0 + \widetilde{I}_0\widetilde{R}_1 + \widetilde{I}_0\widetilde{I}_1\widetilde{R}_2 + \cdots + \widetilde{I}_0\widetilde{I}_1 \cdots \widetilde{I}_{n-1}\widetilde{R}_n + \cdots ,$$

where again for any $i$, $\widetilde{I}_i$ is independent of $\widetilde{R}_j$ for $j \neq i$. Now a similar geometric series to (3.10) is applied. The expression for $M$ is computed in a similar manner by observing

$$M =^d \widetilde{I}_0 + \widetilde{I}_0\widetilde{I}_1 + \widetilde{I}_0\widetilde{I}_1\widetilde{I}_2 + \ldots .$$

□

We can now prove Theorem 8 for $n = 2$

*Proof:* Combining and manipulating the explicit expressions from the lemmas above we obtain that under a policy $\pi^{(\tau^*)}$ (alt. $\pi^{(\sigma^*)}$), the reward as in (3.6) as a function of $\tau$ (alt. $\sigma$) is,

$$g(\tau) = \frac{A_1(\gamma,\tau) - c\, A_2(\gamma,\tau)}{A_3(\gamma,\tau)}. \tag{3.11}$$

where $A_i(\cdot,\cdot)$, $i = 1,2,3$ are as defined in Theorem 8. We consider first the case $c \geq \gamma^2$. Observe that at $c = \gamma^2$,

$$g(\tau)\Big|_{c=\gamma^2} = \gamma - \frac{2e^{\frac{\tau}{\gamma}}(\gamma-1)^2\gamma^2}{A_3(\gamma,\tau)} < \gamma$$

and hence the $\pi^{(s)}$ policy is preferable to the $\pi^{(\tau)}$ policy for any finite $\tau$ (note that as $\tau \to \infty$ the inequality above becomes an equality). Further $g(\tau)$ is monotonically decreasing in $c$ and hence for $c > \gamma^2$ it remains optimal to use $\pi^{(s)}$.

Moving to the $c < \gamma^2$ case, we optimize the (continuous and differentiable) function $g(\tau)$ over $(0,\infty)$ to obtain equation (3.3) from Theorem 8. We do this by representing the derivative as $g'(\tau) = h(\tau)f(\tau)$ where,

$$h(\tau) = \frac{(\gamma-1)^2(e^{\frac{2\tau}{\gamma}}(2-\gamma)+\gamma)}{(\gamma((\gamma-2)\gamma+\tau(1-\gamma))+e^{\frac{2\tau}{\gamma}}(2-\gamma)(\gamma^2+\tau-\tau\gamma))^2},$$

and,

$$f(\tau) = e^{\frac{2\tau}{\gamma}}(\gamma^2 - c)(\gamma - 2) + 2e^{\frac{\tau}{\gamma}}\gamma(\gamma - \tau(\gamma - 1)) - \gamma(\gamma^2 - c).$$

It can be shown that for $c < \gamma^2$ the derivative $f'(\tau) < 0$. Now since $h(\cdot) > 0$ we have that, $g'(\tau) = 0$ if and only if $f(\tau) = 0$ and equation (3.3) is $f(\tau^*) = 0$. Since $f(0) > 0$ and $\lim_{\tau\to\infty} f(\tau) < 0$, it is evident that for $c < \gamma^2$ there is a single root to $f(\tau) = 0$ and further at the solution $\tau^*$ the second order conditions hold,

$$g''(\tau^*) = 2h(\tau^*)\Big((1 - e^{\frac{\tau^*}{\gamma}})\gamma^2 - c - e^{\frac{\tau^*}{\gamma}}\tau^*(1-\gamma)\Big) < 0.$$

□

Note that at the extremes of $\tau$, the reward $g(\tau)$ as in (3.11) yields the expected results. With zero switching costs,

$$\lim_{\tau\to 0} g(\tau)\Big|_{c=0} = \lim_{\tau\to 0}\frac{A_1(\gamma,\tau)}{A_3(\gamma,\tau)} = 1 - (1-\gamma)^2,$$

and further without switching,

$$\lim_{\tau\to\infty}\frac{A_1(\gamma,\tau) - cA_2(\gamma,\tau)}{A_3(\gamma,\tau)} = \lim_{\tau\to\infty}\frac{A_1(\gamma,\tau)}{A_3(\gamma,\tau)} - c\,0 = \gamma.$$

We can derive the results for $n = \infty$ using a similar (yet simpler) analysis. As described in Section 3.2 the system may be viewed as a two channel system where every transition is always to a steady state channel. The usage of the transient probabilities $p(\tau\ ;\ i)$ as in (3.7) is then replaced by $\gamma$. This results in correspondently simpler expressions in all the cases where transient probabilities are used ($n = 2$) and can now be replaced by $\gamma$. The resulting reward expression is then,

$$g(\tau) = \frac{(\gamma-1)c + \gamma(\gamma - \gamma\tau + \tau)}{\gamma^2 - \gamma\tau + \tau},$$

with derivative,

$$g'(\tau) = \frac{(\gamma-1)^2\left(c - \gamma^2\right)}{\left(\gamma^2 - \gamma\tau + \tau\right)^2}.$$

This shows that $g(\cdot)$ is monotonic decreasing in $\tau$ if $c < \gamma^2$ and monotonic increasing in $\tau$

if $c > \gamma^2$. Hence for $c < \gamma^2$ the optimal switching parameter is $\tau^* = 0$ and yields reward as in Theorem 8. Further, since $\lim_{\tau\to\infty} g(\tau) = \gamma$, for $c > \gamma^2$ it is not optimal to use $\pi^{(\tau)}$ for any finite $\tau$, and $\pi^{(s)}$ is preferable.

## 3.4 Numerical Results

We now carry out numerical experiments for our system under various policies, focusing on Case II. We first describe the numerical computations and simulation experiment used for Figure 3.1 and then expand with further numerical results.

The Case I curves in the figure are obtained directly from Theorem 7. The Case II curves for $n = \infty$ are created directly using Theorem 8 and the $n = 2$ curves are created by numerically solving equation (3.3) of Theorem 8 (using the bisection method). The remaining curves for $n = 3$ and $n = 4$ using both the call-gapping and the cool-off policies are obtained via Monte-Carlo simulation and optimization to find the optimal $\tau$ and $\sigma$ parameters for these policies.

This Monte-Carlo simulation (as well as other code) can be found in the GitHub repository [85]. We simulated the system over a grid of $c$ in the range $[0, \gamma^2]$ where $\gamma^2 = 0.16$, with a spacing of 0.005. Then for each value of $c$ we simulated both the call-gapping and cool-off policies over respective grids of $\tau$ and $\sigma$ in the range $[0, 1.55]$, each with a spacing of 0.001. In each of these individual simulation runs we kept a constant seed to reduce variability in the curves using common random numbers, and simulated for a time horizon of $T = 10^5$, with an arbitrary fixed initial condition. We then obtained the average reward $g$ using a crude Monte Carlo estimator over the continuous time $[0, T]$ and then plotted the reward for the optimal $\tau$ and $\sigma$ in each case.

In addition to the simulations leading to Figure 3.1 we also carried out extensive additional numerical experiments for further parameter values. Key results from these experiments are summarized in Table 3.1. Our purpose is to compare the cool-off and call-gapping policies for finite $n > 2$. As we observe from Figure 3.1, the cool-off policy is able to achieve a slightly higher average reward for $\gamma = 0.4$. We then considered the system for several additional $\gamma$ values and several values of $n$.

In each case, the cool-off policy is indeed slightly better than call-gapping. The numerical results in Table 3.1 report the maximal gap, as well as error estimates, over a range of $c$ values between the rewards of the policies. We also present an estimate of the $c$ value in which the gap is maximal and highlight in bold the values of $n$ for which the gap is largest[1]. As an overarching summary, we see that while the cool-off policy is slightly better, the difference between the rewards is never high and hence in practice using the simpler call-gapping policy is probably preferable. As expected from Figure 3.1, for each value of $\gamma$ there is some finite $n$ in which the maximal gap is highest (for $n = 2$ and $n \to \infty$ there is no gap as the policies are identical). It is also evident that for low $\gamma$ values, the worst case $n$ is higher than that of higher $\gamma$ values. This observation informed our selection of the range of $n$ for which to run the experiments.

The computational experiments required non-negligible computation time because for any system setting $(\gamma, n, c)$ optimization over the best $\tau$ (resp. $\sigma$) parameter was required. We thus relied on an empirical observation associated with the nature of the system. We observed that for any fixed $\gamma$ and $c$, as $n$ increases, the optimal parameter ($\tau^*$ or $\sigma^*$) decreases with $n$ (with the optimal value tending to 0 as $n \to \infty$). This allowed us to use Theorem 8 to first compute the optimal parameter for $n = 2$ and use it as an upper bound for the (stochastic) search for the optimal parameter for $n = 3$. Further for any $n = k \geq 4$, we used the estimated optimal parameter of $n = k - 1$ to determine an upper bound. This allowed us to use a fixed size search grid of size 20 for the optimal $\tau$ (resp. $\sigma$) of each level $n \geq 3$. Other elements of the simulation involved the search grid over $c$. For this, after initial experimentation indicating the location of the maximal gap, we always considered $c \in [0, \gamma^2/2]$ and searched over 25 points within this grid. Each sample path involved a time horizon of $10^3$ time units, and for each parameter setting $(\gamma, n, c)$, we simulated 100 repetitions, allowing us to obtain 95% error bounds using a standard normal approximation. The simulation duration was in the order of 24 hours on a standard laptop using the (compiled) Julia language.

In addition to the simulations, we also carried out a robustness analysis for Case II with $n = 2$. In this case, the optimal call-gapping (and cool-off) parameter $\tau$ is easily

[1] For $\gamma = 0.2$ the maximal gap is at either $n = 14$, $n = 15$, or $n = 16$ and in considering the error estimates, the exact value is not determined based on the simulations we ran.

Table 3.1: Evaluating the difference between the cool-off and call-gapping policy.

| System | Number of channels | Maximal Gap | Worst cost |
|---|---|---|---|
| $\gamma = 0.2$ | $n = 3$ | $0.00385 \pm 0.00029$ | $c = 0.0048$ |
| | $n = 4$ | $0.00823 \pm 0.00033$ | $c = 0.0048$ |
| | $n = 5$ | $0.01295 \pm 0.00045$ | $c = 0.0056$ |
| | $n = 6$ | $0.01767 \pm 0.00045$ | $c = 0.0056$ |
| | $n = 7$ | $0.02188 \pm 0.00044$ | $c = 0.0056$ |
| | $n = 8$ | $0.02619 \pm 0.00047$ | $c = 0.0064$ |
| | $n = 9$ | $0.02879 \pm 0.00051$ | $c = 0.0072$ |
| | $n = 10$ | $0.03136 \pm 0.00057$ | $c = 0.0072$ |
| | $n = 11$ | $0.03277 \pm 0.00047$ | $c = 0.0072$ |
| | $n = 12$ | $0.03414 \pm 0.0006$ | $c = 0.008$ |
| | $n = 13$ | $0.03554 \pm 0.00048$ | $c = 0.008$ |
| | $\mathbf{n = 14}$ | $\mathbf{0.03609 \pm 0.0006}$ | $c = 0.0088$ |
| | $\mathbf{n = 15}$ | $\mathbf{0.03611 \pm 0.00072}$ | $c = 0.0096$ |
| | $\mathbf{n = 16}$ | $\mathbf{0.03611 \pm 0.00061}$ | $c = 0.0104$ |
| | $n = 17$ | $0.03532 \pm 0.00064$ | $c = 0.0112$ |
| | $n = 18$ | $0.0339 \pm 0.00066$ | $c = 0.0112$ |
| | $n = 19$ | $0.0332 \pm 0.00064$ | $c = 0.0112$ |
| | $n = 20$ | $0.03208 \pm 0.0007$ | $c = 0.0128$ |
| $\gamma = 0.3$ | $n = 3$ | $0.00679 \pm 0.00035$ | $c = 0.009$ |
| | $n = 4$ | $0.01318 \pm 0.00044$ | $c = 0.009$ |
| | $n = 5$ | $0.01927 \pm 0.00063$ | $c = 0.018$ |
| | $n = 6$ | $0.02351 \pm 0.00055$ | $c = 0.0144$ |
| | $n = 7$ | $0.0268 \pm 0.00074$ | $c = 0.018$ |
| | $n = 8$ | $0.02829 \pm 0.00073$ | $c = 0.0198$ |
| | $\mathbf{n = 9}$ | $\mathbf{0.03012 \pm 0.00072}$ | $c = 0.0216$ |
| | $n = 10$ | $0.02924 \pm 0.00056$ | $c = 0.0216$ |
| | $n = 11$ | $0.0293 \pm 0.00084$ | $c = 0.0252$ |
| | $n = 12$ | $0.02798 \pm 0.00096$ | $c = 0.0288$ |
| | $n = 13$ | $0.026 \pm 0.00078$ | $c = 0.0288$ |
| | $n = 14$ | $0.02524 \pm 0.00078$ | $c = 0.0306$ |
| | $n = 15$ | $0.02216 \pm 0.00074$ | $c = 0.027$ |
| $\gamma = 0.4$ | $n = 3$ | $0.00899 \pm 0.0005$ | $c = 0.0224$ |
| | $n = 4$ | $0.01606 \pm 0.00066$ | $c = 0.032$ |
| | $n = 5$ | $0.02114 \pm 0.00064$ | $c = 0.0256$ |
| | $n = 6$ | $0.02289 \pm 0.00072$ | $c = 0.032$ |
| | $\mathbf{n = 7}$ | $\mathbf{0.02426 \pm 0.00071}$ | $c = 0.0416$ |
| | $n = 8$ | $0.02294 \pm 0.0007$ | $c = 0.0352$ |
| | $n = 9$ | $0.0213 \pm 0.00074$ | $c = 0.0448$ |
| | $n = 10$ | $0.01849 \pm 0.00076$ | $c = 0.0544$ |
| $\gamma = 0.6$ | $n = 3$ | $0.01001 \pm 0.00053$ | $c = 0.0648$ |
| | $\mathbf{n = 4}$ | $\mathbf{0.01274 \pm 0.00055}$ | $c = 0.0648$ |
| | $n = 5$ | $0.01267 \pm 0.00064$ | $c = 0.1008$ |
| $\gamma = 0.8$ | $\mathbf{n = 3}$ | $\mathbf{0.00452 \pm 0.00042}$ | $c = 0.1792$ |
| | $n = 4$ | $0.00389 \pm 0.0005$ | $c = 0.2432$ |
| | $n = 5$ | $0.00236 \pm 0.00035$ | $c = 0.2048$ |

obtained via equation (3.3). However we also wished to see how misspecification of the system parameter $\gamma$, leading to a misspecification of the optimal $\tau$, affects performance. For this we again considered the case where $\gamma = 0.4$, however we now assumed that the system is perceived to operate at a potentially different system parameter, namely $\hat{\gamma}$. The controller then optimizes $\tau$ based on (3.3) and the perceived value, $\hat{\gamma}$. We then compared the ratio of the actual reward, $\hat{g}$ and the ideal reward, $g$ obtained using $\gamma = 0.4$. Both were obtained via the expressions in Theorem 8. The ratio is plotted in Figure 3.2 where we considered cost values $c = 0.04, 0.08, 0.12$, and $0.16$.

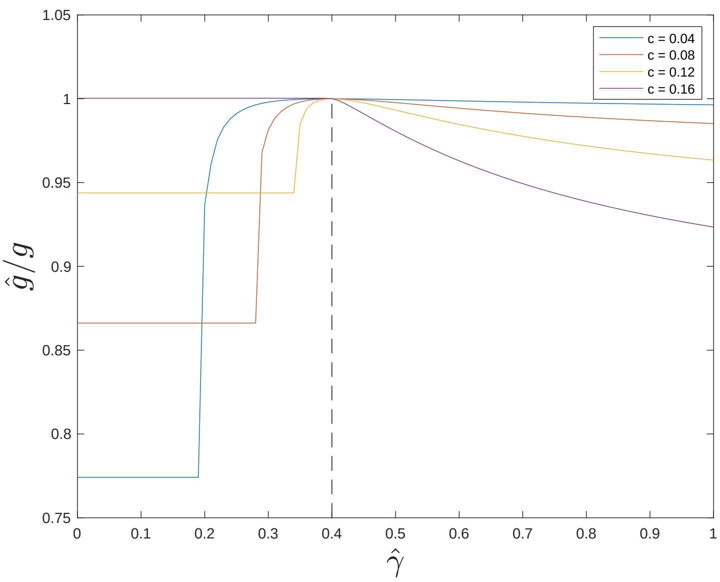



Figure 3.2: Robustness analysis when $\gamma = 0.4$ for Case II and $n = 2$ under call-gapping. The system is controlled with a perceived value of $\hat{\gamma}$ and the ratio of the reward and the optimal reward is plotted for several values of $c$.

As we see in Figure 3.2, as we might expect, for larger deviations of $\hat{\gamma}$ from $\gamma = 0.4$ we saw larger relative losses of the reward. However, the relative loss was not more than a few percentage points in most cases. The plateaus observed on each of the curves are due to values of $\hat{\gamma}$ during which $c > \hat{\gamma}^2$ in which case the $\pi^{(s)}$ policy was employed.

## 3.5 Conclusion and Extensions

Our aim with this study was to obtain a qualitative view on the interaction of the number of channels, the available information, and the switching costs in a channel selection context. In such a setting, system designers can consider the value of information as well as the effect of switching costs (the value of efficient switching) in lieu of various control policies. Through a simple model, we obtained a qualitative relationship between the various system factors and in certain cases we were able to explicitly analyze the system.

The case of full observation is straightforward to analyze, however for the case of partial observation, to the best of our knowledge, explicit analysis is only attainable with $n = 2$ and $n = \infty$ as we have done here. This is for the call-gapping and cool-off policies that we introduced. We carried out further numerical analysis for other small finite $n$ illustrating that in practice there are only minor efficiency gains to be had by using a cool-off instead of a call-gapping. This is despite the fact that the call-gapping policy only requires local information of the current state of the channel while the cool-off policy requires information for all channels. We complemented our analysis with a numerical robustness experiment hinting that inexact knowledge of system parameters would not hinder performance greatly when using a call-gapping policy.

Our work did not focus on optimality of the policies discussed within the class of all possible policies. We conjecture that for $n = 2$ call-gapping (equivalent to cool-off) is optimal in the case of partial observation. However proving this remains a challenge for a future study. Further, real systems will typically exhibit a more complicated structure than two-state Markov chains. For such systems, adaptations of the call-gapping and cool-off policies may still be employed, however analysis is more complicated. Nevertheless, we believe that our qualitative and quantitative results based on two state Markov chains may help serve as a guide for the tradeoffs between information, switching costs, policy complexity, and the number of channels in a system.

# Chapter 4

# The Value of Communication and Cooperation in a Two-Server Service System

## 4.1 Introduction

Numerous workplaces require servers to complete crucial and highly technical tasks. These tasks may be subject to service failure and incur operational fees, so a server may not be willing to complete them. For example, controlling the spread of highly infectious diseases or managing the disposal of radioactive waste always demands highly skilled staff, and there is the possibility of service failure. The system manager aims to accomplish as much work as possible while minimising the overall operating cost, in a situation where servers have a tendency to avoid a job being allocated to them and to let their co-workers undertake the task. Such a situation has been studied in telecommunications analysis, in particular for a network layer with distributed routing selections [89], a data link layer with multiple access techniques [16], and devices powered by distributed energy sources [64]. Game theory, which has already been applied to study telecommunication problems [34], provides a mathematical framework with which to model and analyse this situation since it considers the behaviour of individuals given that everyone's best strategy is affected by the strategies adopted by the other participants in the system. In this chapter we shall take another look at the system that was studied by Guglielmi and Badia [33] and derive somewhat different results.

The chapter is organised as follows. In Section 4.2 we describe the strategic server system proposed by Guglielmi and Badia [33], and compare our analysis with theirs. In Section 4.3 we introduce some preliminaries on game theory and how we use them to obtain the main results. In Section 4.4 we derive the optimal strategy if each server has complete knowledge of the other's probability of successful transmission $p_i$ and they are willing to cooperate. In Section 4.5 we obtain the unique threshold type Nash equilibrium strategy [40, Section 1.1.6] under the scenario in which each server is informed only that the other server is using a threshold strategy, and what the threshold is. In Section 4.6 we describe the Nash equilibrium strategy in the case when both servers know each other's probability of successful transmission $p_i$, but individuals maximise their own payoffs without considering the effect on the other server. The chapter concludes with Section 4.7 where we summarise our results and propose some directions for future research.

## 4.2 The strategic server system of Guglielmi and Badia

Guglielmi and Badia [33] proposed and analysed a transmission system of two servers. Each server has to make a decision whether to be *active*, that is available for the service, or *inactive*, that is not available for the service. A server who is available for the service will be chosen to undertake the service via a rule that we shall discuss below.

For $i = 1, 2$, we shall use $-i$ to denote server $j \neq i$. For $i = 1, 2$, server $i$'s probability of successfully transmitting a packet when it is active is $p_i$. If $p_i$ is unobservable to server $-i$, then from the point view of server $-i$, $p_i$ is a random variable. we assume that $p_i$s are assumed to be independent and identically distributed according to a continuous uniform distribution on $[0, 1]$, and each server knows its own $p_i$. We call $(p_1, p_2)$ the *state* of the system, and so $[0, 1] \times [0, 1]$ is the *state space* of the system. The model assumes that a successful transmission gains a reward of 1 for each server, whether or not they were the one to undertake the service. The one-off cost of being active is $c \in [0, 1]$. If both servers are active, the service is allocated to the one with the higher probability of successful transmission.

The servers' expected payoffs, which depend on their decisions and $(p_1, p_2)$, have four different cases. If both servers are inactive, both payoffs are 0 because the service is not

Table 4.1: Expected payoff matrix of the game.

| | | server 2 | |
|---|---|---|---|
| | | *active* | *inactive* |
| server 1 | *active* | $(\max\{p_1, p_2\} - c, \max\{p_1, p_2\} - c)$ | $(p_1 - c, p_1)$ |
| | *inactive* | $(p_2, p_2 - c)$ | $(0, 0)$ |

completed. If server $i$ is active and server $-i$ is inactive, the expected payoff to server $i$ is $p_i - c$ and to server $-i$ is $p_i$. This is because the probability of successful transmission is $p_i$ in this case, so both servers gain a reward of 1, while server $i$ needs to pay a cost $c$ for being active; if both servers are active, both expected payoffs are $\max\{p_1, p_2\} - c$, because now the probability of successful transmission is $\max\{p_1, p_2\}$ and both servers need to pay $c$ for being active. The expected payoff matrix is shown in Table 4.1, where the expected payoffs to servers 1 and 2 are given by the first and second coordinates, respectively.

Guglielmi and Badia [33] applied game theory to analyse four scenarios.

- In Scenario 1, each server does not know the other's probability of successful transmission. They claimed that the Nash equilibrium strategy is of threshold type. That is, server $i$ chooses to be active when $p_i \geq p_i^*$, $i = 1, 2$, and the threshold values $p_1^*$ and $p_2^*$ must satisfy $p_1^* p_2^* = c$.

- In Scenario 2, they examined two special cases of Scenario 1 where one of the two servers is always inactive. They claimed that the two cases are the worst case allocation with respect to social welfare.

- In Scenario 3, both servers have full information about each other's $p_i$. They obtained results similar to those in Scenario 1. Each server chooses to be active only if $p_i \geq p_i^*$, $i = 1, 2$, but server 1 chooses to be inactive when $p_1^* \leq p_1 \leq \frac{p_1^* - 1}{p_2^* - 1} p_2 - \frac{p_1^* - p_2^*}{p_2^* - 1}$, and server 2 chooses to be inactive when $p_2^* \leq p_2 \leq \frac{p_2^* - 1}{p_1^* - 1} p_1 - \frac{p_2^* - p_1^*}{p_1^* - 1}$.

- In Scenario 4, coordinated servers have complete knowledge about each other's $p_i$. They derived the unique Nash equilibrium strategy which is of threshold type with threshold value $c$. That is, each server chooses to be active if and only if $p_i \geq c\,, i = 1, 2$.

Unfortunately, the analysis of [33] contained errors. In this chapter, we correct the errors and change the order so that we look at the most straightforward case first:

- In Case I, which corresponds to Scenario 4, both servers have full knowledge of the other's probability of successful transmission $p_i$ and both try to optimise social welfare. The best strategy is of threshold type, but the threshold value is $\frac{c}{2}$ instead of $c$. That is, each server chooses to be active if and only if $p_i \geq \frac{c}{2}\,, i = 1, 2$.

- In Case II, which corresponds to Scenario 1, each server does not know the other's probability of successful transmission. The threshold strategy with threshold value $\sqrt{c}$ is a Nash equilibrium strategy. That is, if both servers choose to be active when $p_i \geq \sqrt{c}\,, i = 1, 2$, neither server has an incentive to deviate. In addition, we prove this is the only threshold Nash equilibrium strategy. This is different from the conclusion stated by Guglielmi and Badia [33] that all pairs satisfying $p_1^* p_2^* = c$ are Nash equilibria. Also, we calculate the expression for social welfare for the threshold strategies and obtain the best case allocation. It is clear from the expression that the worst case is not necessarily when one of the two servers is always inactive as stated in Scenario 2.

- In Case III, which corresponds to Scenario 3, each server knows the other's probability of successful transmission. We show that server $i$ chooses to be active only if $p_i \geq c\,, i = 1, 2$, and there is a region in the parameter space where there are multiple Nash equilibria. The regions where each server is active in our analysis is different from that in [33], and we also obtain a mixed Nash equilibrium strategy in a specific region.

- For both Case II and Case III, we propose regulations and prove that by imposing these regulations, we can eliminate the social inefficiency caused by noncooperation.

## 4.3 Preliminaries

In game theory, an action profile $(a_1, a_2)$ represents the actions adopted by both servers, and it could be pure or mixed. If, for $i = 1, 2$, we let $A_i = \{active, inactive\}$, then for server $i$, a pure strategy is $a_i \in A_i$, and a mixed strategy is $a_i = \sigma_i$ where $\sigma_i$ is the probability that server $i$ is active. A Nash equilibrium is an action profile where no server has an incentive to deviate unilaterally. We use $s = (s_1, s_2)$ to denote the strategy on the whole state space, and $BR_i(s_{-i})$ to denote server $i$'s best response given server $-i$ adopts strategy $s_{-i}$. In many queueing models, $s_i$ can be represented by a single numerical value [40, Section 1.1.6]. In this chapter, we let $(\underline{p_1^*}, \underline{p_2^*})$ denote a threshold strategy where server 1 chooses to be active if and only if $p_1 \geq p_1^*$, and server 2 is active if and only if $p_2 \geq p_2^*$, where $0 \leq p_1^*, p_2^* \leq 1$. In Section 4.5, we use $(a_1, \underline{p_2^*})$ to represent the situation where server 1 takes action $a_1$ and server 2 adopts the threshold strategy with threshold value $p_2^*$. For a given strategy $s$, we define the *social welfare* $S_O(s)$ to be the expected value of the sum of the two servers' payoffs. We define the social optimum as the maximum social welfare under certain assumptions.

A system that is in a Nash equilibrium is not necessarily at a social optimum, but we can sometimes regulate the game to adjust a Nash equilibrium to match the strategy of the optimal social welfare. For example, a queue could be regulated by announcing an admission fee [66] or imposing a toll on waiting [40, p24]. Li [57] generalised the setup in Guglielmi and Badia [33] by introducing an incentive parameter $b$ to make the payment system fairer. When a service is completed, the server who performs the service is awarded 1, and the server who does not perform the service is awarded $b$ $(\leq 1)$. So the setup of Guglielmi and Badia [33] had $b = 1$. Li [57] showed that by introducing $b$, the servers are encouraged to stay in the system.

We apply game theory to analyse this model. More specifically, we employ *Bayesian* *game theory* [72, Section 2.6] to examine Case II where both servers are not informed of the other's probability of successful transmission, but each is aware that the other is adopting a threshold strategy and what the threshold value is. That is, for $i = 1, 2$, server $i$ does not know $p_{-i}$, but knows server $-i$ is active if and only if $p_{-i} > p^*_{-i}$ and what $p^*_{-i}$ is. A Bayesian game is designed to analyse the situation with imperfect information and models servers' information about the state of nature by prior belief and its type. The type is the signal that each server observes. We let $p_i$ denote server $i$'s type, and $U_i((a_i, \underline{p^*_{-i}}), p_i)$ denote its payoff. That is, server $i$'s payoff depends on its type $p_i$ and the strategy $(a_i, \underline{p^*_{-i}})$. We prove that in this situation, the Nash equilibrium strategy is unique and also of threshold type. For the case in which both servers have full information about each other but act selfishly, the signal for each server is the same, that is, each server knows its own $p_i$ and the other server's $p_{-i}$. We show that a Nash equilibrium strategy is not necessarily of threshold type in the full information case. We also propose regulations and prove that by imposing these regulations, we can eliminate the social inefficiency caused by noncooperation.

## 4.4 Case I: Cooperative servers with communication

Case I represents the situation in which both servers have full knowledge of the other's probability of successful transmission $p_i$ and both try to optimise social welfare. This models the perspective of the system manager, and the social welfare in this case is maximal under the environment described in Section 4.1. The best strategy for both servers to maximise social welfare is described in Theorem 9.

**Theorem 9.** *If $p = (p_1, p_2)$ is observable to both servers when they make decisions, the best*

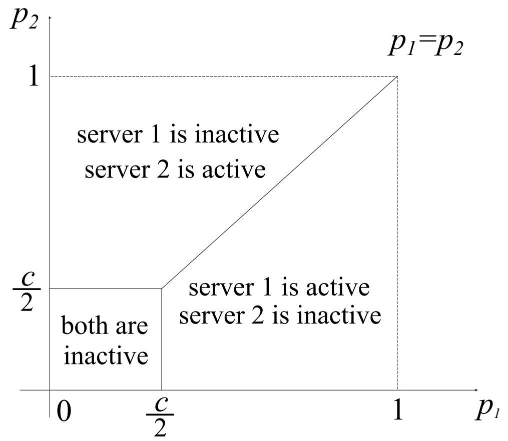



Figure 4.1: Optimal action profile for cooperative servers with communication.

*strategy to maximise overall welfare is*

$$s^* = \begin{cases} (inactive, inactive), & \max\{p_1, p_2\} < \frac{c}{2} \\ (\sigma_1, inactive), & \max\{p_1, p_2\} = p_1 = \frac{c}{2} \\ (inactive, \sigma_2), & \max\{p_1, p_2\} = p_2 = \frac{c}{2} \\ (active, inactive), & \max\{p_1, p_2\} = p_1 > \frac{c}{2} \\ (inactive, active), & \max\{p_1, p_2\} = p_2 > \frac{c}{2}. \end{cases} \tag{4.1}$$

*where $\sigma_i$ is any mixed strategy for server $i$.*

**Proof:** If $\max\{p_1, p_2\} < \frac{c}{2}$ and either server is active, a negative expected payoff is incurred. Thus the best strategy is $(inactive, inactive)$ and the sum of the expected payoffs is 0.

If $\max\{p_i, p_{-i}\} = p_i > \frac{c}{2}$, server $i$ could make its expected payoff positive by choosing to be active, while server $-i$ should remain inactive to avoid an additional cost $c$. In this case, the sum of the expected payoffs is $2p_i - c$.

If $\max\{p_i, p_{-i}\} = p_i = \frac{c}{2}$, server $i$ is indifferent between being inactive and being active, and any mixed strategy $\sigma_i$ results in the same expected payoff of 0. After simple calculations and comparisons, we obtain the results in Equation (4.1). □

The best strategy $s^*$ is shown in Figure 4.1. The social welfare (see Section 1.1) under strategy $s^*$ is

$$S_O(s^*) = \int_0^{\frac{c}{2}} \int_0^{\frac{c}{2}} 0 \, dp_1 \, dp_2 + \int_{\frac{c}{2}}^1 \int_0^{p_1} (2p_1 - c) \, dp_2 \, dp_1 + \int_{\frac{c}{2}}^1 \int_0^{p_2} (2p_2 - c) \, dp_1 \, dp_2$$
$$= \frac{1}{12} c^3 - c + \frac{4}{3}. \tag{4.2}$$

This is the best social welfare under an operating cost $c$ and a uniform distribution for both servers' probabilities of successful transmission.

## 4.5 Case II: Uncooperative servers without communication

Case II studies the scenario in which server $i$ is informed that server $-i$ is using a threshold strategy such that it will be active if and only if $p_{-i} \geq p^*_{-i}$. In this section, we first calculate the Nash equilibrium strategy and the social welfare for the values $p^*_i$ and $p^*_{-i}$ of the thresholds. We then propose a regulation and conclude that the social welfare under the threshold strategy is maximised by this regulation.

### 4.5.1 Nash equilibrium strategy in Case II

Guglielmi and Badia[33] stated that for a given $p^*_{-i}$, if server $i$ is active when its probability of successful transmission is $p_i$, then it should also be active for any other $\tilde{p}_i > p_i$. In this section, we prove that the best response for server $i$ is a threshold strategy. In addition, we calculate a Nash equilibrium strategy and prove this is the only one. This is expressed in Theorem 10.

**Theorem 10.** *For $i = 1, 2$, server $i$'s best response is of threshold type regardless of server $-i$'s strategy and $p_{-i}$'s distribution. Assume server $i$ knows that server $-i$ is active if and only if $p_{-i} \geq p^*_{-i}$, server $i$ knows $p^*_{-i}$ and that $p_{-i}$ follows a uniform distribution on $[0, 1]$, then server* *i's best response is also a threshold strategy with*

$$BR(p^*_{-i}) = \begin{cases} \sqrt{2c - p^{*2}_{-i}}, & if\ p^*_{-i} \le \sqrt{c} \\ \dfrac{c}{p^*_{-i}}, & if\ p^*_{-i} > \sqrt{c}\,. \end{cases} \tag{4.3}$$

*Also, the unique Nash equilibrium strategy is* $(\underline{\sqrt{c}}, \underline{\sqrt{c}})$ *which means both servers adopt a threshold strategy with* $p^*_1 = p^*_2 = \sqrt{c}$ *.*

**Proof:** Assume that server 2 is adopting any strategy $s_2$. For server 1, when it is active, its payoff is either $p_1 - c$ or $\max\{p_1, p_2\} - c$ when server 2 is inactive or active, respectively. Both $p_1$ and $\max\{p_1, p_2\}$ are increasing with $p_1$, thus the expected payoff of server 1 when it is active is increasing with $p_1$. When server 1 is inactive, its expected payoff is not a function of $p_1$. thus, if server 1's best response is to be active at $p_1$, then this is also its best response at any $\tilde{p}_1 \ge p_1$. So the best response of server 1 is of threshold type no matter what strategy the other server adopts. An identical analysis holds for server 2. In the Nash equilibrium, each server plays a best response against the other server simultaneously, so the Nash equilibrium strategy is of threshold type.

Next, we assume that server $i$ knows its own $p_i$, and that server 2 adopts a strategy with threshold $p^*_2$, which is known to server 1. We discuss the expected payoff to server 1 in two cases: $p_1 < p^*_2$ and $p_1 \ge p^*_2$.

When $p_1 < p^*_2$,

$$\begin{aligned} \mathbb{E}[U_1((active, \underline{p^*_2}), p_1)] &= \int_0^{p^*_2} (p_1 - c)dp_2 + \int_{p^*_2}^1 (p_2 - c)dp_2 \\ &= \frac{1 - p^{*2}_2}{2} + p_1 p^*_2 - c\,, \\ \mathbb{E}[U_1((inactive, \underline{p^*_2}), p_1)] &= \int_{p^*_2}^1 p_2 dp_2 = \frac{1 - p^{*2}_2}{2}\,. \end{aligned}$$

Then

$$\mathbb{E}[U_1((active, \underline{p^*_2}), p_1)] \ge \mathbb{E}[U_1((inactive, \underline{p^*_2}), p_1)] \Longleftrightarrow p_1 \in \left[\frac{c}{p^*_2}, p^*_2\right)\,.$$

When $p_1 \geq p_2^*$,

$$\begin{aligned}
\mathbb{E}[U_1((active, \underline{p_2^*}), p_1)] &= \int_0^{p_2^*} (p_1 - c) dp_2 + \int_{p_2^*}^{p_1} (p_1 - c) dp_2 + \int_{p_1}^{1} (p_2 - c) dp_2 \\
&= \frac{p_1^2 + 1}{2} - c\,, \\
\mathbb{E}[U_1((inactive, \underline{p_2^*}), p_1)] &= \int_{p_2^*}^{1} p_2 dp_2 = \frac{1 - p_2^{*2}}{2}\,.
\end{aligned}$$

Then

$$\mathbb{E}[U_1((active, \underline{p_2^*}), p_1)] \geq \mathbb{E}[U_1((inactive, \underline{p_2^*}), p_1)] \Longleftrightarrow p_1 \in \left[\max\{p_2^*, \sqrt{2c - p_2^{*2}}\}, 1\right].$$

In summary, if $0 \leq p_2^* \leq \sqrt{c}$, then $\left[\frac{c}{p_2^*}, p_2^*\right)$ is empty and server 1 never becomes active when $p_1 < p_2^*$. Furthermore, $\max\{p_2^*, \sqrt{2c - p_2^{*2}}\} = \sqrt{2c - p_2^{*2}}$, so server 1 becomes active when $p_1 \in [\sqrt{2c - p_2^{*2}}, 1]$. On the other hand, if $\sqrt{c} < p_2^* \leq 1$, then $\frac{c}{p_2^*} < p_2^*$ and $\max\{p_2^*, \sqrt{2c - p_2^{*2}}\} = p_2^*$, with the result that $p_1 \in \left[\frac{c}{p_2^*}, 1\right]$. Thus the best response for server 1 is threshold type with threshold value

$$BR_1(p_2^*) = \begin{cases} \sqrt{2c - p_2^{*2}}, & 0 \leq p_2^* \leq \sqrt{c} \\ \frac{c}{p_2^*}, & \sqrt{c} < p_2^* \leq 1\,. \end{cases} \tag{4.4}$$

which is the same as (4.3) when $i = 1$. An identical analysis holds when $i = 2$.

For the Nash equilibrium strategy, we notice that when $p_{-i}^* \in [0, \sqrt{c}]$,

$$p_i^* = \sqrt{2c - p_{-i}^{*2}} \geq \sqrt{c},$$

thus a Nash equilibrium should also satisfy

$$p_{-i}^* = \frac{c}{p_i^*}.$$

By solving the two equations simultaneously, we obtain the threshold type Nash equilibrium strategy $(\underline{\sqrt{c}}, \underline{\sqrt{c}})$; when $p^*_{-i} \in [\sqrt{c}, 1]$, the analysis is similar, and $p^*_i = \frac{c}{p^*_{-i}} \in [0, \sqrt{c}]$, thus $p^*_{-i} = \sqrt{2c - p^{*2}_i}$. The resulting Nash equilibrium strategy is also $(\underline{\sqrt{c}}, \underline{\sqrt{c}})$. Thus $(\underline{\sqrt{c}}, \underline{\sqrt{c}})$ is the unique Nash equilibrium strategy, that is, each server chooses to be active if and only if its successful transmission probability is greater than or equal to $\sqrt{c}$. □

### 4.5.2 Social welfare of uncooperative servers without communication

We assume both servers adopt threshold strategies of $p^*_1$ and $p^*_2$ and discuss the social welfare for the two cases: $p^*_1 < p^*_2$ and $p^*_1 \geq p^*_2$. We denote the social welfare gained from server 1 and server 2 by $T_1(\underline{p^*_1}, \underline{p^*_2})$ and $T_2(\underline{p^*_1}, \underline{p^*_2})$ if they adopt threshold strategy $(\underline{p^*_1}, \underline{p^*_2})$, respectively. Our computation is based on the partitioned regions shown in Figure 4.2 when $p^*_1 < p^*_2$, and Figure 4.3 when $p^*_1 \geq p^*_2$.

When $p^*_1 < p^*_2$, in region $A_1$, both servers are inactive, and both expected payoffs are 0. In region $B_1$, server 1 is active and server 2 is inactive, and the expected payoffs of server 1 and server 2 are $p_1 - c$ and $p_1$, respectively. Region $C_1$ is similar to $B_1$, server 1 is inactive and server 2 is active, and the expected payoffs to server 1 and server 2 are $p_2$ and $p_2 - c$, respectively. In $D_1$ and $E_1$, both servers are active. In $D_1$ where $p_1 \leq p_2$, both expected payoffs are $p_2 - c$, while in $E_1$ where $p_1 > p_2$, both expected payoffs are $p_1 - c$. The expected payoffs for server 1 in the five regions in Figure 4.2 and Figure 4.3 are

$$
\begin{aligned}
A_1 &= \int_0^{p^*_1} \int_0^{p^*_2} 0 \, dp_2 \, dp_1 = 0 \\
B_1 &= \int_{p^*_1}^1 \int_0^{p^*_2} (p_1 - c) \, dp_2 \, dp_1 = \left( \frac{1}{2} - c\,(1 - p^*_1) - \frac{p^{*2}_1}{2} \right) p^*_2 \\
C_1 &= \int_0^{p^*_1} \int_{p^*_2}^1 p_2 \, dp_2 \, dp_1 = p^*_1 \left( \frac{1}{2} - \frac{p^{*2}_2}{2} \right) \\
D_1 &= \int_{p^*_2}^1 \int_{p^*_1}^{p_2} (p_2 - c) \, dp_1 \, dp_2 = \frac{1}{3} - \frac{c}{2} - p^*_1 \left( \frac{1}{2} - c + c\, p^*_2 \right) + \left( \frac{c}{2} + \frac{p^*_1}{2} - \frac{p^*_2}{3} \right) p^{*2}_2 \\
E_1 &= \int_{p^*_2}^1 \int_{p_2}^1 (p_1 - c) dp_1 dp_2 = \frac{1}{6} \left( p^*_2 - 3c + 2 \right) \left( p^*_2 - 1 \right)^2 .
\end{aligned}
$$

Thus the expected payoff for server 1 if $p_1^* < p_2^*$ is

$$T_1(\underline{p_1^*}, \underline{p_2^*}) = A_1 + B_1 + C_1 + D_1 + E_1 = \frac{1}{6}\left(4 + 6\,c\,(p_1^* - 1) - 3\,p_1^{*2}\,p_2^* - p_2^{*3}\right).$$

A similar procedure follows for server 2. Server 2's expected payoff if $p_1^* < p_2^*$ is

$$T_2(\underline{p_1^*}, \underline{p_2^*}) = \frac{1}{6}\left(4 + 6\,c\,(p_2^* - 1) - 3\,p_1^{*2}\,p_2^* - p_2^{*3}\right).$$

Thus, the social welfare

$$S_O(\underline{p_1^*}, \underline{p_2^*}) = T_1(\underline{p_1^*}, \underline{p_2^*}) + T_2(\underline{p_1^*}, \underline{p_2^*}) = \frac{4}{3} + c\,(p_1^* + p_2^* - 2) - \frac{1}{3}\left(3\,p_1^{*2} + p_2^{*2}\right)\,p_2^*.$$

When $p_1^* \geq p_2^*$, by symmetry, the social welfare

$$S_O(\underline{p_1^*}, \underline{p_2^*}) = T_1(\underline{p_1^*}, \underline{p_2^*}) + T_2(\underline{p_1^*}, \underline{p_2^*}) = \frac{4}{3} + c\,(p_1^* + p_2^* - 2) - \frac{1}{3}\,p_1^*\left(p_1^{*2} + 3\,p_2^{*2}\right).$$

$S_O(\underline{p_1^*}, \underline{p_2^*})$ attains its maximum at $\left(\underline{\sqrt{\frac{c}{2}}}, \underline{\sqrt{\frac{c}{2}}}\right)$, while the unique Nash equilibrium strategy is $(\underline{\sqrt{c}}, \underline{\sqrt{c}})$ and

$$S_O\left(\underline{\sqrt{\frac{c}{2}}}, \underline{\sqrt{\frac{c}{2}}}\right) = \frac{2\sqrt{2}}{3}\,c\sqrt{c} - 2\,c + \frac{4}{3}$$
$$S_O(\underline{\sqrt{c}}, \underline{\sqrt{c}}) = \frac{2}{3}\,c\sqrt{c} - 2\,c + \frac{4}{3}.$$

Therefore, $S_O(\underline{\sqrt{c}}, \underline{\sqrt{c}}) \leq S_O\left(\underline{\sqrt{\frac{c}{2}}}, \underline{\sqrt{\frac{c}{2}}}\right)$. Hence $(\underline{\sqrt{c}}, \underline{\sqrt{c}})$ is not the social optimal threshold strategy.

If we impose a regulation that whichever server chooses to be active, the other server pays it $\frac{c}{2}$ as a subsidy, then the Nash equilibrium becomes the socially optimal solution. For example, if server 1 is active while server 2 is inactive, the previous payoff $(p_1 - c, p_1)$ becomes $\left(p_1 - \frac{c}{2}, p_1 - \frac{c}{2}\right)$; if both servers are active, the payoffs remain unchanged. The regulation is essentially making the original game fairer. If we use $\tilde{\mathbb{E}}[U_i((active, \underline{p_{-i}^*}), p_i)]$

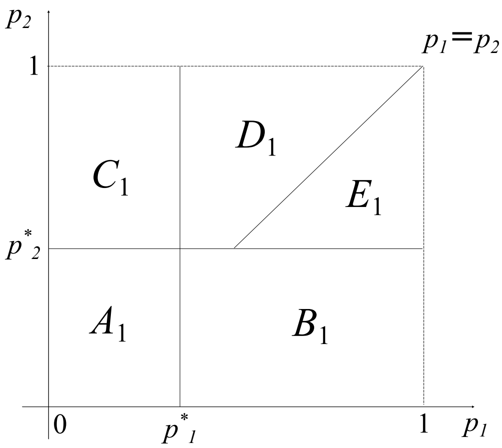



Figure 4.2: Areas for expected payoff computation when $p_1^* < p_2^*$.

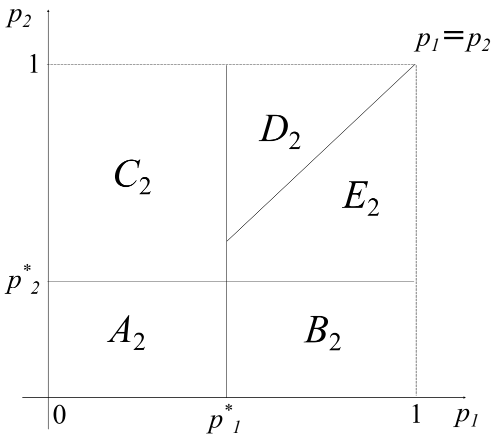



Figure 4.3: Areas for expected payoff computation when $p_1^* \geq p_2^*$.

to denote the expected payoff of server $i$ after regulation, then when $p_1 < p_2^*$,

$$\begin{aligned}\tilde{\mathbb{E}}[U_1((active, \underline{p_2^*}), p_1)] &= \int_0^{p_2^*} (p_1 - c) dp_2 + \int_{p_2^*}^1 (p_2 - c) dp_2 + \int_0^{p_2^*} \left(\frac{c}{2} - c\right) dp_2 \\ &= \frac{1 - p_2^{*2}}{2} + p_1 p_2^* + \frac{c}{2}\, p_2^* - c \\ \tilde{\mathbb{E}}[U_1((inactive, \underline{p_2^*}), p_1)] &= \int_{p_2^*}^1 \left(p_2 - \frac{c}{2}\right) dp_2 = \frac{1 - p_2^{*2}}{2} - \frac{c}{2}\,(1 - p_2^*)\,,\end{aligned}$$

hence

$$\tilde{\mathbb{E}}[U_1((active, \underline{p_2^*}), p_1)] \geq \tilde{\mathbb{E}}[U_1((inactive, \underline{p_2^*}), p_1)] \iff p_1 \in \left[\frac{c}{2\, p_2^*}, p_2^*\right).$$

When $p_1 \geq p_2^*$,

$$\begin{aligned}\tilde{\mathbb{E}}[U_1((active, \underline{p_2^*}), p_1)] &= \int_0^{p_2^*} p_1 dp_2 + \int_{p_2^*}^{p_1} p_1 dp_2 + \int_{p_1}^1 p_2 dp_2 + \int_0^{p_2^*} \frac{c}{2} dp_2 - c \\ &= \frac{1 + p_1^2}{2} + \frac{c}{2}\, p_2^* - c \\ \tilde{\mathbb{E}}[U_1((inactive, \underline{p_2^*}), p_1)] &= \int_{p_2^*}^1 \left(p_2 - \frac{c}{2}\right) dp_2 = \frac{1 - p_2^{*2}}{2} - \frac{c}{2}\,(1 - p_2^*)\,,\end{aligned}$$

hence

$$\tilde{\mathbb{E}}[U_1((active, \underline{p_2^*}), p_1)] \geq \tilde{\mathbb{E}}[U_1((inactive, \underline{p_2^*}), p_1)] \iff p_1 \in \left[\max\{p_2^*, \sqrt{c - p_2^{*2}}\}, 1\right].$$

Thus

$$\tilde{\mathbb{E}}[U_1((active, \underline{p_2^*}), p_1)] \geq \tilde{\mathbb{E}}[U_1((inactive, \underline{p_2^*}), p_1)]$$

$$\iff p_1 \in \left[\frac{c}{2\, p_2^*}, p_2^*\right] \cup \left[\max\{p_2^*, \sqrt{c - p_2^{*2}}\}, 1\right]$$

$$\iff BR_1(p_2^*) \in \begin{cases} \sqrt{c - p_2^{*2}}, & 0 \leq p_2^* \leq \sqrt{\frac{c}{2}} \\ \frac{c}{2\, p_2^*}, & \sqrt{\frac{c}{2}} < p_2^* \leq 1\,. \end{cases}$$

A similar analysis as in the proof of Theorem 10 establishes that $\left(\underline{\sqrt{\frac{c}{2}}}, \underline{\sqrt{\frac{c}{2}}}\right)$ is the unique Nash equilibrium strategy.

We remark here that if $p_1$ and $p_2$ independently follow the same distribution on $[0, 1]$ whose cumulative distribution function is $F(\cdot)$, then the threshold Nash equilibrium strategy is unique and is of type $(\underline{h(c)}; \underline{h(c)})$, where $h(c)$ satisfies $h(c)F(h(c)) = c$.

## 4.6 Case III: Uncooperative servers with communication

Case III analyses the situation where both servers have full information about each other's $p_i$, but each server maximises its own payoff. We first derive the Nash equilibria for the whole state space, and then calculate the best and worst social welfare within the Nash equilibrium strategies. Finally, we apply a regulation and obtain the social welfare based on a regulated Nash equilibrium strategy which is exactly the same as $S_O(s^*)$ in Case I.

### 4.6.1 Nash equilibrium strategy in Case III

The Nash equilibrium in this case is similar to that of Case I, but both servers are more conservative in their decisions to be active. The Nash equilibrium strategy is expressed in Theorem 11.

**Theorem 11.** *If $p = (p_1, p_2)$ is observable to both servers when they make decisions and both servers care only about their own payoffs, then the Nash equilibrium strategy on the whole state space is given by*

$$
s_N^* = \begin{cases}
(inactive, inactive), & \textit{if } \max\{p_1, p_2\} < c \\
(\sigma_1, inactive), & \textit{if } \max\{p_1, p_2\} = p_1 = c \\
(inactive, \sigma_2), & \textit{if } \max\{p_1, p_2\} = p_2 = c \\
(inactive, active), & \textit{if } \{p_1 < c < p_2\} \textit{ or } \{p_2 - p_1 > c\} \\
(active, inactive), & \textit{if } \{p_2 < c < p_1\} \textit{ or } \{p_1 - p_2 > c\} \\
(inactive, active) \text{ or } (active, inactive) \text{ or } \left(\frac{p_2 - c}{p_2}, \frac{p_1 - c}{p_2}\right), & \\
\qquad \textit{if } \{0 \le p_1 - p_2 \le c\} \textit{ and } \{p_1 > c\} \textit{ and } \{p_2 > c\} & \\
(inactive, active) \text{ or } (active, inactive) \text{ or } \left(\frac{p_2 - c}{p_1}, \frac{p_1 - c}{p_1}\right), & \\
\qquad \textit{if } \{0 < p_2 - p_1 \le c\} \textit{ and } \{p_1 > c\} \textit{ and } \{p_2 > c\}. &
\end{cases} \tag{4.5}
$$

*where $\sigma_i$ is any mixed strategy for server i.*

**Proof:** When $\max\{p_1, p_2\} < c$, *inactive* is the dominant strategy for both servers, so $(inactive, inactive)$ is the Nash equilibrium. On the boundary $\max\{p_i, p_{-i}\} = p_i = c$, as server $i$ is indifferent between being active and being inactive, any mixed strategy $\sigma_i$ is a Nash equilibrium and achieves zero payoff.

When $p_2 - c > p_1$, then server 2 has a higher expected payoff if it chooses to be active irrespective of what server 1 chooses to do. So server 2 should be active. Then server 1 should choose to be inactive.

When $p_1 < c < p_2$, if server 2 is inactive, server 1 has the choice between an expected payoff of 0 if it is inactive and $p_1 - c < 0$ if it is active. So it should choose to be inactive. Alternatively, if server 2 chooses to be active then server 1 has the choice between expected payoffs of $p_2$ if it is inactive, and $p_2 - c$ if it is active. So again it should choose to be inactive. Given that server 1 is inactive, server 2 has a choice between expected payoffs of $p_2 - c > 0$ if it chooses to be active and 0 if it chooses to be inactive. So it should choose to be active. This gives us $(inactive, active)$ as the optimal strategy if $p_1 < c < p_2$.

Similarly, if $p_2 < c < p_1$ or $p_1 - p_2 > c$, then the optimal strategy is $(active, inactive)$.

When $|p_1 - p_2| \le c$ and $\min\{p_1, p_2\} > c$, if server 1 is active, the expected payoff for server 2 if it is active is $\max\{p_1, p_2\} - c$ which is less than or equal to $p_1$, so the best response for server 2 is to choose to be inactive. On the other hand, if server 1 is inactive, the best response of server 2 is to choose to be active because its expected payoff $p_2 - c$ is positive. So the conclusion is that if $|p_1 - p_2| \le c$ and $\min(p_1, p_2) > c$, it is a Nash equilibrium for server 1 to be active precisely when server 2 is inactive and vice versa. This means that any partition of this region into disjoint sets where the strategy is $(\textit{active}, \textit{inactive})$ and $(\textit{inactive}, \textit{active})$ corresponds to a Nash equilibrium strategy. If $p_1 \ge p_2$, when server 1 is active with probability $\frac{p_2 - c}{p_2}$, the expected payoffs of server 2 when it is active and inactive are both $\frac{p_1(p_2 - c)}{p_2}$, that is, server 2 is indifferent among any mixed strategy; when server 2 is active with probability $\frac{p_1 - c}{p_2}$, the expected payoffs of server 1 when it is active and inactive are both $p_1 - c$, so server 1 is indifferent among any mixed strategy. Thus, if the strategy profile $\left(\frac{p_2 - c}{p_2}, \frac{p_1 - c}{p_2}\right)$ is adopted by both servers, neither side has an incentive to deviate, and we conclude that $\left(\frac{p_2 - c}{p_2}, \frac{p_1 - c}{p_2}\right)$ is a mixed Nash equilibrium strategy when $p_1 \ge p_2$. In this case, if server 1's probability of being active changes from $\frac{p_2 - c}{p_2}$ to a larger value (smaller value), then for server 2, it pays to stay inactive (active); if server 2's probability of being active changes from $\frac{p_1 - c}{p_2}$ to a larger value (smaller value), then for server 1, it pays to stay inactive (active). Thus, $\left(\frac{p_2 - c}{p_2}, \frac{p_1 - c}{p_2}\right)$ is not a stable Nash equilibrium. Similarly, $\left(\frac{p_2 - c}{p_1}, \frac{p_1 - c}{p_1}\right)$ is a mixed Nash equilibrium strategy when $p_1 < p_2$ and it is not a stable Nash equilibrium either. □

The Nash equilibrium strategy $s_N^*$ is shown in Figure 4.4 when $0 < c < 0.5$, and Figure 4.5 when $0.5 \le c < 1$.

### 4.6.2 Social welfare of uncooperative servers with communication

It follows from the analysis above that there are multiple Nash equilibria corresponding to the set $\{|p_1 - p_2| \le c\} \cap \{p_1 > c\} \cap \{p_2 > c\}$. Any partition of this region into disjoint sets where server 1 is active and server 2 is active results in a Nash equilibrium. The maximum social welfare is obtained if the strategy is $(\textit{active}, \textit{inactive})$ when $\{p_1 - p_2 \le c\} \cap \{p_1 \ge p_2 > c\}$ and $(\textit{inactive}, \textit{active})$ when $\{p_2 - p_1 \le c\} \cap \{p_2 > p_1 > c\}$. The minimum social welfare is attained when the strategy is just the opposite which is $(\textit{inactive}, \textit{active})$ when $\{p_1 - p_2 \le c\} \cap \{p_1 \ge p_2 > c\}$ and $(\textit{active}, \textit{inactive})$ when

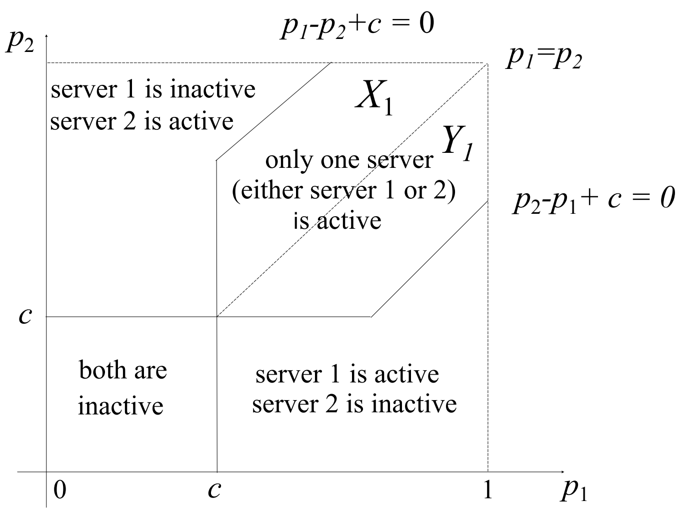



Figure 4.4: Nash equilibrium for uncooperative servers with communication where $0 < c < 0.5$,

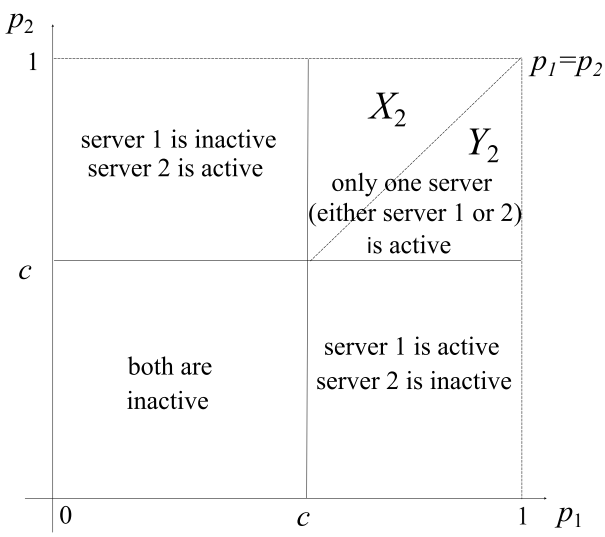



Figure 4.5: Nash equilibrium for uncooperative servers with communication where $0.5 \leq c < 1$.

$\{p_2 - p_1 \leq c\} \cap \{p_2 > p_1 > c\}$.

When $0 \leq c < 0.5$, the case is shown in Figure 4.4. The maximum social welfare Nash equilibrium strategy is $(inactive, active)$ in region $X_1$ and $(active, inactive)$ in region $Y_1$. The sum of the expected payoffs is $2p_2 - c$ in $X_1$ and $2\,p_1 - c$ in $Y_1$. The minimum social welfare Nash equilibrium strategy is $(active, inactive)$ in region $X_1$ and $(inactive, active)$

in region $Y_1$. The sum of the expected payoffs is $2p_1 - c$ in $X_1$ and $2\,p_2 - c$ in $Y_1$. By noticing that the sum of the expected payoffs is symmetric with respect to $p_1 = p_2$, the maximum and minimum social welfare within the set of Nash equilibria are

$$\begin{aligned}
&\max S_O(s_N^*) \\
&= \int_0^c \int_0^c 0\,dp_1\,dp_2 + \int_c^1 \int_0^{p_1} (2p_1 - c)\,dp_2\,dp_1 + \int_c^1 \int_0^{p_2} (2p_2 - c)\,dp_1\,dp_2 \\
&= -\frac{1}{3}c^3 - c + \frac{4}{3},
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
\min S_O(s_N^*) = \int_0^c \int_0^c 0\,dp_1\,dp_2 + 2\left(\int_0^c \int_c^{2c} (2\,p_2 - c) dp_2\,dp_1 \right. \\
+ \int_{2c}^1 \int_0^{p_2 - c} (2\,p_2 - c)\,dp_1\,dp_2 + \int_c^{1-c} \int_{p_1}^{p_1+c} (2\,p_1 - c)\,dp_2\,dp_1 \\
\left. + \int_{1-c}^1 \int_{p_1}^1 (2\,p_1 - c)\,dp_2\,dp_1 \right) = 3\,c^3 - 2\,c^2 - c + \frac{4}{3},
\end{aligned} \tag{4.7}$$

respectively.

When $0.5 \leq c < 1$, the case is shown in Figure 4.5. The maximum social welfare Nash equilibrium strategy is $(inactive, active)$ in region $X_2$ and $(active, inactive)$ in region $Y_2$. The sum of the expected payoffs is $2p_2 - c$ in $X_2$ and $2\,p_1 - c$ in $Y_2$. The minimum social welfare Nash equilibrium strategy is $(active, inactive)$ in region $X_2$ and $(inactive, active)$ in region $Y_2$. The sum of the expected payoffs is $2p_1 - c$ in $X_2$ and $2\,p_2 - c$ in $Y_2$. As above, by noticing that the sum of the expected payoffs is symmetric with respect to $p_1 = p_2$, the maximum and minimum social welfare within the set of Nash equilibria are

$$\begin{aligned}
&\max S_O(s_N^*) \\
&= \int_0^c \int_0^c 0\,dp_1\,dp_2 + \int_c^1 \int_0^{p_1} (2p_1 - c)\,dp_2\,dp_1 + \int_c^1 \int_0^{p_2} (2p_2 - c)\,dp_1\,dp_2 \\
&= -\frac{1}{3}c^3 - c + \frac{4}{3},
\end{aligned} \tag{4.8}$$

and

$$\begin{aligned}&\min S_O(s_N^*) \\ &= \int_0^c \int_0^c 0\, dp_1\, dp_2 + 2\left(\int_0^c \int_c^1 (2\,p_2 - c)\, dp_2\, dp_1 + \int_c^1 \int_c^{p_2} (2\,p_1 - c)\, dp_1\, dp_2\right) \\ &= \frac{1}{3}c^3 - 2\,c^2 + c + \frac{2}{3},\end{aligned} \tag{4.9}$$

respectively.

We compare the action profile of the maximum social welfare with the social optimum strategy in Case I. When $p_1 \le c$ and $p_2 \le c$, uncooperative servers choose to be inactive, but once $\max\{p_1, p_2\} > \frac{c}{2}$, the sum of both servers' expected payoffs is positive, thus, from the system manager's point of view, the server with the higher successful transmission probability should be active. The reason why it stays inactive until its probability of successful transmission exceeds $c$ is because of the unfair division of the total payoff. If the server chooses to be active, it will gain a negative expected payoff while the other server will get an advantage.

We suggest imposing a regulation where the inactive server gives the active server $c - \frac{p_1 + p_2}{2}$ if $\min\{p_1, p_2\} > \frac{c}{2}$. Then when $\max\{p_1, p_2\} < \frac{c}{2}$, the game remains unchanged, so the Nash equilibrium is still $(\textit{inactive}, \textit{inactive})$; when $\max\{p_1, p_2\} \ge \frac{c}{2}$, the game has different expected payoffs as shown in Table 4.2. If $p_1 > p_2$, since $\frac{p_1 - p_2}{2} > 0$ and $p_1 -$

Table 4.2: Expected payoff matrix with regulation when $\max\{p_1, p_2\} \ge \frac{c}{2}$.

| | | server 2: *active* | server 2: *inactive* |
|---|---|---|---|
| server 1 | *active* | $\max\{p_1, p_2\} - c$, $\max\{p_1, p_2\} - c$ | $\frac{p_1 - p_2}{2}$, $\frac{3\,p_1 + p_2}{2} - c$ |
| | *inactive* | $\frac{p_1 + 3\,p_2}{2} - c$, $\frac{p_2 - p_1}{2}$ | $0, 0$ |

$c + \frac{p_1 + p_2}{2} > p_1 - c$, but $\frac{p_2 - p_1}{2} < 0$, so $(\textit{active}, \textit{inactive})$ is the only Nash equilibrium. If $p_1 < p_2$, $\frac{p_2 - p_1}{2} > 0$ and $p_2 - c + \frac{p_1 + p_2}{2} > p_2 - c$, but $\frac{p_1 - p_2}{2} < 0$, so $(\textit{inactive}, \textit{active})$ is the only Nash equilibrium. If $p_1 = p_2$, both $(\textit{active}, \textit{inactive})$ and $(\textit{inactive}, \textit{active})$ are Nash equilibria. After this regulation is imposed, the resulting Nash equilibrium strategy is exactly the same as $s^*$ in Case I, which means that the regulation eliminates the effect of noncooperation.

## 4.7 Conclusion

In this chapter we have quantified the value of communication and cooperation, and proposed regulation to increase social welfare by eliminating the loss due to noncooperation in a service system with two strategic servers proposed by Guglielmi and Badia [33].

We have applied game theory to analyse the behaviour of the servers where both servers (I) know each other's $p_i$ and they cooperate to maximise social welfare; (II) do not know each other's $p_i$, but each server knows that the other adopts a threshold strategy and what the threshold value is; (III) know each other's $p_i$ but they only maximise their own expected payoff. We computed Nash equilibrium strategy for Cases II and III, and showed that the unique Nash equilibrium strategy in Case II is $(\underline{\sqrt{c}}, \underline{\sqrt{c}})$, that is, each server will be inactive until its probability of successful transmission is at least $\sqrt{c}$. Furthermore we observed that there are multiple Nash equilibria for Case III.

We compared the social welfare of the Nash equilibrium strategies in the three cases plotted in Figure 4.6. We showed that the social welfare for Case II is the worst. This is reasonable since in Case II, servers lack both communication and cooperation. The best result of Case III is still less than that of Case I. This is caused by noncooperation. After imposing regulation, the Nash equilibrium strategy of Case III can be increased so that its expected payoff is the same as that obtained under the socially optimal strategy. For Case II, due to the lack of communication, there is still a gap between the social welfare and the social optimum, but the maximum is attained under the assumption that both servers adopt threshold strategies. The social welfare of the three cases after regulation is plotted in Figure 4.7.

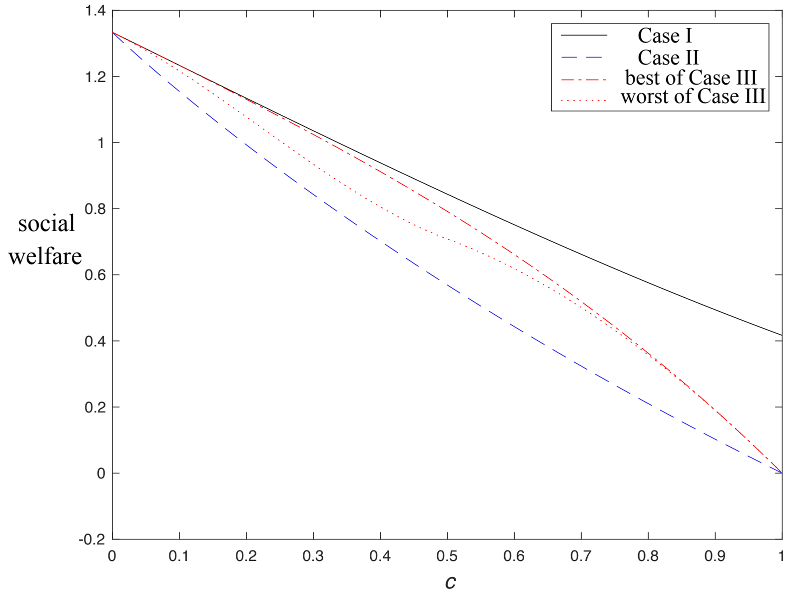



Figure 4.6: Social welfare under Cases I, II, and III without regulation.

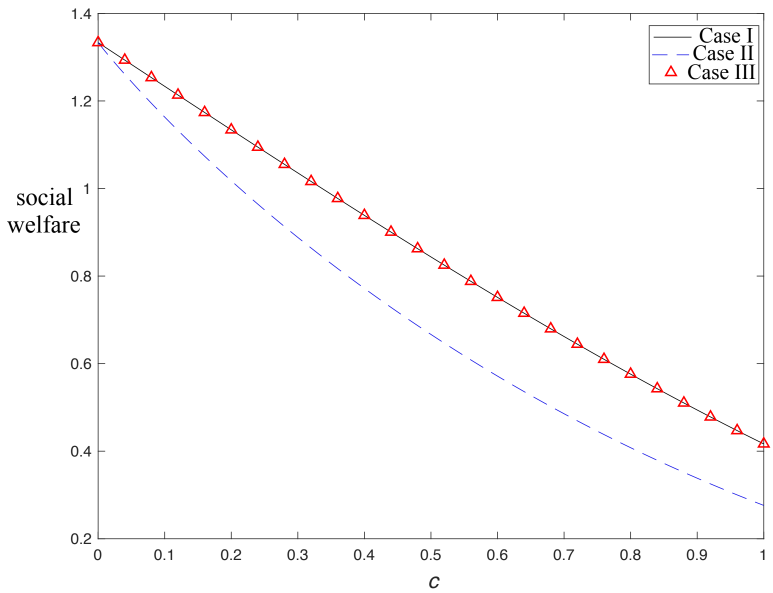



Figure 4.7: Social welfare under Cases I, II, and III with regulation.

# Chapter 5

# Strategic Customer Behavior in an $M/M/1$ Feedback Queue

## 5.1 Introduction

Consider an $M/M/1$ first-come-first-served queue where customers arrive according to a Poisson process with rate $\lambda$ and the service times for each customer are independently and identically distributed according to an exponential distribution with parameter $\mu$. After being served, each customer either successfully completes the service and departs from the system with probability $q$, or the service fails and the customer immediately joins the end of the queue to wait to be served again until she successfully completes.

We define the sojourn time as the total time a customer spends in the system, so it includes both the waiting time and the service time. Upon arriving at the queue, the newly arrived customer observes the number of customers in the system, and by considering the trade-off between her expected sojourn time and the reward due to a successful service completion, she makes a decision to join the queue or balk depending on the number of customers present when she arrives.

The cost is assumed to be linear in the sojourn time with rate $C$. To non-dimensionalise the model, we set $C = 1$ for the rest of the chapter. The reward to a customer when she successfully completes her service, which is assumed to be identical across customers, is denoted by $R_0$. Let $R$ be the reward that a customer actually obtains when she leaves the system. In this first model, customers are not allowed to leave until they successfully complete their service. Hence, the random variable $R$ is equal to the reward $R_0$ with probability one, but the reason that we have introduced it is that the reward is truly random

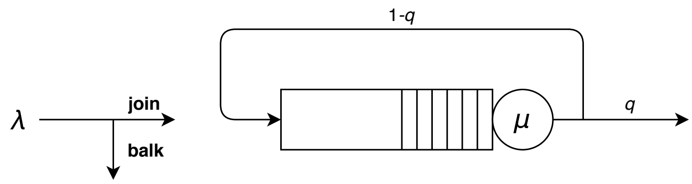



Figure 5.1: An $M/M/1$ feedback queue with strategic customers when reneging is not allowed.

for the system with reneging that we consider later. Indeed, for that system the random variable $R$ is equal to the reward $R_0$ with some probability less than one and equal to zero with some positive probability. Customers decide to join as long as their expected payoff, which is defined as the difference between their expected reward and their expected cost, is positive. See Figure 5.1 for an illustration of the system. The sojourn time of a customer depends on the service times of all customers that are served before she leaves the system and, if she has to repeat her service, it is possible that some of these services are for customers who joined the queue after her. It follows that her expected reward depends on the joining strategy of other customers. As a consequence, the best response of each customer is a function of both the position at which she joins the system and the other customers' strategies. For this reason, it is natural to consider the decision problem in a game theoretic framework and to look into the Nash equilibrium strategy for each customer (see Hassin and Haviv [40]).

The study of the instantaneous Bernoulli feedback queue goes back to Takacs [80], in which he obtained the expected total waiting time for the $M/G/1$ feedback queue by deriving the joint transform of the distribution of the queue length and the remaining service time. Disney, McNickle, and Simon [22], Disney [23], and Disney and König [24] further studied the queue length, the total sojourn time, and the waiting time. Takagi [81] applied the instantaneous Bernoulli feedback queue to study packet transmissions in an error-prone channel with probability of successful transmission $q$. This transmission style is similar to segmented message transmission with the number of segments in a message geometrically distributed with mean $1/q$. Filliger and Hongler [29], and Gallay and Hongler [30] studied queueing networks with feedback loops and intelligent customers. However, the customers in their model are not strategic in the sense that their decisions do not depend on the others' decisions, which is different from our setting.

Altman and Shimkin [4] analysed a system of observable egalitarian processor sharing queues, where customers decide to join or to balk after observing the number of customers in the system upon their arrival, and are not allowed to renege at any stage after joining. They calculated numerically the symmetric threshold equilibrium strategy for the case of Poisson arrivals and proposed a dynamic learning scheme, which converges to the symmetric Nash equilibrium strategy. In the calculation, the optimal threshold of a customer depends on the others' threshold and the number of customers in the system. Adan, Kulkarni, Lee, and Lefeber [2] considered a polling system with two queues where a single server serves the two nodes in a cyclic fashion with exhaustive service. Customers can choose which queue to join upon arrival. They analysed the Nash equilibrium strategies under three scenarios of available information of the queue lengths or the position of the server at decision epochs, and obtained the Nash equilibrium strategies via a new iterative algorithm. In both [4] and [2], customers' best response is affected by future arrivals.

In this chapter, we study a first-come-first-served $M/M/1$ queue with instantaneous Bernoulli feedback, and allow each customer to determine whether to join the queue or not after observing the number of customers in the system. Similar to Altman and Shimkin [4], in our model, a tagged customer's sojourn time is affected by the joining behavior of future arrivals. However, unlike the processor sharing discipline, under the first-come-first-served discipline, the optimal threshold of a tagged customer also depends on not only the others' threshold and the number of customers in the system, but also the position of the tagged customer. This model was first analysed in an unpublished technical report by Brooms and Collins [15]. They considered a first-come-first-served $GI/G/1$ Bernoulli feedback queue with arriving customers observing the number of customers in the system before deciding whether to join or not, but not allowed to renege.

Although arriving customers see the stationary distribution of the number of customers in the system, due to the balking and different joining positions, the distribution observed by feedback customers requires further analysis, which makes the expected sojourn time computation nontrivial. In this chapter, we efficiently obtain the expected payoff of a joining customer for any parameter set using matrix analytic methods (see Neuts [71]), which can also be easily extended to other models. In particular, we compute a customer's conditional expected payoff based on her joining position and the other customers' threshold values, by solving Poisson's equation (see Section 2.1.7 for its definition) for a discrete-time nonhomogeneous quasi-birth-and-death-process. Then we explicitly propose the Nash equilibrium strategies (pure or mixed) of threshold type.

Every time a customer joins at the end of the queue due to a service failure, the time she has already spent becomes a sunk cost. Also, it is possible that her conditions have deteriorated with time. For example, the system could have been empty when the tagged customer first arrived at the queue, but has become overcrowded before she goes to the end of the queue due to a service failure, because of a large number of arrivals during her first service. Thus, such a customer might want to renege if they are allowed. But once they choose to remain, the residual time until their next service has an Erlang distribution, which has an increasing hazard rate. Thus, if it is worth remaining in the system, it is worth waiting until the next service. In the second part of this chapter, we assume that customers are allowed to renege every time they join the end of the queue according to the same threshold strategy with which they choose to join the system. That is, if customers choose to join the system if and only if the number of customers in the system is less than or equal to some threshold value, then they will leave the system after a service failure if and only if the number exceeds the same threshold value. With matrix analytic methods, we can easily compute the Nash equilibrium threshold when reneging is permitted, and compare it with the equilibrium threshold value in the non-reneging case. We show that the customers' equilibrium threshold value when reneging is allowed is greater. However, for some parameter values, their expected payoff can decrease.

The chapter is organised as follows. In Section 5.2 we introduce the basics of the $M/M/1$ feedback queue, and precisely define our threshold joining strategy, which is specified by a real-valued threshold. We also derive an analytical expression for a tagged customer's position-dependent expected sojourn time if the other customers always choose to join. In Section 5.3 we obtain numerically the expected sojourn time and the expected payoff of a tagged customer conditioned on her joining position and the threshold strategy used by others, via matrix analytic methods. Then we propose a threshold Nash equilibrium strategy. In Section 5.4 we assume that customers are allowed to leave after joining and their reneging threshold is the same as the one with which they choose to join the system. We compute the Nash equilibrium threshold when reneging is permitted and compare it with that in the non-reneging case. In Section 5.5 we present two paradoxes observed in the non-reneging and the reneging case. In Section 5.6 we analyse the optimal social welfare in both the non-reneging and the reneging cases, and prove that allowing reneging does not change the socially optimal threshold and optimal social welfare.

## 5.2 Preliminaries

### 5.2.1 Joining strategies

We assume that the queue starts at time 0 with an initial number of customers according to a distribution $\pi(0)$ which is supported on the nonnegative integers. The number of customers in the system is observable to any arriving customer before she decides to join or not to join. For $r = 1, 2, \ldots$, let $u_r$ be a function that maps the numbers $1, 2, \ldots$ to the interval $[0, 1]$ such that $u_r(i)$ is the probability that the $r$th arriving customer chooses to join if there are $i - 1$ customers in front of her (including the one in service), which would mean that she starts in position $i$. We call the function $u_r$ the *joining strategy* for customer $r$ and $\boldsymbol{u}^\infty \equiv (u_1, u_2, \ldots)$ the *joining strategy profile* for the population. If $u_r(i)$ depends only on $i$, then the joining strategy is *symmetric* in which case, $\boldsymbol{u} = \{u, u, \cdots\}$ (see Hassin and Haviv [40, p3]).

Next, we introduce the definition of a threshold strategy. This threshold strategy was first proposed in Hassin [38], and was also used in Hassin and Haviv [39].

**Definition 5.2.1.** (symmetric threshold strategy). *For any $x \in \mathbb{R}^+$, the symmetric threshold* *strategy with threshold value x has components*

$$u^{(x)}(i) \equiv \begin{cases} 1 & \text{if } i \leq n \\ p & \text{if } i = n+1 \\ 0 & \text{if } i \geq n+2\,, \end{cases} \tag{5.1}$$

*where* $n \equiv \lfloor x \rfloor, p \equiv x - n$.

A customer who adopts threshold $x$ always chooses to join at a position which is less than or equal to $x$. She chooses to join at position $\lfloor x \rfloor + 1$ with probability $x - \lfloor x \rfloor$, and refuses to join at any position greater than $\lfloor x \rfloor + 1$. In their unpublished report [15, Theorem 6], Brooms and Collins claimed that any symmetric equilibrium joining strategy must be a threshold strategy. However their proof lacks detail, so we are going to treat this result with caution. If it is correct then our threshold strategy in Theorem 1 is the unique symmetric subgame perfect equilibrium strategy.

### 5.2.2 Basics of a single-server feedback queue

For the single-server feedback queue in Figure 5.1, in the time interval $[0, \infty)$, we denote by $\xi(t)$ and $\tau_r$ the number of customers in the system at time $t$ and the arrival time of the $r$th customer, respectively. Then $\xi_r := \xi(\tau_r)$ is the position at which the $r$th customer joins the system where, when $\xi_r = 1$, the customer immediately goes into service.

To work out the Nash equilibrium strategy, we arbitrarily select a customer as our tagged customer, and calculate her optimal response based on different strategies adopted by others. We are interested in the symmetric Nash equilibrium strategy, that is the strategy which is the best response when others use it too.

We denote the total sojourn time of the tagged customer in the system when the other customers all use threshold $x$ by $W^{(x)}$. Consistent with this notation, $W^{(\infty)}$ is the total sojourn time of a tagged customer in the system when all the other arriving customers always join and are not allowed to renege later.

From Takacs [80, Theorem 1], if $\lambda < q\mu$, then when all customers always join and are not allowed to renege later, the process $\{\xi(t), 0 \leq t < \infty\}$ has a unique stationary distribution

$$\pi_i := \lim_{t\to\infty} \mathbb{P}\{\xi(t) = i\} = \left(1 - \frac{\lambda}{q\mu}\right)\left(\frac{\lambda}{q\mu}\right)^i \ (i = 0, 1, \cdots).$$

Furthermore, Takacs [80, Section VI] gave the Laplace-Stieltjes transform of the unconditional stationary waiting time. We use similar techniques to obtain the conditional expected sojourn time given the joining position of each customer. In the stationary regime, for $i = 1, 2, ...$, let

$$P_i(w) := \mathbb{P}\{W^{(\infty)} \leq w\,,\, \xi_r = i\} \tag{5.2}$$

$$\Pi_i(s) := \int_0^\infty e^{-sw} dP_i(w)\,. \tag{5.3}$$

Then for $|z| \leq 1$, $\mathcal{R}(s) \geq 0$,

$$U(s,z) := \sum_{i=1}^\infty \Pi_i(s) z^i = \left(1 - \frac{\lambda}{q\mu}\right) \sum_{k=1}^\infty \frac{q(1-q)^{k-1}}{a_k(s,z) - b_k(s,z)}\,, \tag{5.4}$$

where

$$\begin{bmatrix} a_k(s,z) \\ b_k(s,z) \end{bmatrix} = \begin{bmatrix} \frac{\mu+\lambda+s}{\mu} & -q \\ \frac{\lambda}{\mu} & (1-q) \end{bmatrix}^k \begin{bmatrix} 1 \\ \frac{\lambda z}{q\mu} \end{bmatrix}. \tag{5.5}$$

To obtain $\int_0^\infty w\, dP_i(w)$, we take the derivative of $U(s,z)$ with respect to $s$ and set $s = 0$.

$$\sum_{i=1}^\infty \left(\int_0^\infty w\, dP_i(w)\right) z^i = -\frac{\partial U(s,z)}{\partial s}\,|_{s=0} \tag{5.6}$$

$$= \left(1 - \frac{\lambda}{q\mu}\right) \frac{q\mu((1-q)\lambda z + (q-2)q\mu)}{(\lambda z - q\mu)^2((q-1)\lambda - (q-2)q\mu)} \tag{5.7}$$

$$= \sum_{i=1}^\infty \left(1 - \frac{\lambda}{q\mu}\right) \frac{i+1-q}{(q\mu)^i((q-1)\lambda - (q-2)q\mu)} z^i\,. \tag{5.8}$$

Hence, the stationary expected sojourn time of a tagged customer if she joins at position

$i$, and all other customers always choose to join upon arrival is

$$w_{i,i}^{(\infty)} := \mathbb{E}\left(W^{(\infty)} \mid \xi_r = i\right) = \frac{\int_0^\infty w\, dP_i(w)}{\pi_i} = \frac{i+1-q}{((q-1)\lambda - (q-2)q\mu)}. \tag{5.9}$$

## 5.3 The Case When Customers Cannot Renege

### 5.3.1 The Expected payoff

In this chapter, we assume that customers are homogeneous which means they value receiving service identically and they place the same per unit time value on their waiting, and we focus on symmetric threshold strategies defined in Definition 5.2.1. When a customer arrives and sees $j-1$ customers already in the system, she will join the queue at the $j$th place. When every customer adopts threshold $x$ and the system starts with less than $\lceil x \rceil + 1$ customers, the tagged customer, upon arrival, can observe at most $\lceil x \rceil$ people in the system. If she chooses to join, her position is at most $\lceil x \rceil + 1$.

Let $w_{i,j}^{(x)}$ be the expected remaining time until the tagged customer departs the system, if there are $j$ customers in the system, she is in position $i \le j$ and all the other customers use threshold $x$. So if a customer joins in position $j$, her expected sojourn time will be $w_{j,j}^{(x)}$. On the other hand, when she leaves the queue she will obtain a reward $R_0$ and her expected payoff when she is in position $i$ there are $j$ customers in total and other customers are using threshold $x$ is thus $z_{i,j}^{(x)} \equiv E(R) - w_{i,j}^{(x)} = R_0 - w_{i,j}^{(x)}$.

We shall show that the vector

$$\boldsymbol{w}^{(x)} = \left(w_{1,1}^{(x)}, w_{1,2}^{(x)}, w_{2,2}^{(x)}, \ldots, w_{1,\lceil x \rceil+1}^{(x)}, \ldots, w_{\lceil x \rceil,\lceil x \rceil+1}^{(x)}, w_{\lceil x \rceil+1,\lceil x \rceil+1}^{(x)}\right)^T$$

satisfies a version of Poisson's equation. In Section 5.4 where we consider a model with reneging, customers do not always get the reward, and we proceed by writing Poisson's equation for the expected payoff $z_{i,j}^{(x)}$ directly.

To compute $\boldsymbol{w}^{(x)}$, we construct a continuous-time quasi-birth-and-death process (QBD) on the state space $\mathcal{S} = \{(i,j) : 1 \le i \le j \le \lceil x \rceil + 1\}$ with its level $j$ denoting the total number of customers including the customer in service in the system, and its phase $i$ denoting the position of the tagged customer. Then we construct the embedded discrete-time QBD obtained by observing this continuous-time Markov chain at its transition points and write $w_{i,j}^{(x)}$ conditioning on the first transition out of state $(i, j)$ in (5.10). Specifically, the expected time until the next transition is $\frac{1}{\lambda+\mu}$. The next transition is an arrival with probability $\frac{\lambda}{\lambda+\mu}$. When $j < \lfloor x \rfloor$, the arriving customer joins the system with probability 1; when $j = \lfloor x \rfloor$, the arriving customer joins the system with probability $p$; when $j = \lfloor x \rfloor + 1$ or $\lfloor x \rfloor + 2$, the arriving customer balks.

The next transition is a service completion with probability $\frac{\mu}{\lambda+\mu}$, after which a customer leaves the system with probability $q$ and joins the end of the system with probability $1 - q$. Hence, if the customer in service is the tagged one ($i = 1$), when she finishes her service, her future sojourn time is 0 with probability $q$, otherwise, her next position is $j$. When the customer in service is not the tagged one, the position of the tagged customer decreases by 1, the total number of customers decreases by 1 with probability $q$ but stays unchanged with probability $1 - q$. From the aforementioned reasoning,

$$\begin{aligned} w_{i,j}^{(x)} =& \frac{1}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \left( w_{i,j+1}^{(x)} \mathbb{1}_{\{j<\lfloor x \rfloor\}} + \left( p\, w_{i,j+1}^{(x)} + (1-p)\, w_{i,j}^{(x)} \right) \mathbb{1}_{\{j=\lfloor x \rfloor\}} + w_{i,j}^{(x)} \mathbb{1}_{\{j=\lfloor x \rfloor+1, \lfloor x \rfloor+2\}} \right) \\ &+ \frac{\mu}{\lambda+\mu} \left( (1-q)\, w_{j,j}^{(x)} \mathbb{1}_{\{i=1\}} + \left( q\, w_{i-1,j-1}^{(x)} + (1-q)\, w_{i-1,j}^{(x)} \right) \mathbb{1}_{\{i>1\}} \right). \end{aligned} \tag{5.10}$$

Thus, we can obtain $\boldsymbol{w}^{(x)}$ by solving Poisson's equation

$$\left( I - P^{(x)} \right) \boldsymbol{w}^{(x)} = \frac{1}{\lambda+\mu} \boldsymbol{e}, \tag{5.11}$$

where $\boldsymbol{e}$ denotes a vector of 1's of the appropriate size, and $P^{(x)}$ is

$$P^{(x)} = \begin{bmatrix} A_0^{(1)} & A_1^{(1)} & 0 & 0 & \cdots & \cdots & \cdots \\ A_{-1}^{(2)} & A_0^{(2)} & A_1^{(2)} & 0 & \cdots & \cdots & \cdots \\ 0 & A_{-1}^{(3)} & A_0^{(3)} & A_1^{(3)} & \cdots & \cdots & \cdots \\ 0 & 0 & A_{-1}^{(4)} & A_0^{(4)} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & A_{-1}^{(\lceil x \rceil+1)} & A_0^{(\lceil x \rceil+1)} \end{bmatrix} \tag{5.12}$$

$$A_{-1}^{(k)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \frac{\mu q}{\lambda+\mu} & 0 & \cdots & 0 \\ 0 & \frac{\mu q}{\lambda+\mu} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\mu q}{\lambda+\mu} \end{bmatrix} \in \mathbb{R}^{k\times(k-1)} \qquad k = 2, \cdots, \lceil x \rceil + 1 \tag{5.13}$$

$$A_{0}^{(k)} = \begin{bmatrix} 0 & 0 & \cdots & \frac{\mu(1-q)}{\lambda+\mu} \\ \frac{\mu(1-q)}{\lambda+\mu} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{\mu(1-q)}{\lambda+\mu} & 0 \end{bmatrix} \in \mathbb{R}^{k\times k} \quad k = 1, \cdots, \lfloor x \rfloor - 1 \tag{5.14}$$

$$A_{0}^{(\lfloor x \rfloor)} = \begin{bmatrix} \frac{\lambda(1-(x-\lfloor x \rfloor))}{\lambda+\mu} & 0 & \cdots & 0 \\ 0 & \frac{\lambda(1-(x-\lfloor x \rfloor))}{\lambda+\mu} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\lambda(1-(x-\lfloor x \rfloor))}{\lambda+\mu} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & \frac{\mu(1-q)}{\lambda+\mu} \\ \frac{\mu(1-q)}{\lambda+\mu} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{\mu(1-q)}{\lambda+\mu} & 0 \end{bmatrix} \in \mathbb{R}^{\lfloor x \rfloor \times \lfloor x \rfloor} \tag{5.15}$$

$$A_{0}^{(k)} = \begin{bmatrix} \frac{\lambda}{\lambda+\mu} & 0 & \cdots & 0 \\ 0 & \frac{\lambda}{\lambda+\mu} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\lambda}{\lambda+\mu} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & \frac{\mu(1-q)}{\lambda+\mu} \\ \frac{\mu(1-q)}{\lambda+\mu} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{\mu(1-q)}{\lambda+\mu} & 0 \end{bmatrix} \in \mathbb{R}^{k\times k} \qquad k = \lfloor x \rfloor + 1,\ \lceil x \rceil + 1 \tag{5.16}$$

$$A_{1}^{(k)} = \begin{bmatrix} \frac{\lambda}{\lambda+\mu} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\lambda}{\lambda+\mu} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\lambda}{\lambda+\mu} & 0 \end{bmatrix} \in \mathbb{R}^{k\times(k+1)} \quad k = 1, \cdots, \lfloor x \rfloor - 1 \tag{5.17}$$

$$A_{1}^{(\lfloor x \rfloor)} = \begin{bmatrix} \frac{\lambda p}{\lambda+\mu} & 0 & \cdots & 0 & 0 \\ 0 & \frac{\lambda p}{\lambda+\mu} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{\lambda p}{\lambda+\mu} & 0 \end{bmatrix} \in \mathbb{R}^{\lfloor x \rfloor \times (\lfloor x \rfloor + 1)} \qquad A_{1}^{(\lceil x \rceil)} = \mathbf{0}_{\lceil x \rceil \times (\lceil x \rceil + 1)}\,. \tag{5.18}$$

We have shown that $w_{i,j}^{(x)}$ can be obtained by solving a system of linear equations. However, the number of equations is quadratic in $\lfloor x \rfloor$. Thus, it is necessary to come up with an efficient way of carrying out the calculation. Equation (5.11) is Poisson's equation for a level dependent QBD. Due to the special structure of QBDs, we propose Algorithm 2 to solve $\boldsymbol{w}^{(x)}$ based on the methodology in Dendievel, Latouche, and Liu [20]. See Latouche and Ramaswami [56, Chapter 12] for a detailed explanation of the matrices $\Gamma^{(j)}, U^{(j)}$ and $G^{(j)}$ for a level dependent QBD which are used in Algorithm 2, noting that the matrix $\Gamma^{(j)}$ in this chapter has the same meaning as matrix $R^{(j)}$ in Section 2.1.6. We use $\Gamma$ to differentiate it from the reward $R$ that is obtained by the customers after they leave the service.

**Algorithm 2**

**procedure** CALCULATE $U^{(j)}, \Gamma^{(j)}, G^{(j)}$ ▷ The $U^{(j)}, \Gamma^{(j)}, G^{(j)}$ of $P^{(x)}$

  $U^{(\lceil x \rceil+1)} \leftarrow A_0^{(\lceil x \rceil+1)}$

  $\Gamma^{(\lceil x \rceil+1)} \leftarrow A_1^{(\lceil x \rceil)} (I - U^{(\lceil x \rceil+1)})^{-1}$

  $G^{(\lceil x \rceil+1)} \leftarrow (I - U^{(\lceil x \rceil+1)})^{-1} A_{-1}^{(\lceil x \rceil+1)}$

  **for** $j = \lceil x \rceil : 2$ **do**

    $U^{(j)} \leftarrow A_0^{(j)} + A_1^{(j)} G^{(j+1)}$

    $\Gamma^{(j)} \leftarrow A_1^{(j-1)} (I - U^{(j)})^{-1}$

    $G^{(j)} \leftarrow (I - U^{(j)})^{-1} A_{-1}^{(j)}$

  **end**

**procedure** POISSON'S EQUATION($U^{(j)}, \Gamma^{(j)}, G^{(j)}, \frac{1}{\lambda+\mu}$)

  $y(1) \leftarrow 0$

  **for** $j = 2 : \lceil x \rceil + 1$ **do**

    $y(j) \leftarrow \frac{1}{\lambda+\mu}(I - U^{(j)})^{-1} (\boldsymbol{e}_j + \sum_{k=j}^{\lceil x \rceil} \Pi_{l=j+1:k+1} \Gamma^{(l)} \boldsymbol{e}_{k+2}) + G^{(j)} y(j-1)$

  **end**

  $y(1) \leftarrow \frac{1}{\lambda+\mu} + A_1^{(1)} y(2)$

  $w(1) = \frac{y(1)}{1-(A_0^{(1)} + A_1^{(1)} G^{(2)})}$ ▷ Expected sojourn time

  **for** $j = 2 : \lceil x \rceil + 1$ **do**

    $w(\frac{j(j-1)}{2} + 1 : \frac{j(j-1)}{2} + j) = y(j) + \Pi_{l=j:2} G^{(l)} w(1)$

  **end**

  **return** $w$

We plot $w_{j,j}^{(x)}$ for $0 \leq x \leq 10$ in Figure 5.2. Several observations can be made. First, $w_{j,j}^{(x)}$ exists only when $\lceil x \rceil \geq j - 1$. The reason follows from the explanation at the beginning of

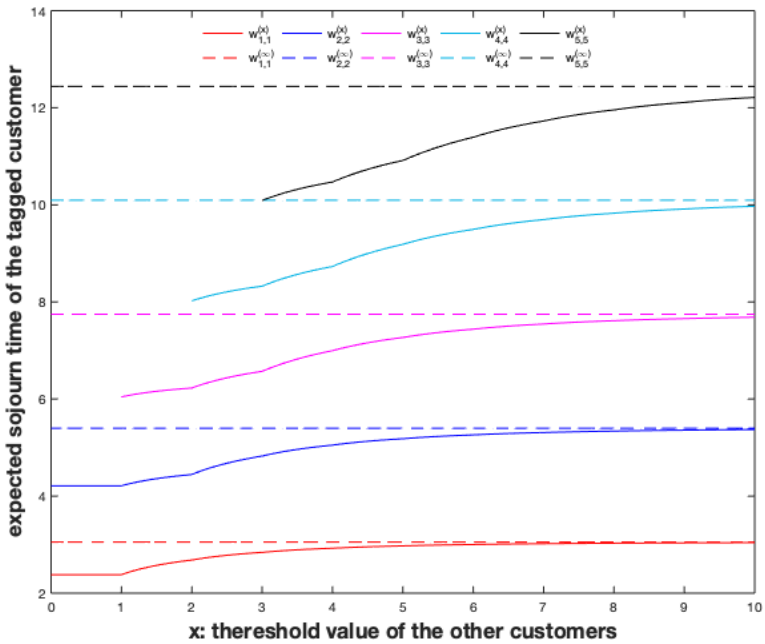



Figure 5.2: Expected sojourn time of the tagged customer ($\lambda = 0.4, \mu = 0.6, q = 0.7$).

this section that the tagged customer cannot be in a position greater than $\lceil x \rceil + 1$. Second, $w_{j,j}^{(x)}$ increases in $j$ for any $1 \leq j \leq \lceil x \rceil + 1$, and increases in $x$ when $x \geq 1$. This property was proved in Brooms and Collins [15] via coupling, and their proof works for $GI/G/1$ feedback queues. For an $M/M/1$ feedback queue, we propose an alternative proof in Lemmas 3 and 4. Third, when $x \in [0, 1)$, as long as the tagged customer is in the system, no newly arriving customer will join the system, hence the expected sojourn time of the tagged customer is independent of $x$. Actually, from (5.10), we explicitly have

$$w_{1,1}^{(x)} = \frac{1}{\mu q} \qquad w_{2,2}^{(x)} = \frac{3-q}{\mu q(2-q)} > \frac{1}{\mu q} \qquad 0 \leq x \leq 1 \,. \tag{5.19}$$

Finally, as expected, $w_{j,j}^{(x)}$ approaches $w_{j,j}^{(\infty)}$ as $x$ increases. Our results are stated in Lemmas 3 and 4 below.

**Lemma 3.** *$w_{j,j}^{(x)}$ is increasing in $j$ for $1 \leq j \leq \lceil x \rceil + 1$.*

*Proof.* For integer $d \geq 1$, we first define

$$ {}^{d}v_{i,j}^{(x)} := \left( (P^{(x)})^{d} \boldsymbol{e} \right)_{\frac{j(j-1)}{2}+i} \qquad 1 \leq i \leq j \leq \lceil x \rceil \,, \tag{5.20} $$

where $V^d$ is the $d$th power of $V$ and $\boldsymbol{v}_k$ is the $k$th element of a vector $\boldsymbol{v}$. We now prove by mathematical induction that ${}^{d}v_{j,j}^{(x)}$ and ${}^{d}v_{i,j}^{(x)}$ are increasing in $j$ for any $d$ and $1 \leq i \leq j \leq \lceil x \rceil$.

When $d = 1$,

$$ {}^{1}v_{1,j}^{(x)} = {}^{1}v_{1,j+1}^{(x)} = 1 - \frac{\mu q}{\lambda + \mu} \quad 1 \leq j \leq \lceil x \rceil \tag{5.21} $$

$$ {}^{1}v_{i,j}^{(x)} = {}^{1}v_{i,j+1}^{(x)} = {}^{1}v_{j+1,j+1}^{(x)} = 1 \quad 1 < i \leq j \leq \lceil x \rceil \,. \tag{5.22} $$

That is, ${}^{1}v_{j+1,j+1}^{(x)} = {}^{1}v_{j,j}^{(x)} > {}^{1}v_{1,1}^{(x)}$ for $1 < j \leq \lceil x \rceil$, and ${}^{1}v_{i,j+1}^{(x)} = {}^{1}v_{i,j}^{(x)}$ for $1 \leq i \leq j \leq \lceil x \rceil$.

Next suppose that the induction assumption is at the $d$th transition,

$$ {}^{d}v_{j+1,j+1}^{(x)} \geq {}^{d}v_{j,j}^{(x)} \qquad {}^{d}v_{i,j+1}^{(x)} \geq {}^{d}v_{i,j}^{(x)} \qquad 1 \leq i \leq j \leq \lceil x \rceil \,. \tag{5.23} $$

Before proving that 5.23 holds for $d+1$, we first prove that ${}^{d}v_{i,j}^{(x)} \geq {}^{d+1}v_{i,j}^{(x)}$. Since ${}^{d}v_{i,j}^{(x)}$ represents the sum of probabilities of being in each state in $\mathcal{S}$ at the $d$th transition, that is the probability that the tagged customer is still in the system in the $d$th transition, if the initial state is $(i, j)$. It follows from this physical interpretation that ${}^{d}v_{i,j}^{(x)} = 1$ when $d < i$. Furthermore, since the event that the tagged customer is still in the system after $d+1$ transitions is a subset of the event that it is still in the system after $d$ transitions, it must be the case that ${}^{d}v_{i,j}^{(x)}$ is decreasing in $d$.

Then by expanding both ${}^{d+1}v_{i,j+1}^{(x)}$ and ${}^{d+1}v_{i,j}^{(x)}$ as in Equation (5.10) and collecting identical terms together, for $1 \leq i \leq j \leq \lceil x \rceil$, we have

$$ \begin{aligned} &\left( {}^{d+1}v_{i,j+1}^{(x)} - {}^{d+1}v_{i,j}^{(x)} \right) = \\ &\frac{\lambda}{\lambda+\mu} \left( \left( {}^{d}v_{i,j+2}^{(x)} - {}^{d}v_{i,j+1}^{(x)} \right) \mathbb{1}_{\{j < \lfloor x \rfloor - 1\}} + p \left( {}^{d}v_{i,j+2}^{(x)} - {}^{d}v_{i,j+1}^{(x)} \right) \mathbb{1}_{\{j = \lfloor x \rfloor - 1\}} + (1-p) \left( {}^{d}v_{i,j+1}^{(x)} - {}^{d}v_{i,j}^{(x)} \right) \mathbb{1}_{\{j = \lfloor x \rfloor\}} \right) \\ &+ \frac{\mu}{\lambda+\mu} \left( (1-q) \left( {}^{d}v_{j+1,j+1}^{(x)} - {}^{d}v_{j,j}^{(x)} \right) \mathbb{1}_{\{i=1\}} + \left( q \left( {}^{d}v_{i-1,j}^{(x)} - {}^{d}v_{i-1,j-1}^{(x)} \right) + (1-q) \left( {}^{d}v_{i-1,j+1}^{(x)} - {}^{d}v_{i-1,j}^{(x)} \right) \right) \mathbb{1}_{\{i>1\}} \right) . \end{aligned} \tag{5.24} $$

Again, by expanding ${}^{d+1}v^{(x)}_{j+1,j+1}$ using Equation (5.10), and collapsing the term $\frac{\mu q}{\lambda+\mu}\,{}^{d}v^{(x)}_{j,j}$, we obtain for $1 \le i \le j \le \lceil x \rceil$,

$$\begin{aligned}
&\left({}^{d+1}v^{(x)}_{j+1,j+1} - {}^{d+1}v^{(x)}_{j,j}\right) \ge \left({}^{d+1}v^{(x)}_{j+1,j+1} - {}^{d}v^{(x)}_{j,j}\right) = \\
&\frac{\mu(1-q)}{\lambda+\mu}\left({}^{d}v^{(x)}_{j,j+1} - {}^{d}v^{(x)}_{j,j}\right) + \frac{\lambda}{\lambda+\mu}\left(\left({}^{d}v^{(x)}_{j+1,j+2} - {}^{d}v^{(x)}_{j,j}\right)\mathbb{1}_{\{j<\lfloor x\rfloor-1\}}\right. \\
&\left.+\left(p\left({}^{d}v^{(x)}_{j+1,j+2} - {}^{d}v^{(x)}_{j,j}\right) + (1-p)\left({}^{d}v^{(x)}_{j+1,j+1} - {}^{d}v^{(x)}_{j,j}\right)\right)\mathbb{1}_{\{j=\lfloor x\rfloor-1\}} + \left({}^{d}v^{(x)}_{j+1,j+1} - {}^{d}v^{(x)}_{j,j}\right)\mathbb{1}_{\{j=\lfloor x\rfloor\}}\right),
\end{aligned} \tag{5.25}$$

where the inequality in (5.25) holds strictly if and only if $d > j-1$. It follows from the induction assumption (5.23) that ${}^{d}v^{(x)}_{j+1,j+2} \ge {}^{d}v^{(x)}_{j+1,j+1} \ge {}^{d}v^{(x)}_{j,j}$, hence for $1 \le i \le j \le \lceil x \rceil$,

$$\left({}^{d+1}v^{(x)}_{i,j+1} - {}^{d+1}v^{(x)}_{i,j}\right) \ge 0\,, \tag{5.26}$$

and

$$\left({}^{d+1}v^{(x)}_{j+1,j+1} - {}^{d+1}v^{(x)}_{j,j}\right) \ge \frac{\mu(1-q)}{\lambda+\mu}\left({}^{d}v^{(x)}_{j,j+1} - {}^{d}v^{(x)}_{j,j}\right) \ge 0\,. \tag{5.27}$$

Finally, from equation (5.11)

$$\boldsymbol{w}^{(x)} = \frac{1}{\lambda+\mu}\left(I - P^{(x)}\right)^{-1}\boldsymbol{e} = \frac{1}{\lambda+\mu}\sum_{d=0}^{\infty}\left(P^{(x)}\right)^{d}\boldsymbol{e}\,. \tag{5.28}$$

It follows that

$$w_{i,j} = \frac{1}{\lambda+\mu}\left(1 + \sum_{d=1}^{\infty}{}^{d}v^{(x)}_{i,j}\right), \tag{5.29}$$

$$w_{j,j} = \frac{1}{\lambda+\mu}\left(1 + \sum_{d=1}^{\infty}{}^{d}v^{(x)}_{j,j}\right) \tag{5.30}$$

are increasing in $j$. □

**Remark** At the expense of making the calculation more intricate, we can prove that $w^{(x)}_{j,j}$ is strictly increasing in $j$. We omit the details.

**Lemma 4.** *For any two threshold policies $x_1$ and $x_2$ with $x_1 < x_2$,*

$$w_{i,j}^{(x_1)} < w_{i,j}^{(x_2)} \qquad 1 \leq i \leq \lceil x_1 \rceil + 1\,. \tag{5.31}$$

*Proof.*

- When $n < x_1 = n + p_1 < n + p_2 = x_2 \leq n + 1$, from equation (5.11),

$$\left(I - P^{(x_1)}\right) \boldsymbol{w}^{(x_1)} = \frac{1}{\lambda + \mu}\boldsymbol{e} \tag{5.32}$$

$$\left(I - P^{(x_2)}\right) \boldsymbol{w}^{(x_2)} = \frac{1}{\lambda + \mu}\boldsymbol{e}\,, \tag{5.33}$$

where the probability transition matrices $P^{(x_1)}$ and $P^{(x_2)}$ have the same dimension. Noting that $P^{(x_2)}$ and $P^{(x_1)}$ only differ in $n$ rows, we have

$$\left(I - P^{(x_2)}\right) (\boldsymbol{w}^{(x_2)} - \boldsymbol{w}^{(x_1)}) = (P^{(x_2)} - P^{(x_1)})\boldsymbol{w}^{(x_1)} = \frac{\lambda(p_2 - p_1)}{\lambda + \mu} \begin{bmatrix} \mathbf{0}_{\frac{n(n-1)}{2} \times 1} \\ w_{1,n+1}^{(x_1)} - w_{1,n}^{(x_1)} \\ w_{2,n+1}^{(x_1)} - w_{2,n}^{(x_1)} \\ \vdots \\ w_{n,n+1}^{(x_1)} - w_{n,n}^{(x_1)} \\ \mathbf{0}_{(2n+3)\times 1} \end{bmatrix}. \tag{5.34}$$

We know from equation (5.29) that $w_{i,n+1}^{(x_1)} - w_{i,n}^{(x_1)} > 0$. Also, since the $((i_1, j_1), (i_2, j_2))$th entry of $(I - P^{(x_2)})^{-1}$ is the tagged customer's expected number of visits to state $(i_2, j_2)$ starting from state $(i_1, j_1)$ in $\mathcal{S}$ before she leaves the system,

$$(I - P^{(x_2)})^{-1} > 0\,.$$

Hence

$$\boldsymbol{w}^{(x_2)} - \boldsymbol{w}^{(x_1)} = (I - P^{(x_2)})^{-1} \left(P^{(x_2)} - P^{(x_1)}\right) \boldsymbol{w}^{(x_1)} > 0\,. \tag{5.35}$$

- When $x_1 = n$ and $x_2 = n + p_2$ or $x_2 = n + 1$, $P^{(x_1)}$ and $P^{(x_2)}$ have different sizes.

However, we can write $P^{(x_2)}$ as

$$P^{(x_2)} = \left[\begin{array}{cccccc|c} A_0^{(1)} & A_1^{(1)} & \cdots & \cdots & \cdots & 0 \\ A_{-1}^{(2)} & A_0^{(2)} & A_1^{(2)} & \cdots & \cdots & 0 \\ \vdots & A_{-1}^{(3)} & A_0^{(3)} & A_1^{(3)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & A_{-1}^{(n+1)} & A_0^{(n+1)} & 0 \\ \hline 0 & \cdots & \cdots & \cdots & A_{-1}^{(n+2)} & A_0^{(n+2)} \end{array}\right] \tag{5.36}$$

$$= \left[\begin{array}{cc|c} \bar{P}^{(x_2)} & & \mathbf{0}_{\frac{(n+1)(n+2)}{2}\times(n+2)} \\ \hline \mathbf{0}_{(n+2)\times\frac{n(n+1)}{2}} & A_{-1}^{(n+2)} & A_0^{(n+2)} \end{array}\right], \tag{5.37}$$

and

$$(I - P^{(x_2)})^{-1} = \left[\begin{array}{c|c} (I - \bar{P}^{(x_2)})^{-1} & \mathbf{0} \\ \hline N\,(I - \bar{P}^{(x_2)})^{-1} & (I - A_0^{(n+2)})^{-1} \end{array}\right], \tag{5.38}$$

where $N = \left[\mathbf{0}_{(n+2)\times\frac{n(n+1)}{2}} \quad (I - A_0^{(n+2)})^{-1}\,A_{-1}^{(n+2)}\right]$. When $x_1 = n$, the position where the tagged customer can join is at most $n+1$, so we compare $\boldsymbol{w}_k^{(x_1)}$ with $\boldsymbol{w}_k^{(x_2)}$ for $k = 1, \cdots, \dfrac{(n+1)(n+2)}{2}$. If we define $\bar{\boldsymbol{w}}^{(x_2)} := \left[I_{\frac{(n+1)(n+2)}{2}} \quad \mathbf{0}\right] \boldsymbol{w}^{(x_2)}$, then

$$\bar{\boldsymbol{w}}^{(x_2)} = \frac{1}{\lambda+\mu}(I - \bar{P}^{(x_2)})^{-1}\,\boldsymbol{e}, \tag{5.39}$$

thus

$$\bar{\boldsymbol{w}}^{(x_2)} - \boldsymbol{w}^{(x_1)} = \frac{\lambda\,p_2}{\lambda+\mu}\,(I - \bar{P}^{(x_2)})^{-1} \begin{bmatrix} \mathbf{0}_{\frac{n(n-1)}{2}\times 1} \\ w_{1,n+1}^{(x_1)} - w_{1,n}^{(x_1)} \\ w_{2,n+1}^{(x_1)} - w_{2,n}^{(x_1)} \\ \vdots \\ w_{n,n+1}^{(x_1)} - w_{n,n}^{(x_1)} \\ \mathbf{0}_{(n+1)\times 1} \end{bmatrix} > 0. \tag{5.40}$$

In (5.40), with an abuse of notation, we include the case $p_2 = 1$.

- When $x_1 = n_1 + p_1, x_2 = n_2 + p_2$, and $n_1 < n_2$, by comparing the expected sojourn time for all the consecutive integers between $x_1$ and $x_2$, it follows from the aforementioned reasoning that

$$w_{i,j}^{(x_1)} < w_{i,j}^{(\lceil x_1 \rceil)} \qquad 1 \le i \le j \le \lceil x_1 \rceil + 1 \tag{5.41}$$

$$\dots$$

$$w_{i,j}^{(\lceil x_2 \rceil - 2)} < w_{i,j}^{(\lceil x_2 \rceil - 1)} \qquad 1 \le i \le j \le \lceil x_2 \rceil - 1 \tag{5.42}$$

$$w_{i,j}^{(\lceil x_2 \rceil - 1)} < w_{i,j}^{(x_2)} \qquad 1 \le i \le j \le \lceil x_2 \rceil . \tag{5.43}$$

Hence $w_{i,j}^{(x_1)} < w_{i,j}^{(x_2)}$, $1 \le i \le j \le \lceil x_1 \rceil + 1$.

□

### 5.3.2 Proof of Lemmas 3 and 4 by coupling

In Section 5.3.1, we proved Lemmas 3 and 4 using transition matrices. We will prove them using coupling in this Section, which can give us a lower bound for $w_{k,k}^{(x_2)} - w_{k,k}^{(x_1)}$ and $w_{k+1,k+1}^{(x)} - w_{k,k}^{(x)}$ for any $k \ge 1$ and $x_2 > x_1 > 0$.

#### Lower bound for $w_{k,k}^{(x_2)} - w_{k,k}^{(x_1)}$

In the coupling setting, all the arrival times, service durations and $U(0,1)$ random variables that drive decisions about the successful service and the joining probabilities are the same. Assume that there are two queues $Q_1$ and $Q_2$ where the tagged customer starts at the same position $k \in \mathbb{Z}^+$ at both queues. Let $0 \le p_1 < p_2 \le 1$, and the other customers in $Q_1$ and $Q_2$ use threshold $x_1 = n + p_1$ and $x_2 = n + p_2$, respectively. The two queues stay coupled until the queue size reaches $n$.

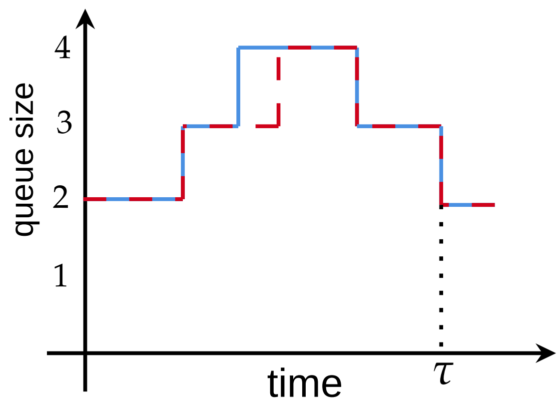
- When both queue lengths are $n$, and a customer arrives, then with probability $p_1$, the arriving customer joins at both queues; with probability $p_2 - p_1$, the arriving customer balks in $Q_1$ but joins in $Q_2$; with probability $1 - p_2$, the arriving customer balks at both queues.

- When the lengths of $Q_1$ and $Q_2$ are $n$ and $n+1$, respectively, and a customer arrives, then the length of $Q_2$ stays at $n+1$ until the next successful service completion;

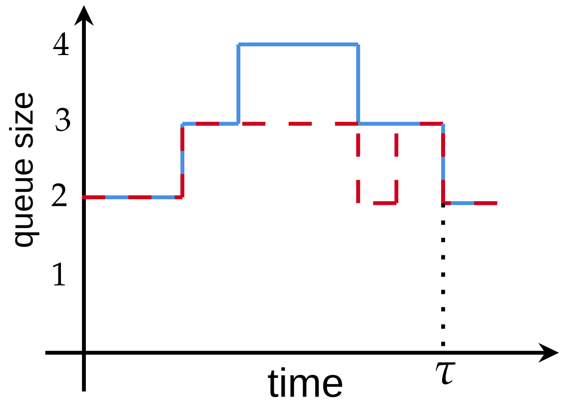
with probability $p_1$, the customer arriving to $Q_2$ is accepted and the two queue are coupled again at length $n+1$, as depicted in Figure 5.3 (a); with probability $1-p_1$, customers arriving to $Q_1$ balks and the queue length of $Q_1$ stays unchanged.

- When the lengths of $Q_1$ and $Q_2$ are $n-1$ and $n$, respectively, and a customer arrives, then the length of $Q_1$ increases by 1; with probability $1-p_2$, customers arriving to $Q_2$ balks and the two queues are coupled again at length $n$, as depicted in Figure 5.3 (b); with probability $p_2$, customers arriving to $Q_2$ joins the queue and the queue length of $Q_2$ increases by 1.

- When the lengths of $Q_1$ and $Q_2$ are $i$ and $i+1$ with $1 \leq i < n-1$, respectively, and a customer arrives, then the arriving customer joins at both queues.




(a) Two queues are coupled again at length $n+1$




(b) Two queues are coupled again at length $n$

Figure 5.3: Coupling.

The coupling continues until the tagged customer finishes her first service, after which the tagged customer either leaves both queues with probability $q$ or joins at the end with probability $1-q$. The queue length difference of $Q_1$ and $Q_2$ is either 0 or 1 and the tagged customer's position is the same in both queues before the coupling ends. Let $\Delta_{i,j}^{(x_1,x_2)}$ be the expected queue length difference when the coupling ends, given that the tagged customer is at position $i$ and the lengths of both queues are $j$, and $\Delta_{i,j^\star}^{(x_1,x_2)}$ be the same quantity but with the lengths of $Q_1$ and $Q_2$ set to be $j$ and $j+1$, respectively. If the tagged customer leaves after her first service or both queues have the same length when she rejoins the queue, the subsequent sample paths start off in the same state; if the length of $Q_2$ is one more than $Q_1$, then the expected waiting time of the tagged customer in $Q_2$ is at least $1/\mu$ more than $Q_1$. Thus,

$$w_{k,k}^{(x_2)} - w_{k,k}^{(x_1)} \geq \frac{(1-q)}{\mu} \Delta_{k,k}^{(x_1,x_2)} , \tag{5.44}$$

where the value of $\Delta_{k,k}^{(x_1,x_2)}$ for $k \geq 1$ can be calculated by Algorithm 3.

**Algorithm 3** (paremeters: $\lambda\,,\mu\,,n\,,p_1\,,p_2\,,q$)

$x_1 = n + p_1$, $x_2 = n + p_2$.
**procedure** CALCULATION OF $\Delta_{k,k}^{(x_1,x_2)}$ BY RECURSION
  **procedure** CALCULATION OF $\Delta_{1,1}^{(x_1,x_2)}, \Delta_{1,1^\star}^{(x_1,x_2)}, \ldots, \Delta_{1,n+1}^{(x_1,x_2)}$
    $\Delta_{1,n+1}^{(x_1,x_2)} \leftarrow 0$
    $\Delta_{1,n^\star}^{(x_1,x_2)} \leftarrow \frac{\lambda p_1 \Delta_{1,n+1}^{(x_1,x_2)} + \mu}{\mu + \lambda p_1}$
    $\Delta_{1,n}^{(x_1,x_2)} \leftarrow \frac{\lambda p_1 \Delta_{1,n+1}^{(x_1,x_2)} + \lambda (p_2 - p_1) \Delta_{1,n^\star}^{(x_1,x_2)}}{\mu + \lambda p_2}$
    $\Delta_{1,(n-1)^\star}^{(x_1,x_2)} \leftarrow \frac{\mu}{\lambda+\mu} + \frac{\lambda p_2}{\lambda+\mu} \Delta_{1,n^\star}^{(x_1,x_2)} + \frac{\lambda(1-p_2)}{\lambda+\mu} \Delta_{1,n}^{(x_1,x_2)}$
    $\Delta_{1,n-1}^{(x_1,x_2)} \leftarrow \frac{\lambda}{\lambda+\mu} \Delta_{1,n}^{(x_1,x_2)}$
    **for** $l = n-2 : -1 : 1$ **do**
      $\Delta_{1,l^\star}^{(x_1,x_2)} \leftarrow \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} \Delta_{1,(l+1)^\star}^{(x_1,x_2)}$
      $\Delta_{1,l}^{(x_1,x_2)} \leftarrow \frac{\lambda}{\lambda+\mu} \Delta_{1,l+1}^{(x_1,x_2)}$
    **end**
  **for** $i = 2 : k \quad (2 \leq k \leq n)$ **do**
    $\Delta_{i,n+1}^{(x_1,x_2)} \leftarrow (1-q)\, \Delta_{i-1,n+1}^{(x_1,x_2)} + q\, \Delta_{i-1,n}^{(x_1,x_2)}$
    $\Delta_{i,n^\star}^{(x_1,x_2)} \leftarrow \frac{\lambda p_1 \Delta_{i,n+1}^{(x_1,x_2)} + \mu(1-q) \Delta_{i-1,n^\star}^{(x_1,x_2)} + \mu q \Delta_{i-1,(n-1)^\star}^{(x_1,x_2)}}{\mu + \lambda p_1}$
    $\Delta_{i,n}^{(x_1,x_2)} \leftarrow \frac{\lambda p_1 \Delta_{i,n+1}^{(x_1,x_2)} + \lambda(p_2-p_1) \Delta_{i,n^\star}^{(x_1,x_2)} + \mu(1-q) \Delta_{i-1,n}^{(x_1,x_2)} + \mu q \Delta_{i-1,n-1}^{(x_1,x_2)}}{\mu + \lambda p_2}$
    **if** $k \leq n-1$ **then**
      $\Delta_{i,(n-1)^\star}^{(x_1,x_2)} \leftarrow \frac{\lambda p_2}{\lambda+\mu} \Delta_{i,n^\star}^{(x_1,x_2)} + \frac{\lambda(1-p_2)}{\lambda+\mu} \Delta_{i,n} + \frac{\mu(1-q)}{\lambda+\mu} \Delta_{i-1,(n-1)^\star}^{(x_1,x_2)} + \frac{\mu q}{\lambda+\mu} \Delta_{i-1,(n-2)^\star}^{(x_1,x_2)}$
      $\Delta_{i,n-1}^{(x_1,x_2)} \leftarrow \frac{\lambda}{\lambda+\mu} \Delta_{i,n-1} + \frac{\mu(1-q)}{\lambda+\mu} \Delta_{i-1,n-1}^{(x_1,x_2)} + \frac{\mu q}{\lambda+\mu} \Delta_{i-1,n-2}^{(x_1,x_2)}$
      **for** $j = n-2 : -1 : i$ **do**
        $\Delta_{i,j^\star}^{(x_1,x_2)} \leftarrow \frac{\lambda}{\lambda+\mu} \Delta_{i,(j+1)^\star}^{(x_1,x_2)} + \frac{\mu(1-q)}{\lambda+\mu} \Delta_{i-1,j^\star} + \frac{\mu q}{\lambda+\mu} \Delta_{i-1,(j-1)^\star}^{(x_1,x_2)}$
        $\Delta_{i,j}^{(x_1,x_2)} \leftarrow \frac{\lambda}{\lambda+\mu} \Delta_{i,j+1} + \frac{\mu(1-q)}{\lambda+\mu} \Delta_{i-1,j}^{(x_1,x_2)} + \frac{\mu q}{\lambda+\mu} \Delta_{i-1,j-1}^{(x_1,x_2)}$
      **end**
    **end**
  **end**

When $x_2 - x_1 > 1$, we write the difference in terms of all the consecutive integers between $x_1$ and $x_2$ and have

$$\begin{aligned} w_{k,k}^{(x_2)} - w_{k,k}^{(x_1)} &= \left(w_{k,k}^{(x_2)} - w_{k,k}^{(\lfloor x_2 \rfloor)}\right) + \left(w_{k,k}^{(\lfloor x_2 \rfloor)} - w_{k,k}^{(\lfloor x_2 \rfloor - 1)}\right) + \ldots + \left(w_{k,k}^{(\lfloor x_1 \rfloor + 1)} - w_{k,k}^{(x_1)}\right) \\ &\geq \frac{1-q}{\mu}\left(\Delta_{k,k}^{(x_2, \lfloor x_2 \rfloor)} + \Delta_{k,k}^{(\lfloor x_2 \rfloor, \lfloor x_2 \rfloor - 1)} + \ldots + \Delta_{k,k}^{(\lfloor x_1 \rfloor + 1, x_1)}\right). \end{aligned} \tag{5.45}$$

**Lower bound of $w_{k+1,k+1}^{(x)} - w_{k,k}^{(x)}$**

Assume that there are two queues $Q_1$ and $Q_2$ with the tagged customer joining at $k$ and $k+1$ in $Q_1$ and $Q_2$, respectively. The coupling for $Q_1$ and $Q_2$ ends when the tagged customer in $Q_1$ finishes her service. Since the tagged customer's positions always differ by 1 in the coupling setting, the tagged customer in $Q_2$ has not finished her service when the coupling ends. The coupling setting for arriving customers is the same as before. that is, all the arrival times, service durations and $U(0,1)$ random variables that drive decisions about the successful service and the joining probabilities are the same. However, when the coupling ends, we *freeze* $Q_1$ until the tagged customer in $Q_2$ finishes her first service.

Let $\Delta_{i^\star,j}^{(x)}$ and $\Delta_{i^\star,j^\star}^{(x)}$ be the same quantities as $\Delta_{i,j}^{(x_1,x_2)}$ and $\Delta_{i,j^\star}^{(x_1,x_2)}$, respectively, but with others using threshold $x$ for both queues and the tagged customer at position $i$ and $i+1$ in $Q_1$ and $Q_2$ respectively. Let $\delta^{(x)}(i,j)$ be the expected queue length difference when the tagged customer in $Q_2$ finishes her first service, given that there are $i$ customers in $Q_2$ and $j$ customers in $Q_2$ when the coupling ended. The calculation of $\Delta_{k^\star,k^\star}^{(x)}$ requires $\delta^{(x)}(i,j)$.

When the tagged customer in $Q_2$ finishes her service, we *thaw* $Q_1$, and for both queues, with probability $q$, the tagged customer leaves; with probability $1-q$, the tagged customer rejoins the queue. Thus,

$$w_{k+1,k+1}^{(x)} - w_{k,k}^{(x)} \geq \frac{q}{\mu} + \frac{1-q}{\mu}\Delta_{k^\star,k^\star}^{(x)}, \tag{5.46}$$

where $\Delta_{k^\star,k^\star}^{(x)}$ can be calculated by Algorithm 4.

**Algorithm 4** (parémeters: $\lambda, \mu, n, p, q$)

**procedure** CALCULATION OF $\Delta^{(x)}_{k^\star,k^\star}$ BY RECURSION
  **procedure** CALCULATION OF $\Delta^{(x)}_{1^\star,1^\star}, \Delta^{(x)}_{1^\star,2}, \dots, \Delta^{(x)}_{1,n+1}$
    $\Delta^{(x)}_{1^\star,n+1} \leftarrow \frac{\mu\, q\, \delta^{(x)}(-1,n)+\mu\,(1-q)\,\delta^{(x)}(0,n+1)}{\mu}$
    $\Delta^{(x)}_{1^\star,n^\star} \leftarrow \frac{\lambda p \Delta^{(x)}_{1^\star,n+1}+\mu\, q\, \delta^{(x)}(0,n)+\mu\,(1-q)\,\delta^{(x)}(1,n+1)}{\mu+\lambda p}$
    $\Delta^{(x)}_{1^\star,n} \leftarrow \frac{\lambda p \Delta^{(x)}_{1^\star,n+1}+\mu\, q\, \delta^{(x)}(-1,n-1)+\mu\,(1-q)\,\delta^{(x)}(0,n)}{\mu+\lambda p}$
    $\Delta^{(x)}_{1^\star,(n-1)^\star} \leftarrow \frac{\mu\, q\, \delta^{(x)}(0,n-1)+\mu\,(1-q)\,\delta^{(x)}(1,n)+\lambda\, p\, \Delta^{(x)}_{1^\star,n^\star}+\lambda\,(1-p)\,\Delta^{(x)}_{1^\star,n}}{\lambda+\mu}$
    $\Delta^{(x)}_{1^\star,n-1} \leftarrow \frac{\mu\, q\, \delta^{(x)}(-1,n-2)+\mu\,(1-q)\,\delta^{(x)}(0,n-1)+\lambda\,\Delta^{(x)}_{1^\star,n}}{\lambda+\mu}$
    **for** $l = n-2 : -1 : 2$ **do**
      $\Delta^{(x)}_{1^\star,l^\star} \leftarrow \frac{\mu\, q\, \delta^{(x)}(0,l)+\mu\,(1-q)\,\delta^{(x)}(1,l+1)+\lambda\,\Delta^{(x)}_{1^\star,(l+1)^\star}}{\lambda+\mu}$
      $\Delta^{(x)}_{1^\star,l} \leftarrow \frac{\mu\, q\, \delta^{(x)}(-1,l-1)+\mu\,(1-q)\,\delta^{(x)}(0,l)+\lambda\,\Delta^{(x)}_{1^\star,l+1}}{\lambda+\mu}$
    **end**
    $\Delta_{1^\star,1^\star} \leftarrow \frac{\mu\, q\, \delta^{(x)}(0,1)+\mu\,(1-q)\,\delta^{(x)}(1,2)+\lambda\,\Delta^{(x)}_{1^\star,2^\star}}{\lambda+\mu}$
  **for** $i = 2 : k \quad (2 \le k \le n)$ **do**
    $\Delta^{(x)}_{i^\star,n+1} \leftarrow \frac{\mu\, q\, \Delta^{(x)}_{(i-1)^\star,n}+\mu\,(1-q)\,\Delta^{(x)}_{(i-1)^\star,n+1}}{\mu}$
    $\Delta^{(x)}_{i^\star,n^\star} \leftarrow \frac{\mu\, q \Delta^{(x)}_{(i-1)^\star,(n-1)^\star}+\mu\,(1-q)\,\Delta^{(x)}_{(i-1)^\star,n^\star}+\lambda\, p \Delta^{(x)}_{i^\star,n+1}}{\mu+\lambda p}$
    $\Delta^{(x)}_{i^\star,n} \leftarrow \frac{\mu\, q \Delta^{(x)}_{(i-1)^\star,n-1}+\mu\,(1-q)\,\Delta^{(x)}_{(i-1)^\star,n}+\lambda\, p \Delta^{(x)}_{i^\star,n+1}}{\mu+\lambda p}$
    **if** $i \le n-1$ **then**
      $\Delta^{(x)}_{i^\star,(n-1)^\star} \leftarrow \frac{\mu\, q\, \Delta^{(x)}_{(i-1)^\star,(n-2)^\star}+\mu\,(1-q)\,\Delta^{(x)}_{(i-1)^\star,(n-1)^\star}+\lambda\, p\, \Delta^{(x)}_{i^\star,n^\star}+\lambda\,(1-p)\,\Delta^{(x)}_{i^\star,n}}{\lambda+\mu}$
      $\Delta^{(x)}_{i^\star,n-1} \leftarrow \frac{\mu\, q\, \Delta^{(x)}_{(i-1)^\star,n-2}+\mu\,(1-q)\,\Delta^{(x)}_{(i-1)^\star,n-1}+\lambda\,\Delta^{(x)}_{1^\star,n}}{\lambda+\mu}$
      **for** $j = n-2 : -1 : i+1$ **do**
        $\Delta^{(x)}_{i^\star,j^\star} \leftarrow \frac{\mu\, q\, \Delta^{(x)}_{(i-1)^\star,(j-1)^\star}+\mu\,(1-q)\,\Delta^{(x)}_{(i-1)^\star,j^\star}+\lambda\,\Delta^{(x)}_{i^\star,(j+1)^\star}}{\lambda+\mu}$
        $\Delta^{(x)}_{i^\star,j} \leftarrow \frac{\mu\, q\, \Delta^{(x)}_{(i-1)^\star,j-1}+\mu\,(1-q)\,\Delta^{(x)}_{(i-1)^\star,j}+\lambda\,\Delta^{(x)}_{i^\star,j+1}}{\lambda+\mu}$
      **end**
      $\Delta^{(x)}_{i^\star,i^\star} \leftarrow \frac{\mu\, q\, \Delta^{(x)}_{(i-1)^\star,(i-1)^\star}+\mu\,(1-q)\,\Delta^{(x)}_{(i-1)^\star,i^\star}+\lambda\,\Delta^{(x)}_{i^\star,(i+1)^\star}}{\lambda+\mu}$
    **end**
  **end**
**function** $y = \delta^{(x)}(i,j)$
  **if** $j = n+1$ **then**
    $y \leftarrow i$
  **else**
    **if** $j = n$ **then**
      $y \leftarrow \frac{\mu\, i+\lambda\, p\, \delta^{(x)}(i+1,j+1)}{\mu+\lambda p}$
    **else**
      $y \leftarrow \frac{\mu}{\lambda+\mu}\, i + \frac{\lambda}{\lambda+\mu}\, \delta^{(x)}(i+1,j+1)$
  **return** $y$

### 5.3.3 The Nash equilibrium threshold

In Lemmas 3 and 4, we have proved that $w^{(x)}_{j,j}$ is increasing in $j$ and $x$, so $z^{(x)}_{j,j}$ is decreasing in $j$ and $x$. We know from the beginning of Section 5.3 that the position where the tagged customer can join is at most $\lceil x \rceil + 1$ if the other customers use threshold $x$ and the system starts with less than $\lceil x \rceil$ customers. If we refer to the highest position that the tagged customer is willing to join, when others use threshold $x$, as the best response, and let $\mathcal{BR}(x)$ denote it, then $\mathcal{BR}(x) = \max\{j : z_{j,j}^{(x)} \geq 0, 1 \leq j \leq \lceil x \rceil + 1\}$.

If $R_0$ is big, then there will be values of $x$ for which $z_{\lceil x \rceil+1,\lceil x \rceil+1}^{(x)} \geq 0$ and so $\mathcal{BR}(x) = \lceil x \rceil + 1$. On this part of the domain, $\mathcal{BR}(x)$ is (obviously) an increasing step function. However, as $x$ increases, there must be a value $x^*$ for which $z_{\lceil x^* \rceil+1,\lceil x^* \rceil+1}^{(x^*)} < 0$. To see this, observe that a customer arriving to position $j$ must wait for at least $j$ services and so $w_{\lceil x \rceil+1,\lceil x \rceil+1}^{(x)} \geq (\lceil x \rceil + 1)/\mu$ and so when $x > R_0\mu - 1$,

$$z_{\lceil x \rceil+1,\lceil x \rceil+1}^{(x)} = R_0 - w_{\lceil x \rceil+1,\lceil x \rceil+1}^{(x)} \tag{5.47}$$

$$\leq R_0 - (\lceil x \rceil + 1)/\mu \tag{5.48}$$

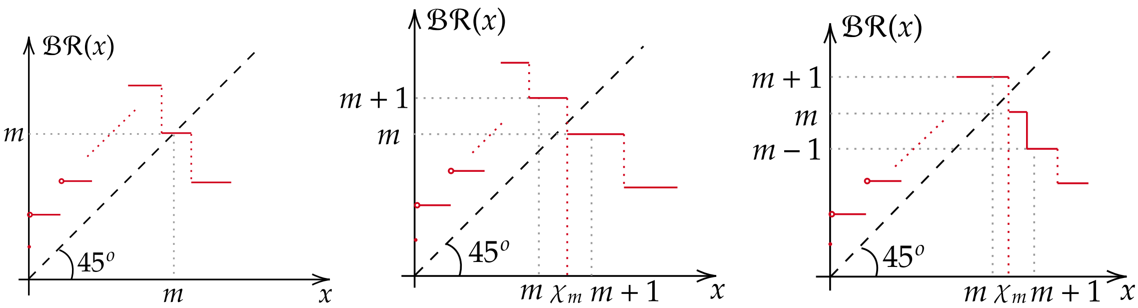
$$< 0. \tag{5.49}$$

For $x > x^*$, $BR(x) < \lceil x \rceil + 1$ and, on this part of the domain Lemma 4 ensures that $BR(x)$ is a monotone decreasing step function.

There are now two possibilities

- there is an integer $m$ such that $BR(m) = m$, or
- there is an integer $m$ such that $BR(m) = m + 1$ and $BR(m + 1) \leq m$,

These are illustrated in Figurei.2(a) and in Figures 3.2(b) and (c) respectively.




(a) $\mathcal{BR}(m) = m$ (b) $\mathcal{BR}(m) = m + 1$, $\mathcal{BR}(m + 1) = m$ (c) $\mathcal{BR}(m) = m + 1$, $\mathcal{BR}(m + 1) < m$

Figure 5.4: Best Response.

For the purpose of presenting the Nash equilibrium, for $m = 1, 2, \cdots$, let $\alpha_m = w^{(m)}_{m,m}$, $\beta_m = w^{(m)}_{m+1,m+1}$ (see Figure 5.5). Let $\chi_m$ be the solution to $w^{(\chi_m)}_{m+1,m+1} = R_0$. We prove in the following lemma that $\chi_m$ exists and is unique.

**Lemma 5.** *If $w^{(m)}_{m+1,m+1} < R_0 < w^{(m+1)}_{m+1,m+1}$, then there exists a unique $\chi_m$ such that $w^{(\chi_m)}_{m+1,m+1} = R_0$.*

*Proof.* For $x \in (m, m+1)$, from Equation (5.11),

$$w^{(x)}_{m+1,m+1} = \frac{1}{\lambda + \mu} \left( \left( I - P^{(x)} \right)^{-1} \boldsymbol{e} \right)_{\frac{(m+1)(m+2)}{2}} . \tag{5.50}$$

The matrix $P^{(x)}$ is substochastic for any $x$, as the sum of the $\left(\frac{j(j+1)}{2} + 1\right)$th row of $P^{(x)}$ is less than 1 for $j = 1, 2 \ldots, \lceil x \rceil + 1$. From a Corollary to the Perron-Frobenius Theorem (Seneta, E. [77, page 8]), $|r^{(x)}| < 1$ for any eigenvalue $r^{(x)}$. Thus, any real eigenvalue $1 - r^{(x)}$ of $I - P^{(x)}$ must be greater than 0. Hence $|I - P^{(x)}| \neq 0$.

Next, we write $x$ as $m + p$. From its expression, $P^{(x)}$ is continuous in $p$, so is $I - P^{(x)}$. Since the entries of the inverse matrix can be written as rational functions of the entries of the original matrix, and the denominators of these rational functions are non-zero for all $x$, $w^{(x)}_{m+1,m+1}$ is continuous in $p$. Hence, $w^{(x)}_{m+1,m+1}$ is continuous in $x$ for $x \in (m, m+1)$. Also, it follows from Lemma 3 and 4 that $\beta_m > \alpha_m$ and $\alpha_{m+1} > \beta_m$, respectively. If $\beta_m < R_0 < \alpha_{m+1}$, then

$$\beta_m = w^{(m)}_{m+1,m+1} < R_0 < w^{(m+1)}_{m+1,m+1} = \alpha_{m+1} .$$

Due to the fact that $w^{(x)}_{m+1,m+1}$ continuous and strictly increasing in $x \in (m, m+1)$, there is a unique $\chi_m \in (m, m+1)$ such that $w^{(\chi_m)}_{m+1,m+1} = R_0$. ◻

We describe the Nash equilibrium strategy for the feedback queueing system in the following theorem.

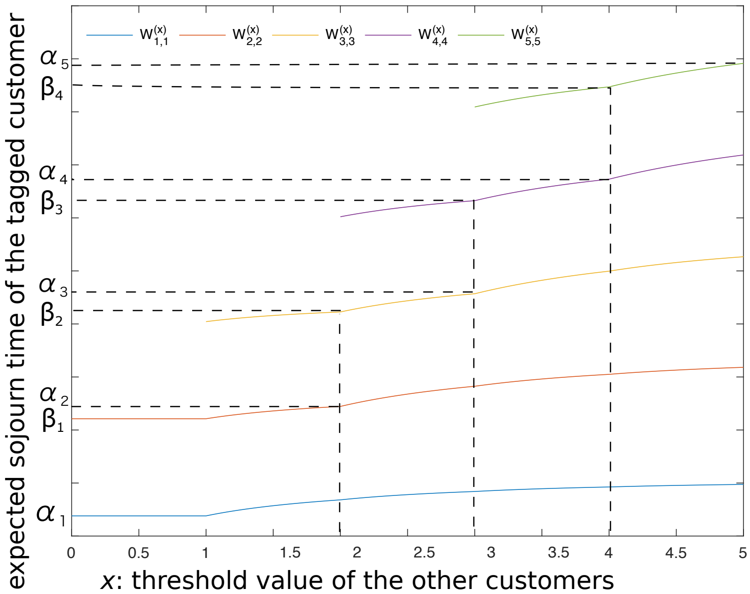



Figure 5.5: Expected sojourn time of the tagged customer.

**Theorem 12.** *There exists an equilibrium threshold strategy with threshold value*

$$x_e = \begin{cases} 0 & \text{if } R_0 < \alpha_1\,, \\ \zeta_0 & \text{if } R_0 = \alpha_1\,, \\ m & \text{if } \alpha_m \leq R_0 \leq \beta_m \quad m = 1, 2, \dots, \\ \chi_m & \text{if } \beta_m < R_0 < \alpha_{m+1} \quad m = 1, 2, \dots\,, \end{cases} \tag{5.51}$$

*where* $\zeta_0 \in [0, 1]$.

Proof. A customer will choose to join the queue if and only if her reward can fully bear her expected sojourn cost.

- When $R_0 < \alpha_1$, even if the tagged customer is the only one in the system, her expected sojourn time $\alpha_1 > R_0$. Thus, her best option is balking. The same analysis works for the other customers. So balking is the Nash equilibrium strategy.

- When $R_0 = \alpha_1$, if the other customers are all using threshold $\chi_0 \in [0, 1]$, there is at most one customer in the system when the tagged customer arrives. For the tagged customer, when she observes one person in the system, her best response is balking as her expected sojourn time $\beta_1 > \alpha_1 = R_0$. When she observes that the system is empty, her expected payoff is zero. Thus, she is indifferent between joining an empty system and balking. Actually, she gains nothing by either strategy. The same analysis works for any other customer, so any threshold strategy with threshold value $\chi_0 \in [0, 1]$ is a Nash equilibrium strategy.

- When $\alpha_m \leq R_0 \leq \beta_m$, the tagged customer's expected sojourn time satisfies $w^{(m)}_{m,m} \leq R_0 \leq w^{(m)}_{m+1,m+1}$, so
$$z^{(m)}_{m+1,m+1} \leq 0 \leq z^{(m)}_{m,m} . \tag{5.52}$$
Hence the tagged customer's best response is $m$ when others adopt threshold $m$. So threshold $m$ is a Nash equilibrium strategy.

- When $\beta_m < R_0 < \alpha_{m+1}$, if other customers all adopt threshold $\chi_m$, the tagged customer gains nothing when she joins at $m+1$, so her best response is any threshold strategy with threshold value between $m$ and $m+1$ (including $\chi_m$). Thus, $\chi_m$ is the Nash equilibrium threshold. □

From Theorem 12, $m$ is either the Nash equilibrium threshold or the integer part of it. The Nash equilibrium threshold is not an integer when $R_0 \in (\beta_m, \alpha_{m+1})$. Figure 5.4(b) represents this case with $\mathcal{BR}(m) = m+1$ and $\mathcal{BR}(m+1) = m$. Figure 5.4(c) depicts the case with $\mathcal{BR}(m) = m+1$ and $\mathcal{BR}(m+1) = m-1$. In both cases, the tagged customer is indifferent between $m$ and $m+1$ when others use a threshold between $m$ and $m+1$, and the conclusion of Theorem 3.1 holds.

Intuitively speaking, a Nash equilibrium is said to be *evolutionarily stable* if it cannot be invaded by any alternative strategy that is initially rare (see Maynard Smith [61]).

**Definition 5.3.1.** ***Evolutionarily stable strategy (ESS).*** *A Nash equilibrium strategy $x$ is said to be an ESS if either (i) $x$ is the unique best response against itself or (ii) for any $x' \neq x$ which is a best response against $x$, $x$ is better than $x'$ as a response to $x'$ itself. That is, with $U(x', x)$*

*denoting a customer's expected payoff when she uses $x'$ and others use $x$, for all $x' \neq x$, either*

$$U(x,x) > U(x',x)\,,\ or \tag{5.53}$$

$$U(x,x) = U(x',x) \quad and \quad U(x,x') > U(x',x')\,. \tag{5.54}$$

To show that the Nash equilibrium startegy with threshold value $x_e$ is an ESS, we first define the total expected payoff of a tagged customer who adopts threshold $x$ when the other customers all adopt threshold $x'$.

$$U(x,x') := \sum_{i=1}^{\lfloor x\rfloor \wedge (\lceil x'\rceil+1)} \pi_{i-1}^{(x')}\, z_{i,i}^{(x')} + (x - \lfloor x\rfloor)\, \pi_{\lfloor x\rfloor}^{(x')}\, z_{\lfloor x\rfloor+1,\lfloor x\rfloor+1}^{(x')}\, \mathbb{1}_{\{\lfloor x\rfloor < \lceil x'\rceil+1\}}\,, \tag{5.55}$$

where $\pi_j^{(x)}$ $0 \le j \le \lceil x\rceil$, is the stationary distribution of the number of customers in the system where everyone adopts threshold $x$. We prove the $x_e$-threshold strategy is an ESS in the following corollary.

**Corollary 2.** *The threshold strategy with threshold value $x_e$ is an ESS when $R \neq \alpha_1$.*

We have already proved that when other customers adopt threshold strategy $x_e$, there is no better strategy than $x_e$ for the tagged customer, that is $U(x_e,x_e) \ge U(x,x_e)$.

- When $R_0 < \alpha_1$, $U(0,0) = 0 > U(x,0)$ for any $x > 0$. Thus, balking is an ESS.
- When $R_0 = \alpha_1$, for any $0 \le \chi_0, \chi_0' \le 1$, $U(\chi_0,\chi_0) = U(\chi_0',\chi_0) = 0$ and $U(\chi_0,\chi_0') = U(\chi_0',\chi_0') = 0$. Thus, $\chi_0 \in [0,1]$ is not an ESS.
- When $\alpha_m \le R_0 \le \beta_m$, $U(m,m) > U(m',m)$ for any $m' \neq m$. Thus, $m$ is an ESS.
- When $\beta_m < R_0 < \alpha_{m+1}$, it follows from the definition of $\chi_m$ that $z_{m+1,m+1}^{(\chi_m)} = R_0 - w_{m+1,m+1}^{(\chi_m)} = 0$, so for any $\chi_m' \in [m, m+1]$

$$U(\chi_m',\chi_m) = U(\chi_m,\chi_m) = \sum_{i=1}^{m} \pi_{i-1}^{(\chi_m)}\, z_{i,i}^{(\chi_m)}\,. \tag{5.56}$$

Furthermore, when $m \le \chi_m' < \chi_m < m+1$, it follows from the fact that $z_{m+1,m+1}^{(x)}$ is decreasing in $x$ that

$$z_{m+1,m+1}^{(\chi_m')} > 0\,. \tag{5.57}$$

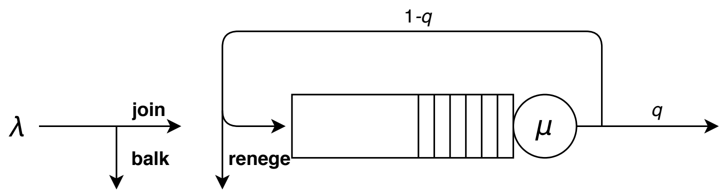



Figure 5.6: An M/M/1 feedback queue with strategic customers when reneging is allowed.

Since $\lfloor \chi'_m \rfloor = \lfloor \chi_m \rfloor = m$, the first summations in $U(\chi'_m, \chi'_m)$ and $U(\chi_m, \chi'_m)$ are equal. However, $(\chi'_m - \lfloor \chi'_m \rfloor) < (\chi_m - \lfloor \chi_m \rfloor)$, hence the second term in $U(\chi'_m, \chi'_m)$ is less than the second term in $U(\chi_m, \chi'_m)$. So $U(\chi'_m, \chi'_m) < U(\chi_m, \chi'_m)$. Similarly, when $\chi_m < \chi'_m \leq m+1$, $z^{(\chi'_m)}_{m+1,m+1} < 0$ but $(\chi_m - \lfloor \chi_m \rfloor) < (\chi'_m - \lfloor \chi'_m \rfloor)$. Following similar lines, we have $U(\chi'_m, \chi'_m) < U(\chi_m, \chi'_m)$. Thus, $\chi_m$ is an ESS. □

## 5.4 The Case When Customers Can Renege

Every time a customer rejoins at the end of the queue due to a service failure, it is possible that her conditions have deteriorated with time. Hence customers might have an incentive to *renege*, that is depart from the queue, when their service fails. Figure 5.6 is an illustration of an $M/M/1$ feedback queue when reneging is allowed. In this section, we focus on the Nash equilibrium threshold when the customers are allowed to renege, and compare it with the equilibrium threshold when they cannot renege. In order to make comparisons between the two cases, we abbreviate the non-reneging case as the $N$-case and the reneging case as the $R$-case.

### 5.4.1 The expected payoff

In our model, every time a customer rejoins the end of the queue, she faces a similar situation as that when she first chooses to join. Thus, we restrict our attention to policies where the customer must use the same threshold when she chooses to balk or renege.

In contrast to Section 5.3, the expected payoff of the tagged customer is affected by her future reneging decisions. In particular, if she chooses to renege, she will not receive the reward $R_0$. So we use $\hat{z}_{i,j}^{(x_{tag},x)}$ to denote the tagged customer's expected payoff, which is the difference between the expected reward and her expected sojourn cost, given that she is at position $i$ and uses threshold strategy $x_{tag}$, there are $j$ customers in the system, and the other customers all adopt threshold $x$. It will turn out that the relevant value of $x_{tag}$ that we need to consider for the purpose of calculating the Nash equilibrium occurs when $x_{tag} = \lfloor x \rfloor + 1$. This satisfies the equation

$$\begin{aligned} \hat{z}_{i,j}^{(\lfloor x \rfloor+1,x)} &= -\frac{1}{\lambda+\mu} \qquad (5.58) \\ &+ \frac{\lambda}{\lambda+\mu}\left(\hat{z}_{i,j+1}^{(\lfloor x \rfloor+1,x)}\,\mathbb{1}_{\{j<\lfloor x \rfloor\}} + \left(p\,\hat{z}_{i,j+1}^{(\lfloor x \rfloor+1,x)} + (1-p)\,\hat{z}_{i,j}^{(\lfloor x \rfloor+1,x)}\right)\mathbb{1}_{\{j=\lfloor x \rfloor\}}\right. \\ &\left.+\hat{z}_{i,j}^{(\lfloor x \rfloor+1,x)}\,\mathbb{1}_{\{j=\lfloor x \rfloor+1\}}\right) + \frac{\mu}{\lambda+\mu}\left(\left(q\,R_0 + (1-q)\,\hat{z}_{j,j}^{(\lfloor x \rfloor+1,x)}\right)\mathbb{1}_{\{i=1\}}\right. \\ &\left.+\left((q+(1-q)\,(1-p)\,\mathbb{1}_{\{j=\lfloor x \rfloor+1\}})\,\hat{z}_{i-1,j-1}^{(\lfloor x \rfloor+1,x)} + (1-q)\,(1-(1-p)\mathbb{1}_{\{j=\lfloor x \rfloor+1\}})\,\hat{z}_{i-1,j}^{(\lfloor x \rfloor+1,x)}\right)\right), \end{aligned}$$

where $p$ is the fractional part of $x$ as defined in Definition 5.2.1. Hence, we can calculate $\hat{z}_{i,j}^{(\lfloor x \rfloor+1,x)}$ via Poisson's equation

$$\left(I - \hat{P}^{(\lfloor x \rfloor+1,x)}\right)\hat{\boldsymbol{z}}^{(\lfloor x \rfloor+1,x)} = \boldsymbol{g}\,, \qquad (5.59)$$

where the matrix $\hat{P}^{(\lfloor x \rfloor+1,x)}$ and the vector $\boldsymbol{g}$ are

$$\hat{P}^{(\lfloor x \rfloor+1,x)} = \begin{bmatrix} A_0^{(1)} & A_1^{(1)} & \cdots & \cdots & \cdots & \cdots & 0 \\ A_{-1}^{(2)} & A_0^{(2)} & A_1^{(2)} & \cdots & \cdots & \cdots & 0 \\ \vdots & A_{-1}^{(3)} & A_0^{(3)} & A_1^{(3)} & \cdots & \cdots & 0 \\ \vdots & \vdots & A_{-1}^{(4)} & A_0^{(4)} & A_1^{(4)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \hat{A}_{-1}^{(\lfloor x \rfloor+1)} & \tilde{A}_0^{(\lfloor x \rfloor+1)} \end{bmatrix}, \qquad (5.60)$$

$$\hat{A}_{-1}^{(\lfloor x\rfloor+1)} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \frac{\mu q+\mu(1-q)(1-(x-\lfloor x\rfloor))}{\lambda+\mu} & 0 & \cdots & 0 \\ 0 & \frac{\mu q+\mu(1-q)(1-(x-\lfloor x\rfloor))}{\lambda+\mu} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\mu q+\mu(1-q)(1-(x-\lfloor x\rfloor))}{\lambda+\mu} \end{bmatrix} \in \mathbb{R}^{(\lfloor x\rfloor+1)\times\lfloor x\rfloor}$$

$$\tilde{A}_{0}^{(\lfloor x\rfloor+1)} = \begin{bmatrix} \frac{\lambda}{\lambda+\mu} & 0 & \cdots & 0 \\ 0 & \frac{\lambda}{\lambda+\mu} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\lambda}{\lambda+\mu} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & \frac{\mu(1-q)}{\lambda+\mu} \\ \frac{\mu(1-q)\,(x-\lfloor x\rfloor)}{\lambda+\mu} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{\mu(1-q)\,(x-\lfloor x\rfloor)}{\lambda+\mu} & 0 \end{bmatrix} \in \mathbb{R}^{(\lfloor x\rfloor+1)\times(\lfloor x\rfloor+1)}$$

$$\boldsymbol{g} = \frac{\mu q\, R_0}{\lambda+\mu} \sum_{j=1}^{\lfloor x\rfloor+1} \boldsymbol{e}_{\frac{j(j-1)}{2}} - \frac{1}{\lambda+\mu}\,\boldsymbol{e}\,, \tag{5.61}$$

and

$$\hat{\boldsymbol{z}}^{(\lfloor x\rfloor+1,x)} = (z_{1,1}^{(\lfloor x\rfloor+1,x)}, z_{1,2}^{(\lfloor x\rfloor+1,x)}, z_{2,2}^{(\lfloor x\rfloor+1,x)}, \cdots, z_{\lfloor x\rfloor,\lfloor x\rfloor+1}^{(\lfloor x\rfloor+1,x)}, z_{\lfloor x\rfloor+1,\lfloor x\rfloor+1}^{(\lfloor x\rfloor+1,x)})^T\,. \tag{5.62}$$

In Section 5.3.3, we derived the Nash equilibrium threshold value by finding the $m$ that satisfies the case in Figure 5.4. In the $N$-case, this means only $z_{\lfloor x\rfloor,\lfloor x\rfloor}^{(\lfloor x\rfloor)}$, $z_{\lfloor x\rfloor+1,\lfloor x\rfloor+1}^{(\lfloor x\rfloor)}$ and $z_{\lfloor x\rfloor+1,\lfloor x\rfloor+1}^{(x)}$ matter in calculating the Nash equilibrium, although the tagged customer can join at position $\lceil x\rceil+1$. Similarly, in the $R$-case we only care about $\hat{z}_{\lfloor x\rfloor,\lfloor x\rfloor}^{(\lfloor x\rfloor,\lfloor x\rfloor)}$, $\hat{z}_{\lfloor x\rfloor+1,\lfloor x\rfloor+1}^{(\lfloor x\rfloor+1,\lfloor x\rfloor)}$, and $\hat{z}_{\lfloor x\rfloor+1,\lfloor x\rfloor+1}^{(\lfloor x\rfloor+1,,x)}$, so we calculate $\hat{z}_{i,j}^{(\lfloor x\rfloor+1,\lfloor x\rfloor)}$ only for $1 \le i \le j \le \lfloor x\rfloor+1$.

In the $R$-case, when others use threshold $x$ and the tagged customer uses threshold $x_{tag} \ge \lceil x\rceil$, the queue size is never greater than $\lceil x\rceil$ at a time point where the tagged customer's service has failed, so the tagged customer will never renege after joining if she uses $x_{tag}$ even though other customers may do so. Hence the calculation of $\hat{\boldsymbol{z}}^{(x_{tag},x)}$ when $x_{tag} \ge \lceil x\rceil$ can be transfered to the calculation of the expected sojourn time. If we define $\hat{w}_{i,j}^{(x_{tag},x)}$ as the expected sojourn time of the tagged customer in the $R$-case, given that she is at position $i$ and uses threshold strategy $x_{tag}$, there are $j$ customers in the system, and the other customers all adopt threshold $x$, then when $x_{tag} = \lfloor x \rfloor + 1$,

$$\hat{\boldsymbol{w}}^{(\lfloor x \rfloor+1,x)} = \left( w_{1,1}^{(\lfloor x \rfloor+1,x)}, w_{1,2}^{(\lfloor x \rfloor+1,x)}, w_{2,2}^{(\lfloor x \rfloor+1,x)}, \cdots, w_{\lfloor x \rfloor,\lfloor x \rfloor+1}^{(\lfloor x \rfloor+1,x)}, w_{\lfloor x \rfloor+1,\lfloor x \rfloor+1}^{(\lfloor x \rfloor+1,x)} \right)^T, \tag{5.63}$$

satisfies a version of Poisson's equation similar to Equation (5.11)

$$\left( I - \hat{P}^{(\lfloor x \rfloor+1,x)} \right) \hat{\boldsymbol{w}} = \frac{1}{\lambda + \mu} \boldsymbol{e}, \tag{5.64}$$

and $\hat{\boldsymbol{z}}^{(\lfloor x \rfloor+1,x)} = R_0 - \hat{\boldsymbol{w}}^{(\lfloor x \rfloor+1,x)}$. Similar to the $N$-case, an equilibrium strategy exists and can be computed using algorithm 2.

We compare $\hat{z}_{j,j}^{(\lfloor x \rfloor+1,x)}$ and $z_{j,j}^{(x)}$ for $j = 1, \ldots, \lfloor x \rfloor + 1$ in Lemma 6.

**Lemma 6.** *When* $\lfloor x \rfloor < x$,

$$\hat{z}_{j,j}^{(\lfloor x \rfloor+1,x)} \geq z_{j,j}^{(x)} \quad for \quad j = 1, \ldots, \lfloor x \rfloor + 1. \tag{5.65}$$

*When* $\lfloor x \rfloor = x$,

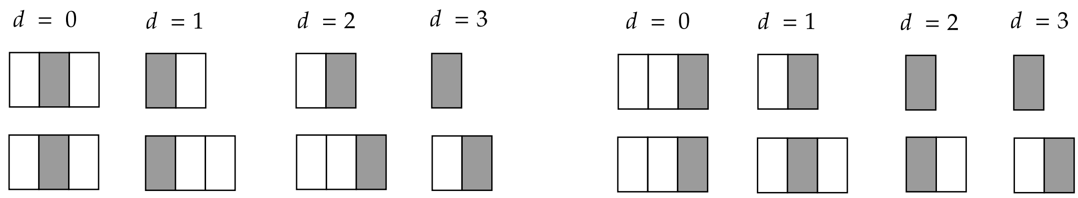
$$\hat{z}_{j,j}^{(\lfloor x \rfloor,x)} = \hat{z}_{j,j}^{(\lfloor x \rfloor+1,x)} = z_{j,j}^{(x)} \quad for \quad j = 1, \ldots, \lfloor x \rfloor. \tag{5.66}$$

*Proof.* It follows from the explanation at the beginning of this section that the tagged customer will never renege after joining if she uses $x_{tag} \geq \lceil x \rceil$ when others uses $x$. Thus the comparison of $z_{i,j}^{(x)}$ and $\hat{z}_{i,j}^{(\lfloor x \rfloor+1,x)}$ is actually the comparison of the respective expected sojourn time $w_{i,j}^{(x)}$ and $\hat{w}_{i,j}^{(\lfloor x \rfloor+1,x)}$. As in Lemma 3, the tagged customer's expected sojourn time is her expected total number of visits to different states until she leaves the system, times $\frac{1}{\lambda+\mu}$. Similar to the proof of Lemma 3, we first define

$${}^{d}\hat{v}_{i,j}^{(x)} := \left( (\hat{P}^{(\lfloor x \rfloor+1,x)})^d \boldsymbol{e} \right)_{\frac{j(j-1)}{2}+i} \qquad 1 \leq i \leq j \leq \lfloor x \rfloor + 1, \tag{5.67}$$

which is the probability that the tagged customer is still in the system at the $d$th transition, if her initial state is $(i, j)$, she never reneges, and others join and renege with threshold $x$. We prove Equation (5.65) by mathematical induction.

First, it follows from the physical interpretation that when $d \leq \lfloor x \rfloor$, ${}^{d}v_{i,j}^{(x)} = {}^{d}\hat{v}_{i,j}^{(\lfloor x \rfloor+1,x)}$.




(a) start with state $(2,3)$ (b) start with state $(3,3)$

Figure 5.7: An example of the comparison between ${}^{d}v^{(x)}_{i,\lfloor x\rfloor+1}$ and ${}^{d}\hat{v}^{(\lfloor x\rfloor+1,x)}_{i,\lfloor x\rfloor+1}$ when $2 < x < 3$.

This is because others' reneging will not affect the tagged customer until she rejoins the queue and is in service for the second time, and the queue size has reached $\lfloor x\rfloor + 1$ before she is in service for the first time. This is possible only when $d > \lfloor x\rfloor$. Indeed,

$$ {}^{\lfloor x\rfloor+1}v^{(x)}_{i,j} > {}^{\lfloor x\rfloor+1}\hat{v}^{(\lfloor x\rfloor+1,x)}_{i,j} \qquad 1 < i \le j = \lfloor x\rfloor + 1\,. \tag{5.68} $$

Figure 5.7 shows an example of Equation (5.68) with $(i,j) = (2,3)$ in (a), $(i,j) = (3,3)$ in (b), and $2 < x < 3$. The gray block represents the tagged customer and the white ones represent other customers in the system. It shows the sample paths where the reneging of others affects the tagged customer at the earliest possible transition. The first row is when others use threshold 2, and the second row is when others use threshold 3. In both Figure 5.7(a) and (b), the reneging of others affects the queue size at the first transition, but will not affect the tagged customer until $d > 3$.

Next, we assume that ${}^{d}v^{(x)}_{i,j} \ge {}^{d}\hat{v}^{(\lfloor x\rfloor+1,x)}_{i,j}$. Then we write the difference between $P^{(x)}$ and $\hat{P}^{(\lfloor x\rfloor+1,x)}$ in the form

$$ P^{(x)} = \hat{P}^{(\lfloor x\rfloor+1,x)} + E\Delta\,, \tag{5.69} $$

where

$$ E = \begin{bmatrix} \mathbf{0}_{\left(\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}+1\right)\times\lfloor x\rfloor} \\ I_n \end{bmatrix} \qquad \Delta = \begin{bmatrix} \mathbf{0}_{\lfloor x\rfloor\times\frac{\lfloor x\rfloor(\lfloor x\rfloor-1)}{2}} & -\frac{\mu(1-q)(1-p)}{\lambda+\mu} I_{\lfloor x\rfloor} & \frac{\mu(1-q)(1-p)}{\lambda+\mu} I_{\lfloor x\rfloor} & \mathbf{0}_{\lfloor x\rfloor\times 1} \end{bmatrix}\,. \tag{5.70} $$

Hence,

$$(P^{(x)})^{d+1}\boldsymbol{e} = P^{(\lfloor x\rfloor+1,x)}\left((P^{(x)})^{d}\boldsymbol{e}\right) + E\Delta\left((P^{(x)})^{d}\boldsymbol{e}\right) \geq P^{(\lfloor x\rfloor+1,x)}\left((P^{(x)})^{d}\boldsymbol{e}\right) \geq (P^{(\lfloor x\rfloor+1,x)})^{d+1}\boldsymbol{e}\,. \tag{5.71}$$

The first inequality follows from the conclusion that ${}^{d}\nu_{i,j}^{(x)}$ is increasing in $j$ for any $d$ and $1 \leq i \leq j \leq \lceil x\rceil$ in Lemma 3. Since ${}^{d}\nu_{i,\lfloor x\rfloor+1}^{(x)} \geq {}^{d}\nu_{i,\lfloor x\rfloor}^{(x)}$ for $i = 1, \ldots, \lfloor x\rfloor$,

$$\Delta\left((P^{(x)})^{d}\boldsymbol{e}\right) = \frac{\mu(1-q)(1-p)}{\lambda+\mu}\begin{bmatrix} {}^{d}\nu_{1,\lfloor x\rfloor+1}^{(x)} - {}^{d}\nu_{1,\lfloor x\rfloor}^{(x)} \\ {}^{d}\nu_{2,\lfloor x\rfloor+1}^{(x)} - {}^{d}\nu_{2,\lfloor x\rfloor}^{(x)} \\ \vdots \\ {}^{d}\nu_{\lfloor x\rfloor,\lfloor x\rfloor+1}^{(x)} - {}^{d}\nu_{\lfloor x\rfloor,\lfloor x\rfloor}^{(x)} \end{bmatrix} \geq 0\,. \tag{5.72}$$

The second inequality is from the induction assumption. Hence,

$${}^{d+1}\nu_{i,j}^{(x)} \geq {}^{d+1}\hat{\nu}_{i,j}^{(\lfloor x\rfloor+1,x)}\,. \tag{5.73}$$

This concludes the proof for Inequality (5.65).

When $x = \lfloor x\rfloor$, $\hat{P}^{(\lfloor x\rfloor,\lfloor x\rfloor)} = P^{(\lfloor x\rfloor)}$, and so $\hat{\boldsymbol{z}}^{(\lfloor x\rfloor,\lfloor x\rfloor)} = \boldsymbol{z}^{(\lfloor x\rfloor)}$. Also, $A_1^{(\lfloor x\rfloor)} = 0$, so

$$\begin{aligned}
&\begin{bmatrix} I_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} & 0_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}\times(\lfloor x\rfloor+1)} \end{bmatrix} P^{(\lfloor x\rfloor)} \\
=&\begin{bmatrix} I_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} & 0_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}\times(\lfloor x\rfloor+1)} \end{bmatrix} P^{(\lfloor x\rfloor)} \begin{bmatrix} I_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} \\ 0_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}\times(\lfloor x\rfloor+1)} \end{bmatrix} \begin{bmatrix} I_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} & 0_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}\times(\lfloor x\rfloor+1)} \end{bmatrix}.
\end{aligned} \tag{5.74}$$

Multiplying equation (5.11) with $\begin{bmatrix} I_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} & 0_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}\times(\lfloor x\rfloor+1)} \end{bmatrix}$ on both sides and applying equation (5.74), we have

$$(I - \bar{P}^{(\lfloor x\rfloor)})\begin{bmatrix} I_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} & 0_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}\times(\lfloor x\rfloor+1)} \end{bmatrix}\boldsymbol{w}^{(\lfloor x\rfloor)} = \frac{1}{\lambda+\mu}\boldsymbol{e} \tag{5.75}$$

where $\bar{P}^{(\lfloor x\rfloor)} := \begin{bmatrix} I_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} & 0_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}\times(\lfloor x\rfloor+1)} \end{bmatrix} P^{(\lfloor x\rfloor)} \begin{bmatrix} I_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} \\ 0_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}\times(\lfloor x\rfloor+1)} \end{bmatrix}$. Hence,

$$(\boldsymbol{w}^{(\lfloor x\rfloor)})_{1:\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} = \left(I - \bar{P}^{(\lfloor x\rfloor)}\right)^{-1} \frac{1}{\lambda+\mu}\boldsymbol{e}\,. \tag{5.76}$$

Similarly,

$$(\hat{\boldsymbol{w}}^{(\lfloor x\rfloor+1,\lfloor x\rfloor)})_{1:\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} = \left(I - \bar{\hat{P}}^{(\lfloor x\rfloor+1,\lfloor x\rfloor)}\right)^{-1} \frac{1}{\lambda+\mu}\boldsymbol{e}\,, \tag{5.77}$$

where $\bar{\hat{P}}^{(\lfloor x\rfloor)} := \left[I_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} \quad 0_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}\times(\lfloor x\rfloor+1)}\right] \hat{P}^{(\lfloor x\rfloor+1,\lfloor x\rfloor)} \begin{bmatrix} I_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} \\ 0_{\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}\times(\lfloor x\rfloor+1)} \end{bmatrix}$. Since $P^{(\lfloor x\rfloor+1,x)}$ and $P^{(x)}$ have the same first $\dfrac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}$ lines and rows, it follows that $\boldsymbol{w}^{(\lfloor x\rfloor)}_{1:\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}} = \hat{\boldsymbol{w}}^{(\lfloor x\rfloor+1,\lfloor x\rfloor)}_{1:\frac{\lfloor x\rfloor(\lfloor x\rfloor+1)}{2}}$. □

One interpretation of Lemma 6 is as follows. When other customers adopt the threshold $x$, for a customer who never reneges, her expected payoff is higher if the other customers are allowed to renege. When $x = \lfloor x\rfloor > 0$, the number of customers in the system never exceeds $\lfloor x\rfloor$ if the tagged customer joins at a position less than $\lfloor x\rfloor + 1$; if the tagged customer joins at $(\lfloor x\rfloor + 1)$th position, the customer who is in service when she joins will leave the system with probability 1: either the service will complete successfully or the customer will renege when the service fails. Thus if the tagged customer joins at position $\lfloor x\rfloor + 1$, then she is better off when others can renege, but there is no difference between the $N$-case and the $R$-case when the position at which the tagged customer joins is less than $\lfloor x\rfloor + 1$.

**Remark**. We can prove that the strict inequality holds in Lemma 6 when $x > \lfloor x\rfloor$. Also, an argument similar to that in Lemmas 3 and 4 can be used to show that for $1 \le i \le j \le \lfloor x\rfloor + 1$, $\hat{z}_{i,j}^{(\lfloor x\rfloor+1,x)}$ is strictly decreasing in $j$ for $1 \le j \le \lceil x\rceil + 1$, and $x$ when $x > 1$.

### 5.4.2 The Nash equilibrium and its comparison with the $N$-case

As in the $N$-case, to work out the Nash equilibrium in the $R$-case, we need to draw the best response plot and investigate the intersection point of $\mathcal{BR}(x)$ and $x$. When $\hat{z}_{m+1,m+1}^{(m+1,m)} \le 0 \le \hat{z}_{m,m}^{(m,m)}$, the tagged customer's best response when others adopt $m$ is also $m$, which is the case in Figure 5.4(a). When $\hat{z}_{m+1,m+1}^{(m+1,x)} = 0$ with $x \in (m, m+1)$, the tagged customer is indifferent between $m$ and $m+1$ when others use threshold $x$, which is the case in Figure 5.4(b).

Before we work out the Nash equilibrium strategy in the $R$-case and compare it with the $N$-case, we first define $Ne(R_0, \lambda, \mu, q)$ and $\hat{Ne}(R_0, \lambda, \mu, q)$ as the Nash equilibrium under the parameter set $R_0, \lambda, \mu, q$ in the $N$-case and the $R$-case, respectively, and use $x_e$ and $\hat{x}_e$ for short if they are from the same $R_0, \lambda, \mu, q$. Similar to our use of $\alpha_m$ and $\beta_m$ in the $N$-case, we let $\gamma_m := \hat{w}^{(m+1,m)}_{m+1,m+1}$ to help explain the Nash equilibrium in the $R$-case which is described in the following.

**Theorem 13.** *The Nash equilibrium threshold value when reneging is allowed is greater than or equal to that when reneging is not allowed.*

Proof. There are three scenarios.

- When $R_0 \in [\alpha_m, \gamma_m]$, then

$$\hat{z}^{(m+1,m)}_{m+1,m+1} = R_0 - \hat{w}^{(m+1,m)}_{m+1,m+1} = R_0 - \gamma_m \leq 0 \tag{5.78}$$

$$z^{(m)}_{m,m} = R_0 - w^{(m)}_{m,m} = R_0 - \alpha_m \geq 0\,. \tag{5.79}$$

  Hence $z^{(m)}_{m+1,m+1} < \hat{z}^{(m+1,m)}_{m+1,m+1} \leq 0 \leq \hat{z}^{(m,m)}_{m,m} = z^{(m)}_{m,m}$ with the first inequality and the equality following from Lemma 6. The tagged customer's best response is $m$ if others' strategy is $m$ in both the $N$-case and the $R$-case, hence $\hat{x}_e = x_e = m$. This case is depicted in Figure 5.8(a).

- When $R_0 \in (\gamma_m, \beta_m]$, then

$$\hat{z}^{(m+1,m)}_{m+1,m+1} = R_0 - \hat{w}^{(m+1,m)}_{m+1,m+1} = R_0 - \gamma_m > 0 \tag{5.80}$$

$$z^{(m)}_{m+1,m+1} = R_0 - w^{(m)}_{m+1,m+1} = R_0 - \beta_m \leq 0\,. \tag{5.81}$$

  Hence $\hat{z}^{(m+1,m+1)}_{m+1,m+1} < z^{(m)}_{m+1,m+1} \leq 0 < \hat{z}^{(m+1,m)}_{m+1,m+1} < \hat{z}^{(m+1,m)}_{m,m} = z^{(m)}_{m,m}$ with the first inequality, the last inequality and the equality following from Lemma 6. The tagged customer's best response is $m$ if others' strategy is $m$ in the $N$-case. In the $R$-case, $\hat{x}_e = \{x : \hat{z}^{(m+1,x)}_{m+1,m+1} = 0\}$ since the tagged customer is indifferent between joining at position $m+1$ and balking if others adopt threshold $\hat{x}_e$. In this case, $\hat{x}_e > x_e = m$, and it is depicted in Figure 5.8(b).

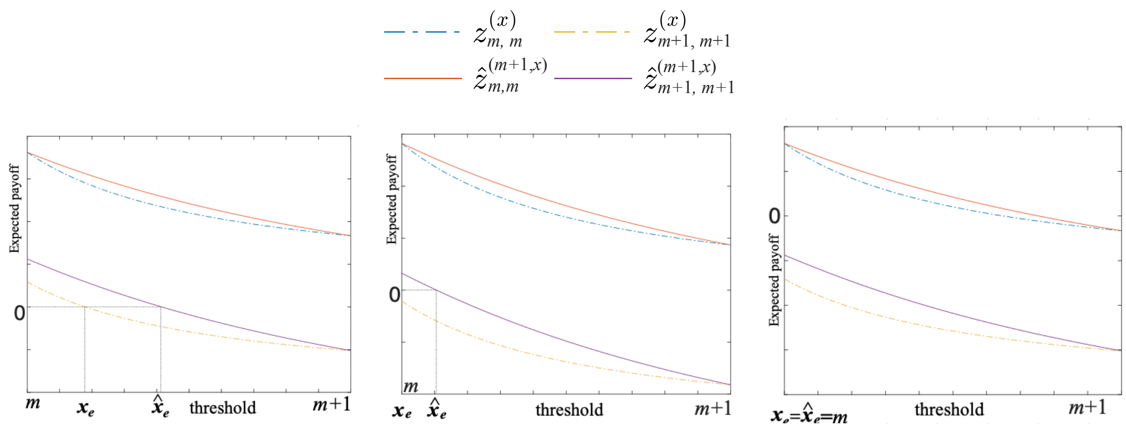
- When $R_0 \in (\beta_m, \alpha_{m+1})$, then

$$\hat{z}_{m+1,m+1}^{(m+1,m+1)} = z_{m+1,m+1}^{(m+1)} < 0 < z_{m+1,m+1}^{(m)} < \hat{z}_{m+1,m+1}^{(m+1,m)}, \tag{5.82}$$

with the first equality and the last inequality following from Lemma 6. The Nash equilibrium strategies are

$$x_e = \{x : z_{m+1,m+1}^{(x)} = 0\} \qquad \hat{x}_e = \{x : \hat{z}_{m+1,m+1}^{(m+1,x)} = 0\},$$

which are mixed in both cases. It follows from Equation (5.65) that $\hat{x}_e > x_e$. Note that $x_e$ here is the same as calculated in Theorem 12. This case is depicted in Figure 5.8(c). □




(a) $R_0 \in [\alpha_m, \gamma_m]$ (b) $R_0 \in (\gamma_m, \beta_m]$ (c) $R_0 \in (\beta_m, \alpha_{m+1})$

Figure 5.8: An illustration of Nash equilibrium threshold comparison.

## 5.5 Two Paradoxes

In the $N$-case, every customer remains in the system until she successfully completes her service and receives reward $R_0$. Increasing $R_0$ can increase customers' incentive to join but also make the system more crowded. In this situation, does everyone become better off when the reward $R_0$ increases? To answer this question, we observe that there are parameter settings where the equilibrium expected payoff can decrease with $R_0$. This paradoxical behaviour is discussed in the following.

**Paradox 1.** *In the N-case, let* $x_k = NE(R_k, \lambda, \mu, q)$, $k = 1, 2$. *Then for* $m = 1, 2$, $z^{(x_1)}_{m,m} > z^{(x_2)}_{m,m}$ *if* $\beta_m < R_1 < R_2 < \alpha_{m+1}$, *where* $m$ *is the integer part of the Nash equilibrium. In other words, if* $R_0 \in (\beta_m, \alpha_{m+1})$, *increasing* $R_0$ *will make everyone joining at position* $m$ *worse off.*

Proof. As in Definition 5.2.1, $p$ is the fractional part of $x$. When $x \in (1, 2)$,

$$w^{(x)}_{2,2} - w^{(x)}_{1,1} = \frac{\mu + \lambda p}{\mu(-\mu q^2 + 2\mu q + \lambda p)} = \frac{1}{\mu\left(1 - \frac{\mu(1-q)^2}{\mu + \lambda p}\right)}, \tag{5.83}$$

which is decreasing in $p$. When $x \in (2, 3)$,

$$w^{(x)}_{3,3} - w^{(x)}_{2,2} = \frac{1}{\mu(1 - \mu^2(1-q)^2 f(p))}, \tag{5.84}$$

where $f(p) = \dfrac{\lambda + 2\lambda pq + \mu q^3 - 3\mu q^2 - \lambda q + 3\mu q}{(\mu + \lambda p)\left(\lambda\mu + \lambda^2 p + \lambda\mu pq + \mu^2 q^3 - 3\mu^2 q^2 + 3\mu^2 q\right)}$. The derivative of the function $f(p)$

$$\begin{aligned}\frac{df(p)}{dp} &= -\frac{\begin{array}{c}\lambda\left(2\lambda^2 p^2 q(\lambda + \mu q) + 2\lambda p(\lambda + \mu q)(\lambda(1-q) + \mu q((q-3)q+3))\right.\\ \left.+\mu(1-q)\left(2\lambda^2 + \mu^2(2-q)q^2((q-3)q+3) + \lambda\mu q((q-5)q+8)\right)\right)\end{array}}{(\mu + \lambda p)^2(\lambda p(\lambda + \mu q) + \mu(\lambda + \mu q((q-3)q+3)))^2} \\ &< 0\,.\end{aligned} \tag{5.85}$$

Thus, $f(p)$ is decreasing in $p$, then $w^{(x)}_{3,3} - w^{(x)}_{2,2}$ is decreasing in $p$.

From Theorem 12, if $\beta_m < R_1 < R_2 < \alpha_{m+1}$, then $x_k$, which is the Nash equilibrium threshold when $R = R_k$, satisfies

$$w^{(x_k)}_{m+1,m+1} = R_k \qquad x_k \in (m, m+1) \qquad k = 1, 2\,. \tag{5.86}$$

Thus, for $m = 1, 2$,

$$z^{(x_1)}_{m,m} = R_1 - w^{(x_1)}_{m,m} = w^{(x_1)}_{m+1,m+1} - w^{(x_1)}_{m,m} > w^{(x_2)}_{m+1,m+1} - w^{(x_2)}_{m,m} = R_2 - w^{(x_2)}_{m,m} = z^{(x_2)}_{m,m}\,. \tag{5.87}$$

□

In Paradox 1, we have proved that if $R_0 \in (\beta_m, \alpha_{m+1})$ for $m = 1, 2$, increasing $R_0$

makes everyone joining at position $m$ worse off. We conjecture that this phenomenon holds for any $m$. Our numerical experience indicates that this is the case. However, the proof has eluded us.

We have proved in the previous section that customers have a higher incentive to join the system if they are allowed to renege later. However, with more customers joining, the system can be more crowded. So we are interested in the question: when customers are given the right to leave, do they become better off?

To answer this question, we first need to work out the equilibrium expected payoff in the $R$-case. In contrast to the $N$-case, customers may renege before they successfully complete the service in the $R$-case, thus their expected payoff cannot be calculated as the difference between $R_0$ and their expected sojourn cost. By similar reasoning to Equation (5.10), it follows that

$$\hat{z}_{i,j}^{(x,x)} = -\frac{1}{\lambda+\mu} \tag{5.88}$$

$$+\frac{\lambda}{\lambda+\mu}\left(\hat{z}_{i,j+1}^{(x,x)}\,\mathbb{1}_{\{j<\lfloor x\rfloor\}}+\left(p\,\hat{z}_{i,j+1}^{(x,x)}+(1-p)\,\hat{z}_{i,j}^{(x,x)}\right)\,\mathbb{1}_{\{j=\lfloor x\rfloor\}}+\hat{z}_{i,j}^{(x,x)}\,\mathbb{1}_{\{j=\lfloor x\rfloor+1\}}\right)$$

$$+\frac{\mu}{\lambda+\mu}\left[\left(q\,R_0+(1-q)\,(1-(1-p)\mathbb{1}_{\{j=\lfloor x\rfloor+1\}})\,\hat{z}_{j,j}^{(x,x)}\right)\,\mathbb{1}_{\{i=1\}}\right. \tag{5.89}$$

$$\left.+\left((q+(1-q)\,(1-p)\,\mathbb{1}_{\{j=\lfloor x\rfloor+1\}})\,\hat{z}_{i-1,j-1}^{(x,x)}+(1-q)\,(1-(1-p)\mathbb{1}_{\{j=\lfloor x\rfloor+1\}})\,\hat{z}_{i-1,j}^{(x,x)}\right)\,\mathbb{1}_{\{i>1\}}\right].$$

Thus, we can obtain $\hat{z}_{i,j}^{(x,x)}$ via Poisson's equation

$$\left(I-\hat{P}^{(x,x)}\right)\,\hat{z}^{(x,x)} = g\,, \tag{5.90}$$

where

$$\hat{P}^{(x,x)} = \begin{bmatrix} A_0^{(1)} & A_1^{(1)} & \cdots & \cdots & \cdots & \cdots & 0 \\ A_{-1}^{(2)} & A_0^{(2)} & A_1^{(2)} & \cdots & \cdots & \cdots & 0 \\ \vdots & A_{-1}^{(3)} & A_0^{(3)} & A_1^{(3)} & \cdots & \cdots & 0 \\ \vdots & \vdots & A_{-1}^{(4)} & A_0^{(4)} & A_1^{(4)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \hat{A}_{-1}^{(\lfloor x\rfloor+1)} & \hat{A}_0^{(\lfloor x\rfloor+1)} \end{bmatrix}, \tag{5.91}$$

$$\hat{A}_0^{(\lfloor x\rfloor+1)} = \begin{bmatrix} \frac{\lambda}{\lambda+\mu} & 0 & \cdots & 0 \\ 0 & \frac{\lambda}{\lambda+\mu} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\lambda}{\lambda+\mu} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & \frac{\mu(1-q)(x-\lfloor x\rfloor)}{\lambda+\mu} \\ \frac{\mu(1-q)(x-\lfloor x\rfloor)}{\lambda+\mu} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \frac{\mu(1-q)(x-\lfloor x\rfloor)}{\lambda+\mu} & 0 \end{bmatrix} \tag{5.92}$$

$$\hat{\mathbf{z}}^{(x,x)} = (z_{1,1}^{(x,x)}, z_{1,2}^{(x,x)}, z_{2,2}^{(x,x)}, z_{1,3}^{(x,x)}, z_{2,3}^{(x,x)}, z_{3,3}^{(x,x)}, \cdots, z_{\lceil x\rceil,\lceil x\rceil+1}^{(x,x)}, z_{\lceil x\rceil+1,\lceil x\rceil+1}^{(x,x)})^T. \tag{5.93}$$

We are interested in the expected payoff when every customer uses $\hat{x}_e$. In Lemma 6, we have proved that if $\hat{x}_e = \lfloor \hat{x}_e \rfloor$, $\hat{\mathbf{z}}^{(\hat{x}_e,\hat{x}_e)} = \mathbf{z}^{(x_e)}$. In the following, we prove that if $\hat{z}_{m+1,m+1}^{(m+1,\hat{x}_e)} = 0$, $\hat{\mathbf{z}}^{(\hat{x}_e,\hat{x}_e)} = \hat{\mathbf{z}}^{(m+1,\hat{x}_e)}$.

**Lemma 7.** *If $\hat{z}_{m+1,m+1}^{(m+1,\hat{x}_e)} = 0$, then $\hat{z}_{i,j}^{(\hat{x}_e,\hat{x}_e)} = \hat{z}_{i,j}^{(m+1,x_e)}$ for any $1 \le i \le j \le m+1$.*

*Proof.* First, when $\hat{z}_{m+1,m+1}^{(m+1,\hat{x}_e)} = 0$, the tagged customer is indifferent between joining or not joining at position $m+1$. In other words, joining with any probability at position $m+1$ will result in a zero expected payoff for her. Hence, $\hat{z}_{m+1,m+1}^{(x,\hat{x}_e)} = 0$ for any $x \in [m, m+1]$ including $\hat{x}_e$.

For a general state $(i,j)$, consider two queues with others using threshold $\hat{x}_e$ and the tagged customer in state $(i,j)$: she uses threshold $m+1$ in queue 1 and $\hat{x}_e$ in queue 2. By coupling the customer arrival processes, their joining decisions, the service processes and the service success probability for every customer including the tagged one, we can see the next customer will arrive, join or not join both queues at the same time, the customer in service in both queues will complete the service and rejoin or not rejoin the queue at the same time, until the first time the tagged customer needs to rejoin the queue and the queue size including her is $m+1$ when she rejoins it. When this is the case, $\hat{z}_{m+1,m+1}^{(x,\hat{x}_e)} = 0$ for any $x \in [m, m+1]$, due to $\hat{z}_{m+1,m+1}^{(m+1,\hat{x}_e)} = 0$. Hence, the tagged customer in both queues either has exactly the same sample path, or reaches the state $(m+1, m+1)$ where her remaining expected payoff is 0 regardless of her decision. So the tagged customer in both queues receives the same expected payoff. □

When $R_0 \in [\alpha_m, \gamma_m]$, then $x_e = \hat{x}_e = m$, thus, $z_{i,i}^{(m)} = \hat{z}_{i,i}^{(m,m)}$, and the stationary distributions in the $N$-case and the $R$-case are the same. Thus there is no difference between the two cases. When $R_0 \in (\gamma_m, \alpha_{m+1}]$, we observe that both the equilibrium expected payoff and the total expected payoff decrease when reneging is allowed. Specifically, when $R_0 \in (\gamma_m, \beta_m)$, we prove this paradoxical behaviour in the following.

**Paradox 2.** *If $R_0 \in (\gamma_m, \beta_m]$, then*

$$\hat{z}_{i,i}^{(\hat{x}_e,\hat{x}_e)} < z_{i,i}^{(x_e)} \quad i = 1, \cdots, m\,.$$

*Furthermore, the total expected payoffs under equilibrium satisfy*

$$\sum_{i=1}^{m+1} \hat{\pi}_{i-1}^{(\hat{x}_e)}\, \hat{z}_{i,i}^{(\hat{x}_e,\hat{x}_e)} < \sum_{i=1}^{m} \pi_{i-1}^{(x_e)} z_{i,i}^{(x_e)}$$

*where $\pi_k^{(x)}$ and $\hat{\pi}_k^{(x)}$ denote the stationary distribution of the number of customers in the system in the N-case and the R-case, respectively.*

Proof. If $R_0 \in (\gamma_m, \beta_m]$, then $\hat{x}_e > x_e = m$. Since the Nash equilibrium threshold in the $R$-case is mixed, it follows from Lemma 7 that $z_{m+1,m+1}^{(\hat{x}_e,\hat{x}_e)} = z_{m+1,m+1}^{(m+1,\hat{x}_e)} = 0$, and $z_{i,i}^{(\hat{x}_e,\hat{x}_e)} = z_{i,i}^{(m+1,\hat{x}_e)}$. In this way,

$$z_{i,i}^{(x_e)} = z_{i,i}^{(m)} = \hat{z}_{i,i}^{(m+1,m)} > \hat{z}_{i,i}^{(m+1,\hat{x}_e)} = \hat{z}_{i,i}^{(\hat{x}_e,\hat{x}_e)} \qquad 1 \le i \le m\,, \tag{5.94}$$

with the second equality following from Lemma 6, and the inequality following from the fact that $\hat{z}_{m+1,m+1}^{(m+1,x)}$ is decreasing in $x$.

Next, we calculate the stationary distribution of the number of customers in the system in the $N$-case and the $R$-case. Figures 5.9 and 5.10 depict the transition rate diagram for both cases, given that each customer uses threshold $x$. Let $\rho = \dfrac{\lambda}{\mu q}$, it follows from the detailed balance equations that for $k = 0, \cdots, \lfloor x \rfloor$,

$$\begin{aligned} \pi_k^{(x)} &= \frac{\rho^k}{(x - \lfloor x \rfloor)\,\rho^{\lfloor x \rfloor + 1} + \dfrac{\rho^{\lfloor x \rfloor + 1} - 1}{\rho - 1}} \\ \hat{\pi}_k^{(x)} &= \frac{\rho^k}{\dfrac{\lambda(x - \lfloor x \rfloor)}{\mu q + \mu(1-q)(1-(x-\lfloor x \rfloor))}\rho^{\lfloor x \rfloor} + \dfrac{\rho^{\lfloor x \rfloor + 1} - 1}{\rho - 1}}\,, \end{aligned} \tag{5.95}$$

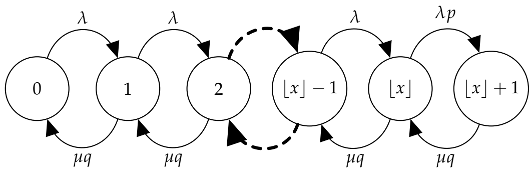



Figure 5.9: Transition rate diagram when the threshold is $x$ when reneging is not allowed.

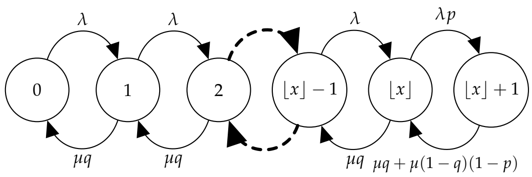



Figure 5.10: Transition rate diagram when the threshold is $x$ when reneging is allowed.

and

$$\pi^{(x)}_{\lfloor x\rfloor+1} = \frac{(x-\lfloor x\rfloor)\,\rho^{n+1}}{(x-\lfloor x\rfloor)\,\rho^{\lfloor x\rfloor+1} + \dfrac{\rho^{\lfloor x\rfloor+1}-1}{\rho-1}} \tag{5.96}$$

$$\hat{\pi}^{(x)}_{\lfloor x\rfloor+1} = \frac{\dfrac{\lambda\,(x-\lfloor x\rfloor)}{\mu q+\mu(1-q)(1-(x-\lfloor x\rfloor))}\rho^{\lfloor x\rfloor}}{\dfrac{\lambda\,(x-\lfloor x\rfloor)}{\mu q+\mu(1-q)(1-(x-\lfloor x\rfloor))}\rho^{\lfloor x\rfloor} + \dfrac{\rho^{\lfloor x\rfloor+1}-1}{\rho-1}}\,.$$

Since $\hat{x}_e > x_e = m$,

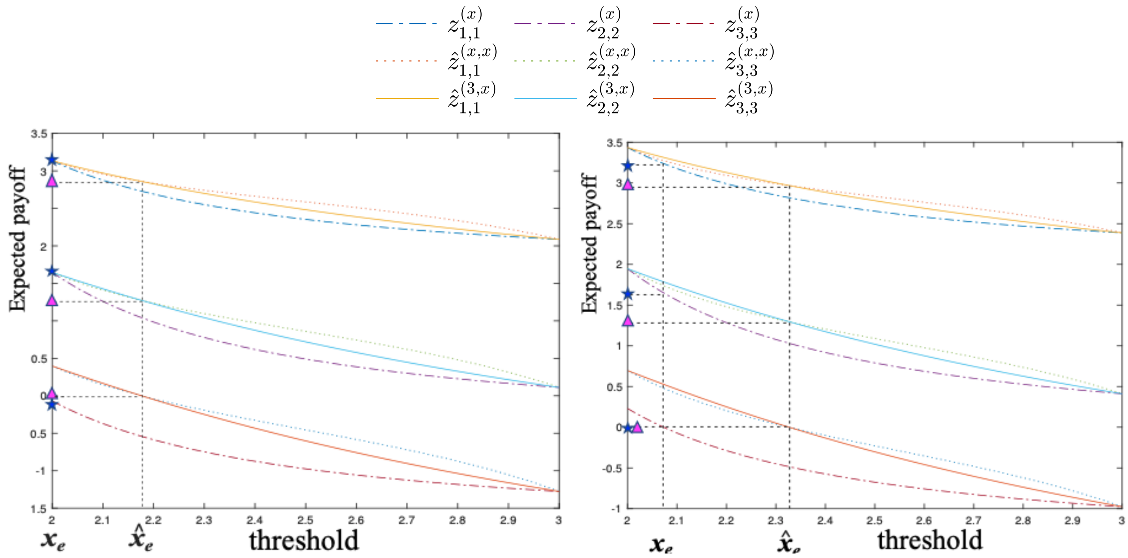
$$\hat{\pi}^{(\hat{x}_e)}_k < \pi^{(x_e)}_k \quad k=0,\cdots,m \qquad \hat{\pi}^{(\hat{x}_e)}_{m+1} > 0 = \pi^{(x_e)}_{m+1}\,. \tag{5.97}$$

Hence

$$\sum_{i=1}^{m+1} \hat{\pi}^{(\hat{x}_e)}_{i-1}\,\hat{z}^{(\hat{x}_e,\hat{x}_e)}_{i,i} = \sum_{i=1}^{m} \hat{\pi}^{(\hat{x}_e)}_{i-1}\,\hat{z}^{(\hat{x}_e,\hat{x}_e)}_{i,i} < \sum_{i=1}^{m} \pi^{(x_e)}_{i-1} z^{(x_e)}_{i,i}\,, \tag{5.98}$$

with the equality following from $z^{(m+1,\hat{x}_e)}_{m+1,m+1} = 0$, and the inequality following from Equations (5.94) and (5.97). □

It can be seen that although the expected payoff $z^{(x_e)}_{m+1,m+1}$ is smaller than $\hat{z}^{(\hat{x}_e,\hat{x}_e)}_{m+1,m+1}$, customers do not really join at position $m+1$ in the $R$-case equilibrium, thus it is not included in the total expected payoff.




(a) $R_0 \in (\gamma_m, \beta_m]$ (b) $R_0 \in (\beta_m, \alpha_{m+1})$

Figure 5.11: Allowing reneging can make everyone worse off.

We illustrate Equation (5.94) in Figure 5.11(a) via an example with $R_0 = 7.5, \lambda = 1, \mu = 0.8, q = 0.4$. The Nash equilibrium threshold is 2 and 2.167 in the $N$-case and the $R$-case, respectively. The blue stars represent $\boldsymbol{z}^{(x_e)}$ with $z_{1,1}^{(x_e)} > z_{2,2}^{(x_e)} > 0 > z_{3,3}^{(x_e)}$, and the triangles represent $\hat{\boldsymbol{z}}^{(\hat{x}_e,\hat{x}_e)}$ with $z_{1,1}^{(\hat{x}_e,\hat{x}_e)} > z_{2,2}^{(\hat{x}_e,\hat{x}_e)} > z_{3,3}^{(\hat{x}_e,\hat{x}_e)} = 0$. It can be observed that and $z_{i,i}^{(x_e)} > \hat{z}_{i,i}^{(\hat{x}_e,\hat{x}_e)}$ for $i = 1, 2$.

In Paradox 2, we proved that when $R_0 \in (\gamma_m, \beta_m]$, allowing reneging makes everyone worse off. Next, we use some numerical examples to show that allowing reneging can make everyone worse off when $R_0 \in (\beta_m, \alpha_{m+1})$. Actually, we observed the paradox in every example.

We first illustrate $z_{i,i}^{(x_e)} > \hat{z}_{i,i}^{(x_e,x_e)}$ for any $1 \leq i \leq \lfloor x_e \rfloor$ in Figure 5.11(b) via an example with $R_0 = 7.8, \lambda = 1, \mu = 0.8, q = 0.4$. The Nash equilibrium is 2.073 and 2.327 in the $N$-case and the $R$-case, respectively. The blue stars represent $\boldsymbol{z}^{(x_e)}$ with $z_{1,1}^{(x_e)} > z_{2,2}^{(x_e)} > z_{3,3}^{(x_e)} = 0$, and the triangles represent $\hat{\boldsymbol{z}}^{(\hat{x}_e,\hat{x}_e)}$ with $z_{1,1}^{(\hat{x}_e,\hat{x}_e)} > z_{2,2}^{(\hat{x}_e,\hat{x}_e)} > z_{3,3}^{(\hat{x}_e,\hat{x}_e)} = 0$. It can be observed that $z_{i,i}^{(x_e)} > \hat{z}_{i,i}^{(\hat{x}_e,\hat{x}_e)}$ for $i = 1, 2$.

Next, we list the Nash equilibrium, the stationary probabilities of having 0 or 1 customers in the queue, and the expected payoff of three examples with different values of

Table 5.1: An numerical example when $R_0 \in (\beta_m, \alpha_{m+1})$.

| $R_0, \lambda, \mu, q$ | | 7.8, 1, 0.8, 0.4 | | 4.4, 1, 0.8, 0.8 | | 13.5, 0.8, 1, 0.2 | |
|---|---|---|---|---|---|---|---|
| $x_e$ | $\hat{x}_e$ | 2.073 | 2.327 | 2.345 | 2.444 | 2.529 | 2.872 |
| $z_{1,1}^{(x_e)}$ | $\hat{z}_{1,1}^{(\hat{x}_e,\hat{x}_e)}$ | 3.245 | 2.964 | 2.599 | 2.591 | 3.740 | 3.546 |
| $z_{2,2}^{(x_e)}$ | $\hat{z}_{2,2}^{(\hat{x}_e,\hat{x}_e)}$ | 1.661 | 1.292 | 1.271 | 1.259 | 1.514 | 1.283 |
| $\pi_0^{(x_e)}$ | $\hat{\pi}_0^{(\hat{x}_e)}$ | 0.063 | 0.053 | 0.158 | 0.154 | 0.018 | 0.017 |
| $\pi_1^{(x_e)}$ | $\hat{\pi}_1^{(\hat{x}_e)}$ | 0.195 | 0.165 | 0.247 | 0.241 | 0.073 | 0.069 |

$R_0, \lambda, \mu$, and $q$ given in Table 5.1, to show that not only $z_{i,i}^{(x_e)} > \hat{z}_{i,i}^{(\hat{x}_e,\hat{x}_e)}$, but also $\pi_{i-1}^{(x_e)} > \hat{\pi}_{i-1}^{(\hat{x}_e)}$ for any $1 \le i \le \lfloor x \rfloor$. It follows from the transition rate diagram in Figure 5.9 and 5.10 that, for $i = 1, \ldots m-1$, the transition rates going from state $i$ to state $i+1$, and vice versa, are identical in the $N$-case and the $R$-case, so the only difference in the stationary distribution for the two cases is the normalisation constant. The greater constant in the $R$-case makes the first $m$ states have less probability mass than the $N$-case, and it is only the final one that compensates. The first example has the same parameters as that in Figure 5.11. When the Nash equilibrium is fractional, $z_{m+1,m+1}^{(x_e)} = \hat{z}_{m+1,m+1}^{(\hat{x}_e,\hat{x}_e)} = 0$ where $m$ is the integer part of the Nash equilibrium, so we omit this in the table. Also, comparing $\sum_{i=1}^{m+1} \pi_{i-1}^{(x_e)} z_{i,i}^{(x_e)}$ and $\sum_{i=1}^{m+1} \hat{\pi}_{i-1}^{(\hat{x}_e)} \hat{z}_{i,i}^{(\hat{x}_e,\hat{x}_e)}$ is the same as comparing $\sum_{i=1}^{m} \pi_{i-1}^{(x_e)} z_{i,i}^{(x_e)}$ and $\sum_{i=1}^{m} \hat{\pi}_{i-1}^{(\hat{x}_e)} \hat{z}_{i,i}^{(\hat{x}_e,\hat{x}_e)}$, hence we only need to calculate $\pi_i^{(x_e)}$ and $\hat{\pi}_i^{(\hat{x}_e)}$ for $i = 0, \cdots, m-1$, so we omit $\pi_k^{(x_e)}$ and $\hat{\pi}_k^{(\hat{x}_e)}$ for $k = m, m+1$. In Table 5.1, the Nash equilibrium thresholds of the three examples are all fractional and have the same integer part, that is, 2. We observe that $z_{i,i}^{(x_e)} > \hat{z}_{i,i}^{(\hat{x}_e,\hat{x}_e)}$ and $\pi_{i-1}^{(x_e)} > \hat{\pi}_{i-1}^{(\hat{x}_e)}$ for $i = 1, 2$.

## 5.6 Social Welfare

In the previous section, we showed that allowing reneging can make every customer worse off. If the goal is to maximise the social welfare which is defined as the total expected net benefit of all customers, how does the reneging affect the social welfare? In this section, we calculate and compare the optimal threshold from the social point of view in the $N$-case and the $R$-case.

### 5.6.1 Social welfare in the $N$-case

When the customers all adopt threshold $x$, the state transition rate diagram in the non-reneging case is shown in Figure 5.9, the social welfare

$$
\begin{aligned}
S^N(x) :&= \lambda \left( \sum_{k=1}^{\lfloor x \rfloor} \pi_{k-1}^{(x)} z_{k,k}^{(x)} + (x - \lfloor x \rfloor) \pi_{\lfloor x \rfloor}^{(x)} z_{\lfloor x \rfloor+1, \lfloor x \rfloor+1}^{(x)} \right) \\
&= \lambda R_0 \left( \sum_{k=0}^{\lfloor x \rfloor -1} \pi_k^{(x)} + (x - \lfloor x \rfloor) \pi_{\lfloor x \rfloor}^{(x)} \right) - \sum_{k=0}^{\lceil x \rceil} k\, \pi_k^{(x)} \\
&= \frac{\begin{array}{c} \lambda R_0 (\rho - 1) \left( (1 + (x - \lfloor x \rfloor) (\rho - 1)) \rho^{\lfloor x \rfloor} - 1 \right) \\ + \rho \left( (1 - \lfloor x \rfloor (1 + (x - \lfloor x \rfloor)(\rho - 1))(\rho - 1) - (x - \lfloor x \rfloor)(\rho - 1)^2 \right) \rho^{\lfloor x \rfloor} - 1) \end{array}}{1 + \rho((1 + (x - \lfloor x \rfloor)(\rho - 1))(\rho - 1)\rho^{\lfloor x \rfloor} - 1)} .
\end{aligned}
\tag{5.99}
$$

where $\rho := \dfrac{\lambda}{\mu q}$, and the second equality follows from Little's law.

**Proposition 1.** *Social welfare $S^N(x)$ is unimodal.*

Proof. We first take the derivative of $S^N(x)$,

$$
\frac{dS^N(x)}{dx} = \begin{cases} \dfrac{\rho^{\lfloor x \rfloor} \left( R_0 \lambda (\rho - 1)^2 - \rho \left( 1 - 2\rho + \lfloor x \rfloor (1 - \rho) + \rho^{\lfloor x \rfloor + 2} \right) \right)}{(1 + \rho^{\lfloor x \rfloor + 1}((x - \lfloor x \rfloor)(1 - \rho) - 1))^2} & \text{when } x > \lfloor x \rfloor \\ \text{undefined} & \text{when } x = \lfloor x \rfloor . \end{cases}
\tag{5.100}
$$

To see that $S^N(x)$ is unimodal, let

$$
f(k) = \left( 1 - 2\rho + k(1 - \rho) + \rho^{2+k} \right) \quad k = 0, 1, 2, \cdots ,
\tag{5.101}
$$

and observe that the numerator in the first Equation of (**??**) can be written as

$$
\rho^{\lfloor x \rfloor} \left( R_0 \lambda (\rho - 1)^2 - \rho f(\lfloor x \rfloor) \right) .
$$

When $\rho \neq 1$

$$
f(k+1) - f(k) = \rho^{2+k}(\rho - 1) + (1 - \rho) = (1 - \rho) \left( 1 - \rho^{2+k} \right) > 0 .
\tag{5.102}
$$

We assume that $R_0 > \frac{1}{\mu q}$ to avoid the trivial case where the reward is smaller than the expected service time even if a customer does not have to wait. Hence

$$f(0) = (-1+\rho)^2 < \frac{R_0\lambda}{\rho}(1-\rho)^2 \tag{5.103}$$

and

$$\lim_{n\to\infty} f(k) = \infty > \frac{R_0\lambda}{\rho}(1-\rho)^2, \tag{5.104}$$

and so,

$$\rho f(0) < \cdots < R_0\lambda(1-\rho)^2 < \cdots < \rho \lim_{n\to\infty} f(k)\,. \tag{5.105}$$

Thus, there exists an integer $n_N^\star$ such that $\frac{dS^N}{dx}$ is increasing when $x \le n_N^\star$; is decreasing when $x > n_N^\star$. That is, $n_N^\star$ is the socially optimal threshold. □

It can be observed that $n_N^\star = \lfloor \nu \rfloor$, where $\nu$ satisfies

$$R_0\mu q - \nu = \frac{\rho}{(1-\rho)^2}\left(\nu(1-\rho) - 1 + \rho^\nu\right)\,. \tag{5.106}$$

This coincides with Naor's result in Equation (2.84) in Section 2.2.1 for the non-feedback $M/M/1$ queue. In other words, from the perspective of society, the feedback parameter $q$ affects the social welfare as it lowers the service rate from $\mu$ to $\mu q$. In addition, the socially optimal threshold value is an integer even though customers are allowed to use fractional thresholds. Figure 5.12 (the blue curve) illustrates how the social welfare varies with the threshold value.

### 5.6.2 Social welfare in the $R$-case

When customers are allowed to renege after they join, the social welfare calculation is more involved, as not every customer who chooses to join contributes $R_0$ to the social welfare. On this account, to calculate the social welfare in the $R$-case, we need to work out the probability that a joining customer reneges before she successfully completes the service. Denote this probability by $\tilde{p}^{(x)}$ when every customer uses threshold $x$. In order to obtain $\tilde{p}^{(x)}$, we first calculate the distribution

$$\tilde{\pi}_k^{(x)} = \frac{\mu(1-q)\,\hat{\pi}_{k+1}^{(x)}}{\sum_{k=0}^{\lfloor x \rfloor} \mu(1-q)\,\hat{\pi}_{k+1}^{(x)}} = \frac{\hat{\pi}_{k+1}^{(x)}}{\sum_{k=0}^{\lfloor x \rfloor} \hat{\pi}_{k+1}^{(x)}} \qquad k = 0, \cdots, \lfloor x \rfloor\,. \tag{5.107}$$

of the number of customers in the system observed by each feedback customer. Each joining customer can only renege when her service fails and there are $\lfloor x \rfloor$ other customers in the system. If this is the case, she reneges with probability $1-p$. So a joining customer reneges at her $k$th feedback with probability

$$\tilde{p}_k^{(x)} := (1-q)(1-p)\,\tilde{\pi}_{\lfloor x \rfloor}^{(x)} \left( (1-q)\left(1-(1-p)\,\tilde{\pi}_{\lfloor x \rfloor}^{(x)}\right)\right)^{k-1}\,. \tag{5.108}$$

Hence, the probability that a joining customer reneges before she successfully completes the service is given by

$$\tilde{p}^{(x)} = \sum_{k=1}^{\infty} \tilde{p}_k^{(x)} = \frac{(1-q)\,(1-p)\,\tilde{\pi}_{\lfloor x \rfloor}^{(x)}}{1-(1-q)\,\left(1-(1-p)\,\tilde{\pi}_{\lfloor x \rfloor}^{(x)}\right)}\,. \tag{5.109}$$

Then the social welfare in the $R$-case is

$$\begin{aligned} S^R(x) :&= \lambda \left( \sum_{k=1}^{\lfloor x \rfloor} \hat{\pi}_{k-1}^{(x)}\,\hat{z}_{k,k}^{(x)} + (x-\lfloor x \rfloor)\,\hat{\pi}_{\lfloor x \rfloor}^{(x)}\,\hat{z}_{\lfloor x \rfloor+1,\lfloor x \rfloor+1}^{(x)} \right) \\ &= \lambda\,R_0 \left( \sum_{k=0}^{\lfloor x \rfloor -1} \hat{\pi}_k^{(x)} + (x-\lfloor x \rfloor)\,\hat{\pi}_{\lfloor x \rfloor}^{(x)} \right) \left(1-\tilde{p}^{(x)}\right) - \sum_{k=0}^{\lceil x \rceil} k\,\hat{\pi}_k^{(x)}\,, \\ &= \frac{\begin{array}{c}\rho\,(R_0\,\mu q(1-\rho)\,(\rho^n((x-\lfloor x \rfloor)(q\rho-1)+1)-(x-\lfloor x \rfloor)(1-q)+1) \\ +\,(\rho^n((n(\rho-1)((x-\lfloor x \rfloor)(\rho-1)+1)+(x-\lfloor x \rfloor)(q(\rho-2)\rho+1)-1))-(x-\lfloor x \rfloor)(1-q)+1))\end{array}}{(1-\rho)\,(\rho^{n+1}((x-\lfloor x \rfloor)(q\rho-1)+1)+(x-\lfloor x \rfloor)(1-q)-1)}\,. \end{aligned} \tag{5.110}$$

If we take the derivative of $S^R(x)$, we have

$$\frac{dS^R(x)}{dx} = \begin{cases} \dfrac{q\,\rho^{\lfloor x \rfloor}\left(R_0\lambda(\rho-1)^2-\rho\,\left(1-2\rho+\lfloor x \rfloor(1-\rho)+\rho^{\lfloor x \rfloor+2}\right)\right)}{(1+\rho^{n+1}((x-\lfloor x \rfloor)\,(1-q\rho)-1))^2} & \text{when } x > \lfloor x \rfloor \\ \text{undefined} & \text{when } x = \lfloor x \rfloor\,. \end{cases} \tag{5.111}$$

Following a similar argument to that in Proposition 1, there exists a socially optimal threshold $n_R^\star$ such that $\dfrac{dS^R(x)}{dx}$ is increasing when $x \le n_R^\star$; is decreasing when $x > n_R^\star$. The part in Equation (5.111) that decides the sign of $\dfrac{dS^R(x)}{dx}$ is

$$R_0\lambda(\rho-1)^2-\rho\,\left(1-2\rho+\lfloor x \rfloor(1-\rho)+\rho^{\lfloor x \rfloor+2}\right)\,,$$

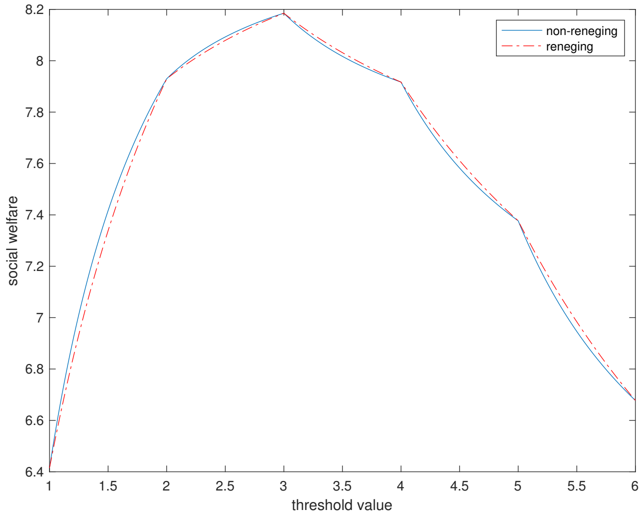



Figure 5.12: Social welfare when $\lambda = 1$, $\mu = 0.8$, $q = 0.8$, $R_0 = 18$.

which is the same as in Equation (5.100), so $n_R^\star = n_N^\star$.

When the threshold is an integer, the joining customers never renege, so there is no difference between the $N$-case and the $R$-case, the socially optimal threshold and the optimal social welfare in both cases are the same. In Figure 5.12, an example of social welfare in the $N$-case (blue) and the $R$-case (red) is plotted. The socially optimal threshold $n_N^\star = n_R^\star = 3$. Also, the figure indicates that the social welfare in the non-reneging case is greater than the reneging case when customers use threshold $x < n_N^*$ but is lower when they use $x > n_N^*$. A possible explanation for this is that, when the customers' threshold is less than the socially-desired value, the reason is that fewer customers use the service, and allowing reneging makes this worse. On the other hand, when the customers' threshold is greater than the socially-desired value, the social welfare is less because joining customers inflict negative externalities on others [44], and allowing reneging makes it easier to leave, which improves the situation.

## 5.7 The long run problem

In this section, we only consider the non-reneging case, but assume that the successful service probability $q$ and service rate $\mu$ can be decided by the service system manager.

We assume that the cost per unit of time associated with operating service at probability $q$ and rate $\mu$ are $c_q$ and $c_\mu$, respectively. In most models it is assumed that this operating cost is independent of the utilization of the server [40].

We derive the maximal social welfare and the maximal profit of the service system We compare the profit-maximizing solution and the social optimality condition with the Nash equilibrium.

**The social planner's problem**

Compared to the original case in Section 5.6, the social planner in the long run problem setting, can not only decide the customers' threshold, but also decide the feedback parameter $q$ and the service rate $\mu$ by paying a price with rate $c_q$ and $c_\mu$, respectively. For $0 \leq i \leq \lfloor x \rfloor$, let $\pi_i^{(x,q,\mu)}$ and $L(x,q,\mu)$ be the stationary probability that there are $i$ people in the system and the expected number of customers in the system, respectively, given that everyone adopts threshold $x$ and the system parameters are $q$ and $\mu$. From the social planner's perspective, her goal is to maximise

$$S(x,q,\mu) \equiv \left[\lambda R \left(\sum_{k=0}^{\lfloor x \rfloor -1} \pi_i^{(x,q,\mu)} + (x - \lfloor x \rfloor)\pi_{\lfloor x \rfloor}^{(x,q,\mu)}\right) - CL(x,q,\mu) - c_q\, q - c_\mu\, \mu\right], \tag{5.112}$$

by deciding the triplet $(x,q,\mu)$.

**The system manager's problem**

Another problem of interest is considered from the system manager's point of view. As a system manager, she cannot decide customers' behavior, but she can affect their behavior by deciding the feedback parameter $q$, the service rate $\mu$, and an admission fee $r$. Thus a system manager's goal is to maximise

$$Re(r,q,\mu) \equiv \left[\lambda r \left(\sum_{k=0}^{\lfloor x \rfloor -1} \pi_i^{(x,q,\mu)} + (x - \lfloor x \rfloor)\pi_{\lfloor x \rfloor}^{(x,q,\mu)}\right) - c_q\, q - c_\mu\, \mu\right], \tag{5.113}$$

Table 5.2: The optimal social welfare in the long run problem with $R = 10, \lambda = 0.4$ and different $c_q$ and $c_\mu$.

| | | |
|---|---|---|
| $c_q = 1,\ c_\mu = 0.5$ | 1st run | $(x^*, q^*, \mu^*) = (8.4616, 0.7724, 1.5309)$<br>$NE(R, \lambda, q^*, \mu^*) = 13$<br>$S(x_e^*, q^*, \mu^*) = 1.9510$ |
| | 2nd run | $(x^*, q^*, \mu^*) = (8.4415, 0.7664, 1.55289)$<br>$NE(R, \lambda, q^*, \mu^*) = 13$<br>$S(x_e^*, q^*, \mu^*) = 1.9510$ |
| $c_q = 0.5,\ c_\mu = 1$ | 1st run | $(x^*, q^*, \mu^*) = (6.4364, 1.0000, 1.0238)$<br>$NE(R, \lambda, q^*, \mu^*) = 10$<br>$S(x_e^*, q^*, \mu^*) = 1.8360$ |
| | 2nd run | $(x^*, q^*, \mu^*) = (4.5867, 0.9999, 1.0210)$<br>$NE(R, \lambda, q^*, \mu^*) = 10$<br>$S(x_e^*, q^*, \mu^*) = 1.8298$ |
| $c_q = 1.4,\ c_\mu = 1.4$ | 1st run | $(x^*, q^*, \mu^*) = (5.1740, 0.9577, 0.9406)$<br>$NE(R, \lambda, q^*, \mu^*) = 9$<br>$S(x_e^*, q^*, \mu^*) = 0.5513$ |
| | 2nd run | $(x^*, q^*, \mu^*) = (5.13880.94150.9525)$<br>$NE(R, \lambda, q^*, \mu^*) = 9$<br>$S(x_e^*, q^*, \mu^*) = 0.5513$ |

where

$$x = NE(R - r, \lambda, q, \mu) . \tag{5.114}$$

We adopted the simulated annealing in Algorithm 5 to calculate solutions for (5.112) and (5.113) for the social planner and the system manager, respectively.

The solution found by simulated annealing depends on the set of random variables generated, hence it is appropriate to run the algorithm multiple times. But running the algorithm once took around three days, so we run the algorithm twice for each parameter setting. The results are displayed in the table 5.2 and 5.3. In table 5.3, we also display the corresponding social welfare under the $\mu_m$, $q_m$ and $r_m$ that makes the revenue optimal.

It can be seen that under the same parameter setting, the system manager, compared with the social planner, prefers fewer people to join. But given the socially optimal $q$ and $\mu$, the Nash equilibrium threshold is greater than the socially optimal one. Another interesting observation is that the system manager prefers to offer a slightly better service (greater $p$ and $\mu$) not to attract more customers but to charge more, and the threshold preferred by the system manager is always $x_e = 2$ for the three parameter settings according to the two runs' results.

Table 5.3: The optimal revenue in the long run problem with $R = 10, \lambda = 0.4$ and different $c_q$ and $c_\mu$.

| | | | |
|---|---|---|---|
| $c_q = 1,\ c_\mu = 0.5$ | 1st run | $(r_m, q_m, \mu_m)$ | $= (8.5912, 0.7686, 1.7156)$ |
| | | $NE\,(R - r_m, \lambda, q_m, \mu_m)$ | $= 2$ |
| | | $Re(r_m, q_m, \mu_m)$ | $= 1.5835$ |
| | | $S(NE\,(R - r_m, \lambda, q_m, \mu_m)\,, q_m, \mu_m)$ | $= 1.7605$ |
| | 2nd run | $(r_m, q_m, \mu_m)$ | $= (8.6096, 0.8259, 1.6385)$ |
| | | $NE\,(R - r_m, \lambda, q_m, \mu_m)$ | $= 2$ |
| | | $Re(r_m, q_m, \mu_m)$ | $= 1.5811$ |
| | | $S(NE\,(R - r_m, \lambda, q_m, \mu_m)\,, q_m, \mu_m)$ | $= 1.7620$ |
| $c_q = 0.5,\ c_\mu = 1$ | 1st run | $(r_m, q_m, \mu_m)$ | $= (8.0162, 0.9965, 1.0111)$ |
| | | $NE\,(R - r_m, \lambda, q_m, \mu_m)$ | $= 2$ |
| | | $Re(r_m, q_m, \mu_m)$ | $= 1.3721$ |
| | | $S(NE\,(R - r_m, \lambda, q_m, \mu_m)\,, q_m, \mu_m)$ | $= 1.6270$ |
| | 2nd run | $(r_m, q_m, \mu_m)$ | $= (7.8768, 0.9949, 0.9462)$ |
| | | $NE\,(R - r_m, \lambda, q_m, \mu_m)$ | $= 2$ |
| | | $Re(r_m, q_m, \mu_m)$ | $= 1.3527$ |
| | | $S(NE\,(R - r_m, \lambda, q_m, \mu_m)\,, q_m, \mu_m)$ | $= 1.6169$ |
| $c_q = 1.4,\ c_\mu = 1.4$ | 1st run | $(r_m, q_m, \mu_m)$ | $= (7.6815, 0.8556, 0.9649)$ |
| | | $NE\,(R - r_m, \lambda, q_m, \mu_m)$ | $= 2$ |
| | | $Re(r_m, q_m, \mu_m)$ | $= 0.1043$ |
| | | $S(NE\,(R - r_m, \lambda, q_m, \mu_m)\,, q_m, \mu_m)$ | $= 0.3502$ |
| | 2nd run | $(r_m, q_m, \mu_m)$ | $= (7.5784, 0.8368, 0.9438)$ |
| | | $NE\,(R - r_m, \lambda, q_m, \mu_m)$ | $= 2$ |
| | | $Re(r_m, q_m, \mu_m)$ | $= 0.0974$ |
| | | $S(NE\,(R - r_m, \lambda, q_m, \mu_m)\,, q_m, \mu_m)$ | $= 0.3469$ |

**Algorithm 5** Simulated Annealing

Set parameters: $\lambda$, $c_\mu$, $c_q$, $R$, $iter_{max} = 1000$, $\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix}$

**procedure** OPTIMAL SOCIAL WELFARE
  Initialize $[x^{(0)}, q^{(0)}, \mu^{(0)}]$
  $SW^{(0)} = SWelfareObservable(x^{(0)}, q^{(0)}, \mu^{(0)})$
  **for** $k = 1 : iter_{max}$ **do**
    generate $z \sim N(\mathbf{0}, \Sigma)$
    $[x^{(k)}, q^{(k)}, \mu^{(k)}] = [x^{(0)}, q^{(0)}, \mu^{(0)}] + z$
    **if** $r < 0$ or $q \notin [0,1]$ or $\mu < 0$ **then**
      $SW^{(k)} = -\infty$
    **else**
      $SW^{(k)} = SWelfareObservable(x^{(k)}, q^{(k)}, \mu^{(k)})$
    $accept\ rate = min(e^{(Revenue^{(k)} - Revenue^{(k-1)})\,k}, 1)$
    generate $u \sim U(0,1)$
    **if** $u \geq accept\ rate$ **then**
      $[r^{(k)}, q^{(k)}, \mu^{(k)}] = [r^{(k-1)}, q^{(k-1)}, \mu^{(k-1)}]$
      $SW^{(k)} = SW^{(k-1)}$
    $k = k + 1$
  **return** $[x^{(k)}, q^{(k)}, \mu^{(k)}, SW^{(k)}]$

**procedure** MAXIMAL REVENUE
  Initialize $[r^{(0)}, q^{(0)}, \mu^{(0)}]$
  $x_e^{(0)} = NE(R - r^{(0)}, \lambda, q^{(0)}, \mu^{(0)})$
  $Revenue^{(0)} = RevObservable(r^{(0)}, q^{(0)}, \mu^{(0)}, x^{(0)})$
  **for** $k = 1 : iter_{max}$ **do**
    generate $z \sim N(\mathbf{0}, \Sigma)$
    $[r^{(k)}, q^{(k)}, \mu^{(k)}] = [r^{(0)}, q^{(0)}, \mu^{(0)}] + z$
    **if** $r < 0$ or $q \notin [0,1]$ or $\mu < 0$ **then**
      $Revenue^{(k)} = -\infty$
    **else**
      $x_e^{(k)} = NE(R - r^{(k)}, \lambda, q^{(k)}, \mu^{(k)})$
      $Revenue^{(k)} = RevObservable(r^{(k)}, q^{(k)}, \mu^{(k)}, x^{(k)})$
    $accept\ rate = min(e^{(Revenue^{(k)} - Revenue^{(k-1)})\,k}, 1)$
    generate $u \sim U(0,1)$
    **if** $u \geq accept\ rate$ **then**
      $[r^{(k)}, q^{(k)}, \mu^{(k)}] = [r^{(k-1)}, q^{(k-1)}, \mu^{(k-1)}]$
      $Revenue^{(k)} = Revenue^{(k-1)}$
    $k = k + 1$
  **return** $[r^{(k)}, q^{(k)}, \mu^{(k)}, Revenue^{(k)}]$

**function** $y = SWelfareObservable(x, q, \mu)$
  $y \leftarrow \left[\lambda R \left(\sum_{k=1}^{\lfloor x \rfloor - 1} \pi_i^{(x,q,\mu)} + (x - \lfloor x \rfloor)\pi_n^{(x,q,\mu)}\right) - CL(x,q,\mu) - c_q q - c_\mu \mu\right]$
  **return** $y$

**function** $y = RevObservable(r, q, \mu, x)$
  $y \leftarrow \left[\lambda r \left(\sum_{k=1}^{\lfloor x \rfloor - 1} \pi_i^{(x,q,\mu)} + (x - \lfloor x \rfloor)\pi_n^{(x,q,\mu)}\right) - c_q q - c_\mu \mu\right]$
  **return** $y$

# Chapter 6

# Extension of the $M/M/1$ Feedback Queue with a General Payoff

In Chapter 5, we investigated the strategic behavior of customers in an $M/M/1$ feedback queue whose cost is linear in their sojourn time. In this chapter, we continue discussing the same model but assume that waiting does not incur a cost but, instead, the reward is discounted with rate $\alpha$ by the time that the customer spends in the system (see Chen and Frank [18] for a similar setting).

For $i = 1, 2, \ldots$, and $\alpha, x \geq 0$, we denote by $R_{i,i}^{(x)}(\alpha)$ the discounted reward of the tagged customer if she joins in position $i$, and the other customers use threshold $x$. We assume that there is an outside opportunity whose value is $v$. When an customer arrives at the system, she can observe the number of customers and then decides either to join the system or take the outside opportunity. The outside opportunity $v$ can also be viewed as an admission fee. In both cases, if there are $i-1$ customers in the system when an customer arrives, she will choose to join if $\mathbb{E}\left(R_{i,i}^{(x)}(\alpha)\right)$ is greater than $v$, decide to balk if $\mathbb{E}\left(R_{i,i}^{(x)}(\alpha)\right)$ is lower than $v$, and be indifferent between joining and balking if $\mathbb{E}\left(R_{i,i}^{(x)}(\alpha)\right)$ equals $v$. Thus, instead of having $R, C$ in the payoff expression as in Chapter 5, now we have $R, \alpha$, and $v$ in the payoff setting.

We calculated the symmetric Nash equilibrium thresholds in the $N$-case and the $R$-case, and compare them. Similar to Chapter 5, in the $R$-case, under the Nash equilibrium, customers have a greater incentive to join. However, their equilibrium expected payoff either stays unchanged or decreases when reneging is allowed.

This Chapter is organised as follows. In Section 6.1 we focus on the $N$-case, and calculate the expected discounted reward of a tagged customer conditioned on her joining position and the threshold strategy used by others, via matrix analytic methods. Then we calculate a Nash equilibrium threshold. In Section 6.2 we allow customers to renege every time their service fails, and assume that the reneging and joining strategies are of the same threshold. We compute the Nash equilibrium and compare it with the Nash equilibrium in the $N$-case. In Section 6.3, we present a paradox we observed when comparing the Nash equilibrium in the $N$-case and the $R$-case. In Section 6.4, we numerically calculate the distribution for the sojourn time of a tagged customer, given her joining position and the other customers' threshold, by inverting the Laplace transform of it.

## 6.1 The Case When Customers Cannot Renege

### 6.1.1 The expected payoff

Let $W_{i,j}^{(x)}$ be same as in Chapter 5, that is the remaining time of the tagged customer until she departs the system, if there are $j$ customers in the system, she is in position $i$ where $1 \leq i \leq j \leq \lceil x \rceil + 1$, and all the other customers use threshold $x$. The discounted reward of a customer who joins in position $i$, and all the other customers use threshold $x$, is then $R_{i,i}^{(x)}(\alpha) \equiv R_0\, e^{-\alpha W_{i,i}^{(x)}}$. Let $z_{i,i}^{(x)}(\alpha) \equiv \mathbb{E}\left(R_{i,i}^{(x)}(\alpha)\right) - v$. When $v$ is an admission fee, then $z_{i,i}^{(x)}(\alpha)$ can be regarded as a customer's expected payoff.

We will show that the vector

$$z^{(x)}(\alpha) = \left(z_{1,1}^{(x)}(\alpha), z_{1,2}^{(x)}(\alpha), z_{2,2}^{(x)}(\alpha), \ldots, z_{1,\lceil x \rceil+1}^{(x)}(\alpha), \ldots, z_{\lceil x \rceil,\lceil x \rceil+1}^{(x)}(\alpha), z_{\lceil x \rceil+1,\lceil x \rceil+1}^{(x)}(\alpha)\right)^T$$

can be obtained by solving a version of Poisson's equation. To compute $z^{(x)}$, we assume that all the other customers use threshold $x$ and the tagged customer joins the queue at time 0. For $t \geq 0$, let $X(t)$ denote the state of the tagged customer at time $t$, and let $\mathcal{S} \equiv \{(i,j), 1 \leq i \leq j \leq \lfloor x \rfloor + 1\} \cup \{0\}$ be the state space of process $\{X(t), t \geq 0\}$ where $i$ denotes the position of the tagged customer, $j$ denotes the total number of customers in the system, and state 0 represents the state when the tagged customer has left the system. Then $\{X(t), t \geq 0\}$ forms a continuous-time QBD on $\mathcal{S}$ with 0 an absorbing state and $\{(i,j), 1 \leq i \leq j \leq \lfloor x \rfloor + 1\}$ being transient. Let $Y(t)$ be the time until the first transition after time $t$, then $Y(t)$ is exponentially distributed with rate $\lambda+\mu$. The event is an arrival with probability $\lambda/(\lambda+\mu)$. When $j<\lfloor x\rfloor$, the arriving customer joins the system with probability 1; when $j=\lfloor x\rfloor$, the arriving customer joins the system with probability $p$; when $j=\lfloor x\rfloor+1$ or $\lfloor x\rfloor+2$, the arriving customer balks.

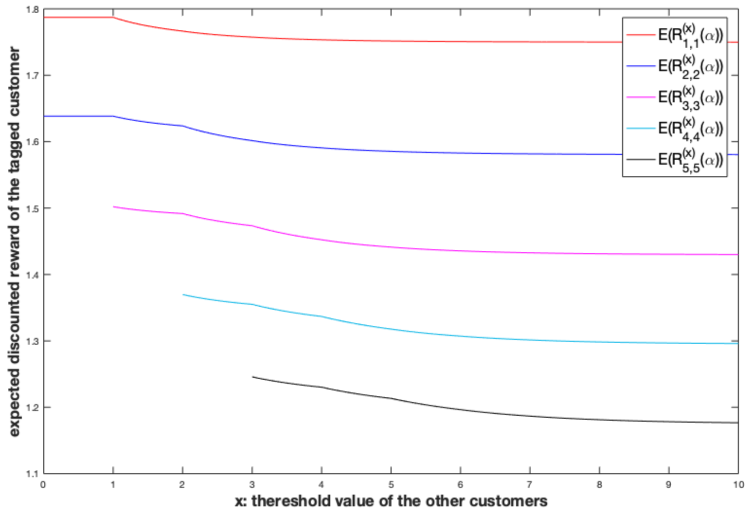



Figure 6.1: Expected discounted reward of the tagged customer.

On the other hand, the next transition is a service completion with probability $\mu/(\lambda+\mu)$, after which a customer leaves the system with probability $q$ and joins the end of the system with probability $1-q$. Hence, if the customer in service is the tagged one ($i=1$), when she finishes her service, her future sojourn time is 0 with probability $q$, otherwise, her next position is $j$. When the customer in service is not the tagged one, the position of the tagged customer decreases by 1, the total number of customers decreases by 1 with probability $q$ but stays unchanged with probability $1-q$. Hence for $1\le i\le j\le\lceil x\rceil+1$ and $t\ge 0$, given that $X(t)=(i,j)$, the conditional probability

$$P\{X(Y(t))=(i',j')\mid X(t)=(i,j)\}=\begin{cases}\frac{\lambda}{\lambda+\mu} & \text{if } \{j<\lfloor x\rfloor,\ (i',j')=(i,j+1)\} \text{ or } \{j>\lfloor x\rfloor,\ (i',j')=(i,j)\}\\ \frac{\lambda p}{\lambda+\mu} & \text{if } j=\lfloor x\rfloor,\ (i',j')=(i,j+1)\\ \frac{\lambda(1-p)}{\lambda+\mu} & \text{if } j=\lfloor x\rfloor,\ (i',j')=(i,j)\\ \frac{\mu q}{\lambda+\mu} & \text{if } \{i=1,\ (i',j')=0\} \text{ or } \{i=1,\ (i',j')=0\}\\ \frac{\mu(1-q)}{\lambda+\mu} & \text{if } \{i>1,\ (i',j')=(i-1,j)\} \text{ or } \{i=1,\ (i',j')=(j,j)\}.\end{cases} \tag{6.1}$$

Assume that the state is $(i,j)$ at time $t$, then conditioning on the first transition out of state $(i,j)$, the expected discounted reward $\mathbb{E}\left(R_0\, e^{-\alpha W_{i,j}^{(x)}}\right)$ is given by

$$\begin{aligned}
&\mathbb{E}\left(R_0\, e^{-\alpha W_{i,j}^{(x)}}\right) \qquad (6.2)\\
=&R_0\,\mathbb{E}\left(\mathbb{E}\left(e^{-\alpha W_{i,j}^{(x)}} \mid T(t), X(T(t))\right)\right)\\
=&R_0 \int_0^\infty (\lambda+\mu)\, e^{-(\lambda+\mu)y} \sum_{(i',j')} P(X(T(t))=(i',j') \mid X(t)=(i,j))\,\mathbb{E}\left(e^{-\alpha W_{i,j}^{(x)}} \mid T(t)=y, X(T(t))=(i',j')\right) dy\\
=&R_0 \int_0^\infty (\lambda+\mu)\, e^{-(\lambda+\mu)y} \sum_{(i',j')} P(X(T(t))=(i',j') \mid X(t)=(i,j))\,\mathbb{E}\left(e^{-\alpha\left(y+W_{i',j'}^{(x)}\right)}\right) dy\\
=&R_0 \left(\sum_{(i',j')} P(X(T(t))=(i',j') \mid X(t)=(i,j))\,\mathbb{E}\left(e^{-\alpha W_{i',j'}^{(x)}}\right)\right) \int_0^\infty (\lambda+\mu)\, e^{-(\lambda+\mu+\alpha)y}\, dy\\
=&R_0 \left(\frac{\lambda}{\lambda+\mu}\left(\mathbb{E}\left(e^{-\alpha W_{i,j+1}^{(x)}}\right)\mathbb{1}_{\{i<\lfloor x\rfloor\}} + \left(p\,\mathbb{E}\left(e^{-\alpha W_{i,j+1}^{(x)}}\right) + (1-p)\,\mathbb{E}\left(e^{-\alpha W_{i,j}^{(x)}}\right)\right)\mathbb{1}_{\{i=\lfloor x\rfloor\}} + \mathbb{E}\left(e^{-\alpha W_{i,j}^{(x)}}\right)\mathbb{1}_{\{i>\lfloor x\rfloor\}}\right)\right.\\
&\left.+\frac{\mu}{\lambda+\mu}\left(\left(q\,\mathbb{E}\left(e^{-\alpha W_{i-1,j-1}^{(x)}}\right) + (1-q)\,\mathbb{E}\left(e^{-\alpha W_{i-1,j}^{(x)}}\right)\right)\mathbb{1}_{\{i>1\}} + \left(q + (1-q)\,\mathbb{E}\left(e^{-\alpha W_{j,j}^{(x)}}\right)\right)\mathbb{1}_{\{i=1\}}\right)\right)\frac{\lambda+\mu}{\lambda+\mu+\alpha}\\
=&R_0 \left(\frac{\lambda}{\lambda+\mu+\alpha}\left(\mathbb{E}\left(e^{-\alpha W_{i,j+1}^{(x)}}\right)\mathbb{1}_{\{i<\lfloor x\rfloor\}} + \left(p\,\mathbb{E}\left(e^{-\alpha W_{i,j+1}^{(x)}}\right) + (1-p)\,\mathbb{E}\left(e^{-\alpha W_{i,j}^{(x)}}\right)\right)\mathbb{1}_{\{i=\lfloor x\rfloor\}} + \mathbb{E}\left(e^{-\alpha W_{i,j}^{(x)}}\right)\mathbb{1}_{\{i>\lfloor x\rfloor\}}\right)\right.\\
&\left.+\frac{\mu}{\lambda+\mu+\alpha}\left(\left(q\,\mathbb{E}\left(e^{-\alpha W_{i-1,j-1}^{(x)}}\right) + (1-q)\,\mathbb{E}\left(e^{-\alpha W_{i-1,j}^{(x)}}\right)\right)\mathbb{1}_{\{i>1\}} + \left(q + (1-q)\,\mathbb{E}\left(e^{-\alpha W_{j,j}^{(x)}}\right)\right)\mathbb{1}_{\{i=1\}}\right)\right).
\end{aligned}$$

Hence,

$$\gamma_\alpha^{(x)} \equiv \left[\mathbb{E}\left(e^{-\alpha W_{1,1}^{(x)}}\right), \mathbb{E}\left(e^{-\alpha W_{1,2}^{(x)}}\right), \mathbb{E}\left(e^{-\alpha W_{2,2}^{(x)}}\right), \ldots, \mathbb{E}\left(e^{-\alpha W_{\lfloor x\rfloor+1,\lfloor x\rfloor+1}^{(x)}}\right)\right]^T \tag{6.3}$$

can be obtained by solving the Poisson's equation

$$\left(I - P_\alpha^{(x)}\right)\gamma_\alpha^{(x)} = \boldsymbol{g}_\alpha \tag{6.4}$$

using Algorithm 2, where

$$\boldsymbol{g}_\alpha = \sum_{k=1}^{\lfloor x\rfloor+1} \frac{\mu q}{\alpha+\lambda+\mu}\,\boldsymbol{e}_{\frac{k(k-1)}{2}+1}, \tag{6.5}$$

and $P_\alpha^{(x)}$ is the same as $P^{(x)}$ defined in Equation (5.12), with $\lambda+\mu$ replaced with $\alpha+\lambda+\beta$.

We plot $\mathbb{E}\left(R_{i,i}^{(x)}(\alpha)\right)\left(= R_0\mathbb{E}\left(e^{-\alpha W_{i,i,}^{(x)}}\right)\right)$ for $0 \le x \le 10$ and $1 \le i \le 5$ in Figure 6.1. The line of $\mathbb{E}\left(R_{i,i}^{(x)}(\alpha)\right)$ exists only when $\lceil x\rceil \ge i-1$, since the tagged customer cannot join in a position greater than $\lceil x \rceil + 1$. Moreover, $\mathbb{E}\left(R_{i,i}^{(x)}(\alpha)\right)$ is constant for $0 \le x \le 1$ since no other customers will join when the tagged customer is the system, if their threshold $x$ is less than 1. By solving Equation (6.4) for $0 \le x \le 1$ and $n = 1, 2$ analytically, we have

$$\mathbb{E}\left(R_{1,1}^{(x)}(\alpha)\right) = \frac{\mu q}{s + \mu q} \qquad \mathbb{E}\left(R_{2,2}^{(x)}(\alpha)\right) = \frac{\mu^2 q(s + 2\mu q - \mu q^2)}{(s + \mu q)(s^2 + 2\mu s + 2\mu^2 q - \mu^2 q^2)}. \tag{6.6}$$

To work out the Nash equilibrium threshold, we first need to introduce three lemmas that are about the properties of $\mathbb{E}\left(R_{i,i}^{(x)}(\alpha)\right)$ as a function of $i$ and $x$.

**Lemma 8.** $\mathbb{E}\left(R_{i,i}^{(x)}(\alpha)\right)$ *is decreasing in $i$ for* $1 \le i \le \lceil x \rceil + 1$.

*Proof.* We omit details here since the proof is similar to that for Lemma 3. □

**Lemma 9.** *For any two thresholds $x_1$ and $x_2$ with $x_1 < x_2$,*

$$\mathbb{E}\left(R_{i,j}^{(x_1)}(\alpha)\right) > \mathbb{E}\left(R_{i,j}^{(x_2)}(\alpha)\right) \qquad 1 \le i \le j \le \lceil x_1 \rceil + 1. \tag{6.7}$$

*Proof.* We omit details here since the proof is similar to that for Lemma 4. □

**Lemma 10.** *If* $z_{m+1,m+1}^{(m+1)} < 0 < z_{m+1,m+1}^{(m)}$, *then there exists a unique $\chi_m$ such that* $z_{m+1,m+1}^{(\chi_m)} = 0$.

*Proof.* We omit details here since the proof is similar to that for Lemma 5. □

### 6.1.2 The Nash equilibrium

Let $x_e$ be the Nash equilibrium threshold in the $N$-case. We describe $x_e$ in the following theorem.

**Theorem 14.** *There exists an equilibrium threshold strategy with threshold value*

$$x_e = \begin{cases} 0 & \text{if } \mathbb{E}\left(R^{(0)}_{1,1}(\alpha)\right) < v\,, \\ \zeta_0 & \text{if } \mathbb{E}\left(R^{(0)}_{1,1}(\alpha)\right) = v\,, \\ m & \text{if } \mathbb{E}\left(R^{(m)}_{m+1,m+1}(\alpha)\right) \leq v \leq \mathbb{E}\left(R^{(m)}_{m,m}(\alpha)\right) \quad m = 1,2,\dots, \\ \chi_m & \text{if } \mathbb{E}\left(R^{(m+1)}_{m+1,m+1}(\alpha)\right) < v < \mathbb{E}\left(R^{(m)}_{m+1,m+1}(\alpha)\right) \quad m = 1,2,\dots\,, \end{cases} \tag{6.8}$$

*where $\zeta_0$ is any number that satisfies $0 \leq \zeta_0 \leq 1$.*

*Proof.*

- If $\mathbb{E}\left(R^{(0)}_{1,1}(\alpha)\right) < v$, then $z^{(0)}_{1,1}(\alpha) < 0$, thus even when no customers join the system, it is not optimal for the tagged customer to join the system. Hence, not joining is the best response for any customer given that this is used by the others.

- If $\mathbb{E}\left(R^{(0)}_{1,1}(\alpha)\right) = v$, then $z^{(\zeta_0)}_{1,1}(\alpha) = 0$, $\forall \zeta_0 \in (0,1)$. That is, given that others use a threshold $\zeta_0$, the tagged customer is indifferent between joining and balking, so any threshold in $[0,1]$ is her best response. Thus, any number in $[0,1]$ is the Nash equilibrium threshold.

- If $\mathbb{E}\left(R^{(m)}_{m+1,m+1}(\alpha)\right) \leq v \leq \mathbb{E}\left(R^{(m)}_{m,m}(\alpha)\right)$, then $z^{(m)}_{m+1,m+1}(\alpha) \leq 0 \leq z^{(m)}_{m,m}(\alpha)$. Hence, given all the other customers use threshold $m$, it is the best for the tagged customer to use threshold $m$.

- If $\mathbb{E}\left(R^{(m+1)}_{m+1,m+1}\right) < v < \mathbb{E}\left(R^{(m)}_{m+1,m+1}(\alpha)\right)$, then $z^{(m+1)}_{m+1,m+1}(\alpha) < 0 < z^{(m)}_{m+1,m+1}(\alpha)$. Hence, the best response of the tagged customer is $m$, when all the other customers use threshold $m+1$, and the best response of the tagged customer is $m+1$, when all the others use threshold $m$. When all the others use threhsold $\chi_m$, the tagged customer is indifferent between joining in position $m$ and $m+1$, hence any threshold in $[m, m+1]$, including $\chi_m$, is her best response. Thus $\chi_m$ is the Nash equilibrium threshold.

□

## 6.2 The Case When Customers Can Renege

In this section, we discuss customers' strategic decision if they are allowed to renege after joining. Specifically, arriving customers observe the number of customers in the system, and then decide to join or not using a threshold strategy. If their service fails, they will choose to join at the end of the queue using the same threshold strategy. To work out the symmetric Nash equilibrium threshold, we only need to calculate the tagged customers' expected payoff, given her joining position $1 \le i \le \lfloor x \rfloor + 1$, when she uses threshold $\lfloor x \rfloor + 1$ and the other customers take threshold $x$, given her joining position. See Section 5.4 for a detailed explanation.

### 6.2.1 The expected payoff

In the $R$-case, for $t \ge 0$, let $\widehat{X}(t)$ and $\widehat{T}(t)$ be the same quantities as $X(t)$ and $T(t)$ in the $N$-case. Then for $1 \le i \le j \le \lfloor x \rfloor + 1$, the conditional probability

$$P\{\hat{X}(\hat{Y}(t)) = (i', j') \mid \hat{X}(t) = (i, j)\} \tag{6.9}$$
$$= \begin{cases} \frac{\lambda}{\lambda+\mu} & \text{if} \quad j < \lfloor x \rfloor,\ (i',j') = (i, j+1) \\ \frac{\lambda p}{\lambda+\mu} & \text{if} \quad j = \lfloor x \rfloor,\ (i',j') = (i, j+1) \\ \frac{\lambda(1-p)}{\lambda+\mu} & \text{if} \quad j = \lfloor x \rfloor,\ (i',j') = (i, j) \\ \frac{\lambda}{\lambda+\mu} & \text{if} \quad j > \lfloor x \rfloor,\ (i',j') = (i, j) \\ \frac{\mu q}{\lambda+\mu} & \text{if} \quad i > 1,\ j < \lfloor x \rfloor + 1,\ (i',j') = (i-1, j-1) \\ \frac{\mu(q+(1-q)(1-p))}{\lambda+\mu} & \text{if} \quad i > 1,\ j = \lfloor x \rfloor + 1,\ \chi = (i-1, j-1) \\ \frac{\mu(1-q)}{\lambda+\mu} & \text{if} \quad i > 1,\ j < \lfloor x \rfloor + 1,\ (i',j') = (i-1, j) \\ \frac{\mu(1-q)p}{\lambda+\mu} & \text{if} \quad i > 1,\ j = \lfloor x \rfloor + 1,\ (i',j') = (i-1, j) \\ \frac{\mu q}{\lambda+\mu} & \text{if} \quad i = 1,\ (i',j') = 0 \\ \frac{\mu(1-q)}{\lambda+\mu} & \text{if} \quad i = 1,\ (i',j') = (j, j)\,. \end{cases}$$

let $\widehat{W}_{i,j}^{(x)}$ be the remaining time of the tagged customer until she successfully completes the service or reneges, if there are $j$ customers in the system, she is in position $i$, and all the other customers use threshold $x$. The expected payoff of a customer who joins in position $i$, and all the other customers use threshold $x$, is denoted by $\widehat{z}_{i,i}^{(\lfloor x\rfloor+1,x)}(\alpha) \equiv \mathbb{E}\left(R_0\, e^{-\alpha \widehat{W}_{i,i}^{(\lfloor x\rfloor+1,x)}}\right) - v$. Assume that the state is $(i,j)$ at time $t$, then conditioning on the first transition out of state $(i,j)$, the expected discounted reward $\mathbb{E}\left(R_0\, e^{-\alpha \hat{W}_{i,j}^{(\lfloor x\rfloor+1,x)}}\right)$ is given by

$$
\begin{aligned}
&\mathbb{E}\left(R_0\, e^{-\alpha \hat{W}_{i,j}^{(\lfloor x\rfloor+1,x)}}\right) \qquad (6.10)\\
=&R_0\, \mathbb{E}\left(\mathbb{E}\left(e^{-\alpha \hat{W}_{i,j}^{(\lfloor x\rfloor+1,x)}} \mid Y(0), X(Y(0))\right)\right)\\
=&R_0 \int_0^\infty (\lambda+\mu)\, e^{-(\lambda+\mu)y} \sum_{(i',j')} P(X(Y(0))=(i',j') \mid X(0)=(i,j))\, \mathbb{E}\left(e^{-\alpha \hat{W}_{i,j}^{(\lfloor x\rfloor+1,x)}} \mid Y(0)=y, X(Y(0))=(i',j')\right) dy\\
=&R_0 \int_0^\infty (\lambda+\mu)\, e^{-(\lambda+\mu)y} \sum_{(i',j')} P(X(Y(0))=(i',j') \mid X(0)=(i,j))\, \mathbb{E}\left(e^{-\alpha (y+\hat{W}_{i',j'}^{(\lfloor x\rfloor+1,x)})}\right) dy\\
=&R_0 \left(\sum_{(i',j')} P(X(Y(0))=(i',j') \mid X(0)=(i,j))\, \mathbb{E}\left(e^{-\alpha \hat{W}_{i',j'}^{(\lfloor x\rfloor+1,x)}}\right) \int_0^\infty (\lambda+\mu)\, e^{-(\lambda+\mu+\alpha)y}\, dy\right)\\
=&R_0 \left(\frac{\lambda}{\lambda+\mu}\left(\mathbb{E}\left(e^{-\alpha \hat{W}_{i,j+1}^{(\lfloor x\rfloor+1,x)}}\right)\mathbb{1}_{\{i<\lfloor x\rfloor\}} + \left(p\,\mathbb{E}\left(e^{-\alpha \hat{W}_{i,j+1}^{(\lfloor x\rfloor+1,x)}}\right) + (1-p)\,\mathbb{E}\left(e^{-\alpha \hat{W}_{i,j}^{(\lfloor x\rfloor+1,x)}}\right)\right)\mathbb{1}_{\{i=\lfloor x\rfloor\}} + \mathbb{E}\left(e^{-\alpha \hat{W}_{i,j}^{(\lfloor x\rfloor+1,x)}}\right)\mathbb{1}_{\{i>\lfloor x\rfloor\}}\right)\right.\\
&+\frac{\mu}{\lambda+\mu}\left(q+(1-q)(1-p)\mathbb{E}\left(e^{-\alpha W_{i-1,j-1}^{(x)}}\right)\mathbb{1}_{\{j=\lfloor x\rfloor+1\}} + (1-q)(1-(1-p)\mathbb{1}_{\{j=\lfloor x\rfloor+1\}})\mathbb{E}\left(e^{-\alpha W_{i-1,j}^{(x)}}\right)\right.\\
&\left.\left.+\left(q+(1-q)\,\mathbb{E}\left(e^{-\alpha W_{j,j}^{(\lfloor x\rfloor+1,x)}}\right)\right)\mathbb{1}_{\{i=1\}}\right)\right)\frac{\lambda+\mu}{\lambda+\mu+\alpha}\\
=&R_0 \left(\frac{\lambda}{\lambda+\mu+\alpha}\left(\mathbb{E}\left(e^{-\alpha W_{i,j+1}^{(x)}}\right)\mathbb{1}_{\{i<n\}} + \left(q\,\mathbb{E}\left(e^{-\alpha W_{i,j+1}^{(x)}}\right) + (1-q)\,\mathbb{E}\left(e^{-\alpha W_{i,j}^{(x)}}\right)\right)\mathbb{1}_{\{i=n\}} + \mathbb{E}\left(e^{-\alpha W_{i,j}^{(x)}}\right)\mathbb{1}_{\{i>n\}}\right)\right.\\
&+\frac{\mu}{\lambda+\mu+\alpha}\left(q+(1-q)(1-p)\mathbb{E}\left(e^{-\alpha W_{i-1,j-1}^{(x)}}\right)\mathbb{1}_{\{j=\lfloor x\rfloor+1\}} + (1-q)(1-(1-p)\mathbb{1}_{\{j=\lfloor x\rfloor+1\}})\mathbb{E}\left(e^{-\alpha W_{i-1,j}^{(x)}}\right)\right.\\
&\left.\left.+\left(q+(1-q)\,\mathbb{E}\left(e^{-\alpha W_{j,j}^{(\lfloor x\rfloor+1,x)}}\right)\right)\mathbb{1}_{\{i=1\}}\right)\right).
\end{aligned}
$$

Hence

$$
\hat{\gamma}^{(\lfloor x\rfloor+1,x)}(\alpha) \equiv \left[\mathbb{E}\left(e^{-\alpha \hat{W}_{1,1}^{(\lfloor x\rfloor+1,x)}}\right), \mathbb{E}\left(e^{-\alpha \hat{W}_{1,2}^{(\lfloor x\rfloor+1,x)}}\right), \mathbb{E}\left(e^{-\alpha \hat{W}_{2,2}^{(\lfloor x\rfloor+1,x)}}\right), \ldots, \mathbb{E}\left(e^{-\alpha \hat{W}_{\lfloor x\rfloor+1,\lfloor x\rfloor+1}^{(\lfloor x\rfloor+1,x)}}\right)\right]^T \qquad (6.11)
$$

can be obtained by solving the following Poisson's equation

$$
\left(I - \hat{P}_\alpha^{(\lfloor x\rfloor+1,x)}\right)\hat{\gamma}^{(x)}(\alpha) = \boldsymbol{g}_\alpha \qquad (6.12)
$$

using Algorithm 2, where $\hat{P}_\alpha^{(\lfloor x\rfloor+1,x)}$ is the same as $\hat{P}^{(\lfloor x\rfloor+1,x)}$, with $\lambda+\mu$ replaced with $\alpha+\lambda+\mu$.

Next we compare $\hat{z}_{j,j}^{(x)}(\alpha)$ and $z_{j,j}^{(\lfloor x\rfloor+1,x)}(\alpha)$ in Lemma 11.

**Lemma 11.** *When $x \geq \lfloor x \rfloor$,*

$$\hat{z}_{j,j}^{(\lfloor x\rfloor+1,x)}(\alpha) \geq z_{j,j}^{(x)}(\alpha) \qquad j = 1, 2, \ldots, \lfloor x \rfloor + 1\,. \tag{6.13}$$

*When $x = \lfloor x \rfloor$,*

$$z_{j,j}^{(\lfloor x\rfloor,x)}(\alpha) = z_{j,j}^{(\lfloor x\rfloor+1,x)}(\alpha) = z_{j,j}^{(x)}(\alpha) \qquad j = 1, 2, \ldots, \lfloor x \rfloor\,. \tag{6.14}$$

### 6.2.2 The Nash equilibrium in the $R$-case and it comparison with the $N$-case

Let $\hat{x}_e$ be the Nash equilibrium in the $R$-case. We state $\hat{x}_e$ and its comparison to $x_e$ in the following theorem.

**Theorem 15.** *The Nash equilibrium $\hat{x}_e$ in the R-case is*

$$\hat{x}_e = \begin{cases} 0 & \text{if } \mathbb{E}\left(R_0\, e^{-\alpha \hat{W}_{1,1}^{(1,0)}}\right) < v \\ \hat{\zeta}_0 & \text{if } \mathbb{E}\left(R_0\, e^{-\alpha \hat{W}_{1,1}^{(1,0)}}\right) = v \\ m & \text{if } \mathbb{E}\left(R_0\, e^{-\alpha \hat{W}_{m+1,m+1}^{(m+1,m)}}\right) \leq v \leq \mathbb{E}\left(R_0\, e^{-\alpha \hat{W}_{m,m}^{(m,m)}}\right) \\ \hat{\chi}_m & \text{if } \mathbb{E}\left(R_0\, e^{-\alpha \hat{W}_{m+1,m+1}^{(m+1,m+1)}}\right) < v < \mathbb{E}\left(R_0\, e^{-\alpha \hat{W}_{m+1,m+1}^{(m+1,m)}}\right)\,, \end{cases} \tag{6.15}$$

*where $\hat{\zeta}_0$ is any number in $[0,1]$, and $\hat{\chi}_m$ is the unique number such that $\mathbb{E}\left(R_0\, e^{-\alpha \hat{W}_{m+1,m+1}^{(m+1,\hat{\chi}_m)}}\right) = v$. Moreover, $\hat{x}_e \geq x_e$.*

*Proof.* The derivation of $\hat{x}_e$ is similar to that of $x_e$ in Theorem 14, so we omit the details here. To compare $x_e$ and $\hat{x}_e$, we write down the three possible cases.

- When $\mathbb{E}\left(R_0\, e^{-\alpha \hat{W}_{m+1,m+1}^{(m+1,m)}}\right) \leq v \leq \mathbb{E}\left(R_0\, e^{-\alpha W_{m,m}^{(m)}}\right)$,

$$\hat{z}_{m,m}^{(m,m)}(\alpha) = z_{m,m}^{(m)}(\alpha) = \mathbb{E}\left(R_0\, e^{-\alpha W_{m,m}^{(m)}}\right) - v \geq 0\,, \tag{6.16}$$

and

$$z_{m+1,m+1}^{(m)}(\alpha) < \hat{z}_{m+1,m+1}^{(m+1,m)}(\alpha) = \mathbb{E}\left(R_0\, e^{-\alpha \hat{W}_{m+1,m+1}^{(m+1,m)}}\right) - v \leq 0\,. \tag{6.17}$$

Hence in both the $N$-case and the $R$-case, $x_e = \hat{x}_e = m$. This case is represented in Figure 6.2(a).

- When $\mathbb{E}\left(R_0\, e^{-\alpha W^{(m)}_{m+1,m+1}}\right) < v \leq \mathbb{E}\left(R_0\, e^{-\alpha \hat{W}^{(m+1,m)}_{m+1,m+1}}\right)$,

$$z^{(m)}_{m,m}(\alpha) = \hat{z}^{(m,m)}_{m,m}(\alpha) > \hat{z}^{(m+1,m)}_{(m+1,m+1)}(\alpha) = \mathbb{E}\left(R_0\, e^{-\alpha \hat{W}^{(m+1,m)}_{m+1,m+1}}\right) - v > 0\,, \tag{6.18}$$

and

$$\hat{z}^{(m+1,m+1)}_{m+1,m+1}(\alpha) < z^{(m)}_{m+1,m+1}(\alpha) \leq 0\,. \tag{6.19}$$

Hence, $x_e = m < \hat{x}_e < m+1$. This case is represented in Figure 6.2(b).

- When $\mathbb{E}\left(R_0\, e^{-\alpha \hat{W}^{(m+1,m+1)}_{m+1,m+1}}\right) < v < \mathbb{E}\left(R_0\, e^{-\alpha W^{(m)}_{m+1,m+1}}\right)$,

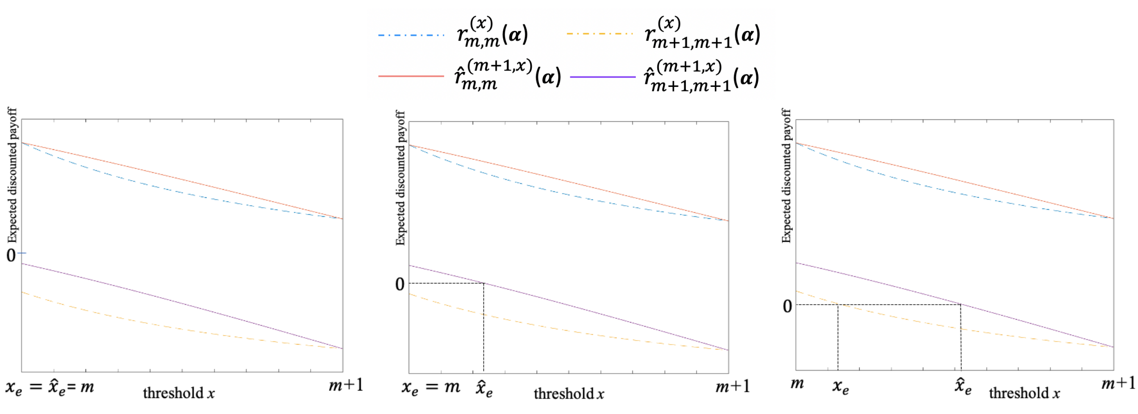
$$z^{(m+1,m+1)}_{m+1,m+1}(\alpha) = \hat{z}^{(m+1,m+1)}_{m+1,m+1}(\alpha) = \mathbb{E}\left(R_0\, e^{-\alpha \hat{W}^{(m+1,m+1)}_{m+1,m+1}}\right) - v < 0 \tag{6.20}$$

and

$$\hat{z}^{(m+1,m)}_{m+1,m+1}(\alpha) > z^{(m)}_{m+1,m+1}(\alpha) > 0\,. \tag{6.21}$$

Hence, $m < x_e < \hat{x}_e < m+1$. This case is represented in Figure 6.2(c).

□



(a) (b) (c)

Figure 6.2: An illustration of Nash equilibrium threshold comparison.

## 6.3 Paradoxes

In Paradox 1 in Chapter 5, we observe that with some parameter setting, increasing the reward $R_0$ can make the expected payoff of customers joining in position $\lfloor x_e \rfloor$ decrease. This paradox arises from the property that $\left(\mathbb{E}\left(W^{(\lfloor x \rfloor + p)}_{\lfloor x \rfloor+1,\lfloor x \rfloor+1}\right) - \mathbb{E}\left(W^{(\lfloor x \rfloor + p)}_{\lfloor x \rfloor,\lfloor x \rfloor}\right)\right)$ for $x > 1$ is decreasing in $p$. That is, the corresponding expected payoff difference is decreasing in $p$.

In the discounted reward setting, When $1 < x < 2$,

$$\frac{\partial\left(z^{(1+p)}_{1,1}(\alpha) - z^{(1+p)}_{1,1}(\alpha)\right)}{\partial p} = \tag{6.22}$$
$$\frac{\begin{aligned}\lambda\mu^3 q^2(1-q)(\alpha^5 + 5\alpha^4\mu - \alpha^3\mu^2q^2 + 2\alpha^3\mu^2 q + 9\alpha^3\mu^2 + 2\alpha^2\mu^3q^3 \\ - 7\alpha^2\mu^3q^2 + 9\alpha^2\mu^3 q + 6\alpha^2\mu^3 + 4\alpha\mu^4q^3 - 11\alpha\mu^4q^2 + 12\alpha\mu^4 q + 2\mu^5q^4 - 5\mu^5q^3 + 4\mu^5q^2)\end{aligned}}{(\alpha^3 + 3\alpha^2\mu + 3\alpha\mu^2 + \mu^3q^3 - 3\mu^3q^2 + 3\mu^3 q)(\alpha^3 + \alpha^2\mu q + 2\alpha^2\mu + \lambda p\alpha^2 - \alpha\mu^2q^2 + 4\alpha\mu^2 q + 2\lambda p\alpha\mu - \mu^3q^3 + 2\mu^3q^2 + \lambda p\mu^2 q)^2}$$
$$> 0\,.$$

When $x \geq 2$, for all the parameter settings we tried, $\left(z^{(\lfloor x \rfloor + p)}_{\lfloor x \rfloor,\lfloor x \rfloor}(\alpha) - z^{(\lfloor x \rfloor + p)}_{\lfloor x \rfloor+1,\lfloor x \rfloor+1}(\alpha)\right)$ is decreasing in $p$. Hence, it still exists that $\left(z^{(\lfloor x \rfloor + p)}_{\lfloor x \rfloor,\lfloor x \rfloor}(\alpha) - z^{(\lfloor x \rfloor + p)}_{\lfloor x \rfloor+1,\lfloor x \rfloor+1}(\alpha)\right)$ for $x > 1$ is decreasing in $p$. However, Paradox 1 vanishes in the discounted reward setting. The reason is when we increase $R_0$, the expected payoffs $\mathbb{E}\left(R^{(x)}_{i,j}(\alpha)\right)$ for $1 \leq i \leq j \leq \lfloor x \rfloor + 1$ all increase by the same factor, the Nash equilibrium threshold will increase. However, the property in (6.22) is not enough to tell the change for the expected payoff of a customer joining in position $\lfloor x \rfloor$.

In Theorem 15, we know that the relationship between the Nash equilibrium thresholds in the $N$-case and the $R$-case is similar to that in Theorem 13, althogh the exact values are different. Paradox 2 is from this relationship, and we still have it in the discounted reward setting. Similar to Paradox 2, we can only prove it when $\mathbb{E}\left(e^{-\alpha W^{(m)}_{m+1,m+1}}\right) < v < \mathbb{E}\left(e^{-\alpha \hat{W}^{(m+1,m)}_{m+1,m+1}}\right)$. When $\mathbb{E}\left(R_0\, e^{-\alpha \hat{W}^{(m+1,m+1)}_{m+1,m+1}}\right) < v < \mathbb{E}\left(R_0\, e^{-\alpha W^{(m)}_{m+1,m+1}}\right)$, we illustrate the paradox using numerical examples. We state the paradox in detail below.

**Paradox 3.** *If* $\mathbb{E}\left(e^{-\alpha W^{(m)}_{m+1,m+1}}\right) < v < \mathbb{E}\left(e^{-\alpha \hat{W}^{(m+1,m)}_{m+1,m+1}}\right)$, *then for any* $\alpha$,

$$\hat{z}^{(\hat{x}_e,\hat{x}_e)}_{i,i}(\alpha) < z^{(x_e)}_{i,i}(\alpha) \quad i = 1, \cdots, m\,.$$

*Furthermore, the total expected payoff under equilibrium satisfies*

$$\sum_{i=1}^{m+1} \hat{\pi}_{i-1}^{(\hat{x}_e)}(\alpha)\,\hat{z}_{i,i}^{(\hat{x}_e,\hat{x}_e)}(\alpha) < \sum_{i=1}^{m} \pi_{i-1}^{(x_e)}(\alpha) z_{i,i}^{(x_e)}(\alpha)$$

*where* $\{\pi_k^{(x)}(\alpha), 0 \le k \le m\}$ *and* $\{\hat{\pi}_k^{(x)}(\alpha), 0 \le k \le m\}$ *are the stationary distributions of the number of customers in the system in the N-case and the R-case, respectively.*

Table 6.1: A numerical example when $\mathbb{E}\left(R_0\, e^{-\alpha \hat{W}_{m+1,m+1}^{(m+1,m+1)}}\right) < v < \mathbb{E}\left(R_0\, e^{-\alpha W_{m+1,m+1}^{(m)}}\right)$.

| $R, \alpha, \lambda, \mu, q$ | | $2, 0.05, 0.4, 0.7, 0.2$ | | $2.2, 0.06, 0.4, 0.6, 0.3$ | | $2, 0.04, 0.4, 0.55, 0.2$ | |
|---|---|---|---|---|---|---|---|
| $x_e$ | $\hat{x}_e$ | 2.37 | 2.84 | 2.25 | 2.79 | 2.17 | 2.70 |
| $z_{1,1}^{(x_e)}(\alpha)$ | $\hat{z}_{1,1}^{(\hat{x}_e,\hat{x}_e)}(\alpha)$ | 0.29 | 0.27 | 0.34 | 0.31 | 0.28 | 0.25 |
| $z_{2,2}^{(x_e)}(\alpha)$ | $\hat{z}_{2,2}^{(\hat{x}_e,\hat{x}_e)}(\alpha)$ | 0.12 | 0.09 | 0.15 | 0.11 | 0.13 | 0.09 |
| $z_{3,3}^{(x_e)}(\alpha)$ | $\hat{z}_{3,3}^{(\hat{x}_e,\hat{x}_e)}(\alpha)$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\pi_0^{(x_e)}(\alpha)$ | $\hat{\pi}_0^{(\hat{x}_e)}(\alpha)$ | 0.05 | 0.04 | 0.09 | 0.07 | 0.04 | 0.03 |
| $\pi_1^{(x_e)}(\alpha)$ | $\hat{\pi}_1^{(\hat{x}_e)}(\alpha)$ | 0.14 | 0.12 | 0.20 | 0.16 | 0.14 | 0.11 |
| $\pi_2^{(x_e)}(\alpha)$ | $\hat{\pi}_2^{(\hat{x}_e)}(\alpha)$ | 0.40 | 0.34 | 0.45 | 0.35 | 0.51 | 0.40 |

The proof for Paradox 3 is similar to that for Paradox 2. We omit the details here. Paradox 3 is for the case where $\mathbb{E}\left(e^{-\alpha W_{m+1,m+1}^{(m)}}\right) < v \le \mathbb{E}\left(e^{-\alpha \hat{W}_{m+1,m+1}^{(m+1,m)}}\right)$.

For the case where $\mathbb{E}\left(R_0\, e^{-\alpha \widehat{W}_{m+1,m+1}^{(m+1,m+1)}}\right) < v < \mathbb{E}\left(R_0\, e^{-\alpha W_{m+1,m+1}^{(m)}}\right)$ in Paradox 3, we use some numerical examples to show that $\widehat{z}_{i,i}^{(\widehat{x}_e,\widehat{x}_e)}(\alpha) < z_{i,i}^{(x_e)}(\alpha)$ and $\widehat{\pi}_{i-1}^{(\widehat{x}_e)}(\alpha) < \pi_{i-1}^{(x_e)}(\alpha)$ for $i = 1, 2, \ldots, \lfloor x_e \rfloor$. Actually, the paradox is observed in every example we tried. In Table 6.1, we randomly select some parameter settings of $R, \alpha, \lambda, \mu, q$ with $\lfloor x_e \rfloor = 2$, and list the expected payoffs and the stationary distributions for three examples for the $N$-case and the $R$-case. It can be observed that for $1 \le i \le 2$, $\widehat{x}_e > x_e$, $z_{i,i}^{(\widehat{x}_e,\widehat{x}_e)}(\alpha) < z_{i,i}^{(x_e)}(\alpha)$ and $\widehat{\pi}_{i-1}^{(\widehat{x}_e)}(\alpha) < \pi_{i-1}^{(x_e)}(\alpha)$. Using the property $z_{3,3}^{(\widehat{x}_e,\widehat{x}_e)}(\alpha) = z_{3,3}^{(x_e)}(\alpha) = 0$, we have

$$\sum_{i=1}^{3} \widehat{\pi}_{i-1}^{(\widehat{x}_e)}(\alpha)\,\widehat{z}_{i,i}^{(\widehat{x}_e,\widehat{x}_e)}(\alpha) < \sum_{i=1}^{3} \pi_{i-1}^{(x_e)}(\alpha) z_{i,i}^{(x_e)}(\alpha)\,.$$

## 6.4 The sojourn time distribution

In the above analysis, $\alpha$ is fixed. For the expected discounted factor expressions in (6.3) and (6.12), $\alpha$ can be a variable of a Laplace transform of the sojourn time. In this section, we solve the Laplace transform of the sojourn time, and then take the inverse transform of it numerically to obtain the sojourn time distribution.

The payoff settings in Chapter 5 and 6 are both functions of the sojourn time. To investigate it, first let

$$F_{i,j}^{(x)}(w) \equiv P(W_{i,j}^{(x)} \le w) \qquad 1 \le i \le j \le \lfloor x \rfloor + 1 \, ,$$

be the distribution function of $W_{i,j}^{(x)}$ in the $N$-case, and

$$\hat{F}_{i,j}^{(\lfloor x \rfloor+1,x)}(w) \equiv P(\hat{W}_{i,j}^{(\lfloor x \rfloor+1,x)} \le w) \qquad 1 \le i \le j \le \lfloor x \rfloor + 1 \, ,$$

be that of $\hat{W}_{i,j}^{(\lfloor x \rfloor+1,x)}$ in the $R$-case. For $\mathscr{R}(s) \ge 0$, the Laplace transforms for $W_{i,j}^{(x)}$ and $\hat{W}_{i,j}^{(\lfloor x \rfloor+1,x)}$ are

$$\tilde{F}_{i,j}^{(x)}(s) \equiv \mathcal{L}(F_{i,j}^{(x)}(t)) := \int_{t=0}^{\infty} e^{-st} d\, F_{i,j}^{(x)}(t) = \mathbb{E}\left(e^{-sW_{i,j}^{(x)}}\right) \tag{6.23}$$

$$\tilde{\hat{F}}_{i,j}^{(\lfloor x \rfloor+1,x)}(s) \equiv \mathcal{L}(\hat{F}_{i,j}^{(\lfloor x \rfloor+1,x)}(t)) := \int_{t=0}^{\infty} e^{-st} d\, \hat{F}_{i,j}^{(\lfloor x \rfloor+1,x)}(t) = \mathbb{E}\left(e^{-s\hat{W}_{i,j}^{(\lfloor x \rfloor+1,x)}}\right) \, . \tag{6.24}$$

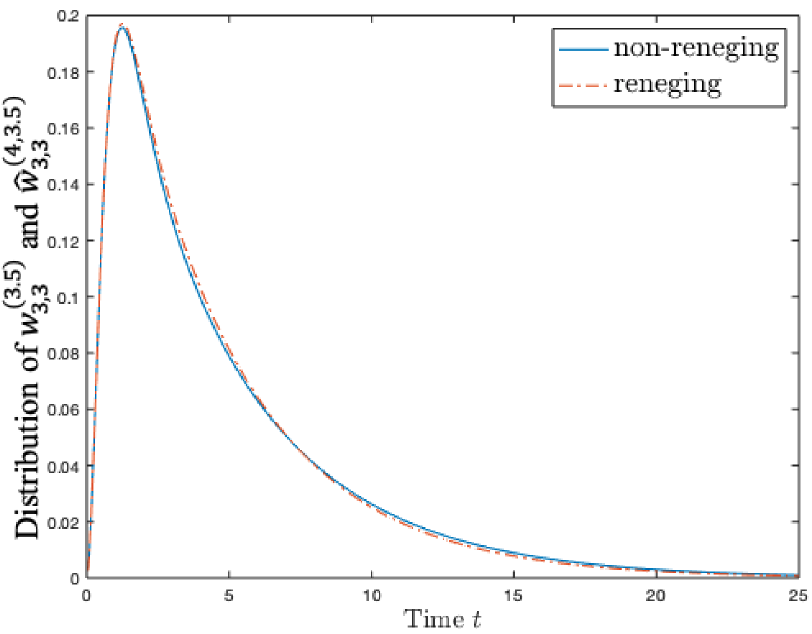
It is readily verified that $\tilde{F}_{i,j}^{(x)}(0) = \tilde{\hat{F}}_{i,j}^{(x)}(0) = 1$. The expected waiting time in Chapter 5 is the first moment of the sojourn time. If we differentiate both sides of Equation (6.23) and set $s$ to be 0, we have

$$\begin{aligned}
\mathbb{E}\left(W_{i,j}^{(x)}\right) &= -\tilde{F}'^{(x)}_{i,j}(0) \qquad (6.25)\\
&= \Bigg[ -\frac{\lambda}{s+\lambda+\mu} \left( \tilde{F}'^{(x)}_{i,j+1}(s) \mathbb{1}_{\{i<n\}} + \tilde{F}'^{(x)}_{i,j}(s) \mathbb{1}_{\{i>n\}} + \left( q\, \tilde{F}'^{(x)}_{i,j+1}(s) + (1-q)\, \tilde{F}'^{(x)}_{i,j}(s) \right) \mathbb{1}_{\{i=n\}} \right) \\
&\quad - \frac{\mu}{s+\lambda+\mu} \left( \left( q\, \tilde{F}'^{(x)}_{i-1,j-1}(s) + (1-q)\, \tilde{F}'^{(x)}_{i-1,j}(s) \right) \mathbb{1}_{\{i>1\}} + (1-q)\, \tilde{F}'^{(x)}_{j,j}(s) \mathbb{1}_{\{i=1\}} \right) \\
&\quad + \frac{\lambda}{(s+\lambda+\mu)^2} \left( \tilde{F}^{(x)}_{i,j+1}(s) \mathbb{1}_{\{i<n\}} + \tilde{F}^{(x)}_{i,j}(s) \mathbb{1}_{\{i>n\}} + \left( q\, \tilde{F}^{(x)}_{i,j+1}(s) + (1-q)\, \tilde{F}^{(x)}_{i,j}(s) \right) \mathbb{1}_{\{i=n\}} \right) \\
&\quad + \frac{\mu}{(s+\lambda+\mu)^2} \left( \left( q\, \tilde{F}^{(x)}_{i-1,j-1}(s) + (1-q)\, \tilde{F}^{(x)}_{i-1,j}(s) \right) \mathbb{1}_{\{i>1\}} + \left( q + (1-q)\, \tilde{F}^{(x)}_{j,j}(s) \right) \mathbb{1}_{\{i=1\}} \right) \Bigg] \Bigg|_{s=0}
\end{aligned}$$

$$= \frac{\lambda}{\lambda+\mu} \left( \mathbb{E}\left(W_{i,j+1}^{(x)}\right) \mathbb{1}_{\{i<n\}} + \mathbb{E}\left(W_{i,j}^{(x)}\right) \mathbb{1}_{\{i>n\}} + \left(q\,\mathbb{E}\left(W_{i,j+1}^{(x)}\right) + (1-q)\,\mathbb{E}\left(W_{i,j}^{(x)}\right)\right) \mathbb{1}_{\{i=n\}} \right)$$
$$+ \frac{\mu}{\lambda+\mu} \left( \left(q\,\mathbb{E}\left(W_{i-1,j-1}^{(x)}\right) + (1-q)\,\mathbb{E}\left(W_{i-1,j}^{(x)}\right)\right) \mathbb{1}_{\{i>1\}} + (1-q)\,\mathbb{E}\left(W_{j,j}^{(x)}\right) \mathbb{1}_{\{i=1\}} \right) + \frac{1}{\lambda+\mu},$$

which is the expression we obtained in (5.10) in Chapter 5. We have a similar argument to Equation (6.25) for the $R$-case.

By observing that Equation (6.23) and (6.24) are the expected discounted factors with the discount rate replaced by $s$ for the $N$-case and the $R$-case, respectively, we can numerically calculate the value of the Laplace transform for any $s > 0$ by solving Poisson's equations in (6.4) and (6.12). The distribution of $W_{i,j}^{(x)}$ and $\hat{W}_{i,j}^{(\lfloor x \rfloor+1,x)}$ can then be obtained by numerically inverting [52] the Laplace transform $\tilde{F}_{i,j}^{(x)}(s)$ and $\tilde{\hat{F}}_{i,j}^{(\lfloor x \rfloor+1,x)}(s)$, respectively. An example of the sojourn time distribution is plotted in Figure 6.3.




(a) $\lambda = 1, \mu = 2, q = 0.3, x = 3.5$

Figure 6.3: The sojourn time distribution.

Once we obtain the distribution of $W_{i,j}^{(x)}, 1 \le i \le j \le \lfloor x \rfloor$, we are able to calculate any expected payoff that is a function of the sojourn time, given the customer's state and the other customer' threshold. The expected payoff in Chapter 5 is the first moment of the sojourn time. Another example is Hassin and Haviv [37], which assumes the reward for an individual reduces to zero if she waits more than a fixed time $T$. Specifically, for each joining customer, if her service is completed in $T$ time, then she receives a reward whose value is $R$, otherwise the reward reduces to zero. The authors analyzed customers' decision to join or not, and if they join, when to renege. In their model, the expected reward is the expected value of a step function of the sojourn time.

# Chapter 7

# Estimating customer delay and tardiness sensitivity from periodic queue length observations

## 7.1 Introduction

When the quality of a product or a service deteriorates with time, an important consideration of customers is to avoid service delay. It occurs quite often that customers arrive at a supermarket earlier in order to obtain fresh products. An expensive managing warehouse always expects items to be picked up as soon as possible (see Elsayed, Lee, Kim, and Scherer [26]). The earlier a customer gets in a concert, the better seat she can take (see Jain, Juneja and Shimkin [48]). In these cases, tardiness, measured from the opening time to the service completion time of a customer, needs to be incorporated as a measure of the cost.

Another issue is that customers try to keep away from congestion. Besides tardiness, a customer hopes to wait for as short a time as possible. Assume that the cost is linear in time with rate $\alpha$ and $\beta$ for the waiting and tardiness, respectively. Then in deciding when to arrive, a customer needs to balance between arriving earlier and waiting less, and the optimal decision is affected by the ratio of $\alpha$ and $\beta$.

While the literature on strategic arrivals to queueing systems with tardiness costs provides considerable methods to describe and obtain the Nash equilibrium, it assumes that the system parameters are known beforehand. The problem of estimation concerning the cost rates has received little attention. In this chapter, we estimate the ratio of $\alpha$ and

$\beta$ from the queue length information.

We consider a single server FCFS queueing system which commences its service at time zero every day. The service time is exponentially distributed with rate $\mu$. It can have a pre-imposed closing time or not. Let $T$ be the service closing time, then the system serves all the customers who arrive prior to time $T$. When $T = \infty$, the system will close after it serves all arriving customers. Each day, a Poisson number, with parameter $\lambda$, of customers decide when to arrive at this system, so as to minimize their expected waiting plus tardiness cost. When $T = \infty$, as $\lambda$ is finite, the system will still serve all the arriving customers in a finite time. Thus, we still call it a day.

The FCFS queueing discipline implies that the expected cost of a customer depends on the number of customers who arrived before her, which is affected by the other customers' decisions. As a consequence, the best response of each customer is a function of the other customers' actions, thus it is appropriate to use a game theory framework to analyze the model and so the resulting solution is a Nash equilibrium strategy profile. If we focus on the symmetric mixed Nash equilibrium, then each customer has the same strategy which is a distribution over $(-\infty, T]$.

We consider both model variations with and without early birds, based on whether there are arriving customers before time 0, where early birds refer to arrivals before service commences (see Haviv [42]). In the with early birds case, customers arriving before time zero are served according to a FCFS discipline. In the case without early birds, customers who arrive before time zero are served in a random order. Hence, arriving before time zero does not bring extra benefit but incurs additional waiting cost, compared with arriving at time zero.

We assume that customers arrive according to the Nash equilibrium every day, and their behaviors are independent for different days. The values of $\alpha$ and $\beta$ are known to the customers but not the system manager. To estimate the ratio $\theta = \beta/(\alpha + \beta)$, the system manager can choose a collection of time instants to observe the queue size every day for many days. Note that although the observations from different days are independent, the observations from different time points in one day are not. We assume that sampling instants on each day are obtained at the same time instants. Based on this sampling procedure we propose an estimator for $\theta$.

### 7.1.1 An outline of methods and our contribution

Our method for estimating $\theta$ applies to both model variations with and without early birds. To work out the estimator, we need to first look into the Nash equilibrium. In this chapter, we focus on symmetric mixed Nash equilibria. The details of deriving such a equilibrium can be found in Haviv [42].

A symmetric Nash equilibrium is an arrival strategy that it is optimal for any customer if it is also used by all others. We denote the Nash equilibrium distribution by $F_e$. In this chapter, we only consider the non-trivial case where $T$ is large enough such that the distribution has one atom at time 0 and a continuous density along some bounded interval $[t_a, t_b]$. For $t \in [t_a, t_b]$, let $f_e(t)$ be the probability density. For $k = 0, 1, \ldots,$ let $Q(t)$ be the queue size at time $t$, and $P_k(t) \equiv P(Q(t) = k)$. Assume that there is no closing time, then it follows from Haviv [42] that in the without early birds case, there exists $t_b > t_a > 0$ and $p_e > 0$ such that

$$f_e(t) = \frac{\mu}{\lambda}(1 - P_0(t) - \theta) \qquad t \in [t_a, t_b] \tag{7.1}$$

$$F_e(0) = p_e \equiv 1 - \int_{t=t_a}^{t_b} f_e(t)\,dt\,. \tag{7.2}$$

That is, there is an atom of size $p_e$ at time zero, a zero density in the interval $(0, t_a)$, and some positive density along $[t_a, t_b]$. Juneja and Shimkin [50] proved the existence and uniqueness of $F_e$. In the with early birds case, there exists $t_a, t_b > 0$ such that

$$f(t) = \begin{cases} \dfrac{\mu}{\lambda}\theta & -t_a \le t < 0 \\ \dfrac{\mu}{\lambda}(1 - P_0(t) - \theta) & 0 \le t \le t_b \end{cases} \tag{7.3}$$

$$\int_0^{t_b} f(t)dt = 1 - t_a\frac{\mu}{\lambda}\theta\,. \tag{7.4}$$

That is, there is a constant density along $[-t_a, 0]$, and some positive density along $[t_a, t_b]$. Note that $P_0(t)$ in (7.1) and (7.3) depends on the Nash equilibrium in the with early birds and without early birds cases, respectively. It can be seen from Equation (7.1), (7.2) (7.3), and (7.4) that the distribution $F_e$ is determined by the ratio $\theta = \beta/(\alpha+\beta)$, not the specific values of $\alpha$ and $\beta$.

In this work, we primarily focus on the model without early birds and assume that there is no service closing time. However, in Section 7.6, we show that the application of our approach to a model with service closing time or with early birds is almost identical. Moreover, the methodology in this work can be extended to other queueing systems with strategic customers.

In the with early birds case, let $W(t) \geq 0$ be the waiting time for a customer who arrives at time $t$, and all the other customers use $F_e$, then the expected cost incurred by this customer is

$$\mathbb{E}_{F_e}[C(t)] = \alpha\, \mathbb{E}_{F_e}[W(t)] + \beta\,(t + \mathbb{E}_{F_e}[W(t)]) \tag{7.5}$$

In equilibrium $C(t) = c$ for any $t \in \{0\} \cup [t_a, t_b]$, where $c$ is a constant, and $C(t) \geq c$ for any $t \notin \{0\} \cup [t_a, t_b]$. These are the conditions for a Wardrop equilibrium, please refer to Wardrop [87] for more details. From Equation (7.5), for $s, t \in \{0\} \cup [t_a, t_b]$, where $t \neq s$, we can express the expected costs $\mathbb{E}_{F_e}[C(s)]$ and $\mathbb{E}_{F_e}[C(t)]$ in terms of $\alpha$, $\beta$ and the expected waiting times of customers arriving at the two times. By setting $\mathbb{E}_{F_e}[C(s)] = \mathbb{E}_{F_e}[C(t)]$ we can obtain an expression for $\theta$ in terms of the expected waiting times of customers arriving at the two times.

To estimate $\theta$, we first need to estimate the expected waiting time which is a function of the expected number of customers at two points in $\{0\} \cup [t_a, t_b]$. Estimation of $[t_a, t_b]$ will be explained in detail in Section 7.3.2. Let $n$ and $m > 2$ be the number of observation days and the number of sampling instants every day, respectively. For $l = 1, \ldots, n$, $i = 1, \ldots, m$, we denote by $t_i^{(l)}$ and $\Xi_i^{(l)}$ the $i$th observation time point and the queue size at that point on the $l$th day. With independent observations from $n$ days, we can estimate the expected number of the queue size by the sample mean.

We demonstrate the performance of the estimator through asymptotic analysis and simulation studies. In particular, in Section 7.4 we prove strong consistency and asymptotic normality of the resulting estimator. The proof of strong consistency is based on the central limit theorem. We also derive an expression for the asymptotic variance of the estimation error as a function of the covariance matrix of the queue lengths at sampling times. We further explain how the asymptotic variance can be numerically approximated.

Our study constructs an estimator of utility parameters of strategic customers in a queueing system by taking advantage of the queueing dynamics brought corresponding to a Nash equilibrium. Moreover, to the best of the authors' knowledge this is the only work to do so for a transient queueing setting. Second, the estimation methodology is applicable for a number of systems with strategic customers whose choice is of the Wardrop equilibria type. The setting that customers are strategic is essential in this work, since our method makes use of the properties of expected cost faced by customers arriving according to the Nash equilibrium distribution. Third, the estimator we propose needs substantially less information than standard queueing inference techniques (see Asanjarani, Nazarathy, and Taylor [7]). Namely, it does not require continuous observations, or an accurate estimate of $[t_a, t_b]$. In fact, with only two points in $\{0\} \cup [t_a, t_b]$, we are able to estimate the ratio $\theta$. We will explain how to choose the two points in detail in Section 7.3. Moreover, in the examples we tried, we observe that the estimator is robust to the number of sampling instants each day.

### 7.1.2 Literature review

Haviv [42] derived the Nash equilibrium arrival distribution for the aforementioned system that this work focuses on. However, the study of the strategic arrival time choice in queueing models goes back to Glazer and Hassin [32], which studied a single server system. A Poisson number of customers decide when to arrive to this system such that their expected waiting time is minimal. Jain, Juneja and Shimkin [48], Juneja and Shimkin [50] and Haviv [42] studied the same queueing system, but took the tardiness cost into consideration, and characterized the symmetric Nash equilibrium arrival distribution in terms of a set of differential equations. Haviv and Ravner [43] studied strategic timing of arrivals to a multi-server loss system, where customers tried to maximize their probability of receiving the service. Comprehensive reviews and summaries of this kind of research can be found in Haviv and Ravner [45] and Hassin [41, Section 4.1].

Most of the literature that deals with queueing process estimation concerns the arrival and service rates, and describes methods that require continuous observation over an interval of time. To cite but a few, see for example Bhat and Rao [12], Armero [6], Armero and Armero [1], Basawa and Prabhu [11], Bingham and Pitts [13]. See Asanjarani, Nazarathy, and Taylor [7] for a comprehensive survey of parameter estimation in queues. In Inoue, Ravner, and Mandjes [46], the authors considered a system with strategic customers who can choose to balk after being aware of information about the delay, and estimated the patience distribution and the corresponding potential arrival rate. Robinson and Chen [76] estimates the relative perceived value of patients waiting time, which is a ratio of the cost of the customers' waiting time to the cost of the server's idle time. Their method relies on constructing an estimation equation from the optimality condition of the stationary expected cost of the system. However, as opposed to our setting, customers are not strategic and the underlying distribution of the queueing process is known.

The chapter is organised as follows. In Section 7.2 we explain the Nash equilibrium arrival distribution and how to numerically approximate it. In Section 7.3 we propose the estimator for $\theta$ for the model without early birds. Section 7.4 presents the asymptotic analysis of the estimator. In Section 7.5 we provide several numerical examples to demonstrate the effect of the number of the observation days and the sampling instants on the estimator. In Section 7.6 we modify the estimator to adapt for the model with early birds or with service closing time.

## 7.2 Preliminaries

### 7.2.1 The Nash equilibrium arrival distribution

Let $\mathbb{E}_{F_e}[C(t)]$ be that defined in Equation (7.5). For the with early birds case, a Wardrop Nash equilibrium $F_e$ satisfies

$$\mathbb{E}_{F_e}[C(t)] = c, \qquad \forall\, t \in \{0\} \cup [t_a, t_b]\,, \tag{7.6}$$

and

$$\mathbb{E}_{F_e}[C(t)] \geq c, \qquad \forall\, t \notin \{0\} \cup [t_a, t_b]\,. \tag{7.7}$$

This is because a customer will only randomize arrival time points if they result in the the same cost. Secondly, the expected cost of a customer arriving at any point outside $\{0\} \cup [t_a, t_b]$ cannot be lower. Otherwise, a customer would deviate to arrive at such a time point and become better off.

If we denote a general arrival distribution by $F$ and its probability density function by $f$ (if it exists), then the arrival process is a non-homogeneous Poisson process with intensity measure $\lambda f(t)$. The queue size dynamics satisfy the Kolmogorov forward equations

$$P_0'(t) = P_1(t)\mu - P_0(t)\lambda f(t) \tag{7.8}$$

$$P_k'(t) = P_{k-1}(t)\lambda f(t) + P_{k+1}(t)\mu - P_k(t)\,\left(\lambda f(t) + \mu\right)\,. \tag{7.9}$$

The expected cost for the case without early birds can then be written as

$$\mathbb{E}_F[C(t)] = \begin{cases} \dfrac{\alpha+\beta}{2\mu}\mathbb{E}_F[Q(0)] & t = 0, \\ \dfrac{\alpha+\beta}{\mu}\mathbb{E}_F[Q(t)] + \beta t & t > 0\,. \end{cases} \tag{7.10}$$

The half fraction when $t = 0$ is from the fact that customers arriving at time zero are served in random order.

### 7.2.2 Discrete approximation

This section explains how $F_e$ can be numerically approximated. The equilibrium arrival distribution $F_e$ satisfies a set of non-linear differential equations, which does not have an analytic expression. To numerically solve the differential equations defined by the queueing dynamics and $F_e$, we adopt the finite-difference method, which was also mentioned in Haviv and Ravner [45, Section 3.1, Algorithm 1] and was termed as *discrete approximation*.

To make the calculation of the expected queue size feasible, we truncate the queue size at $K$. Specifically, we assume that customers can choose to arrive at a time on a discrete grid $\mathcal{T} \equiv \{0, \delta, 2\delta, \ldots\}$, and the queue has a buffer size of $K$. When the value of $\delta$ is very small, with high probability there is at most one event happening in $\delta$, thus for $r = 1, 2, \ldots$ and $k = 0, 1, \ldots, K$, the queue size dynamics $P_k$ on $\mathcal{T}$ satisfy

$$P_0((r+1)\delta) \approx P_0(r\delta) + P_1(r\delta)\mu - P_0(k\delta)\lambda f(r\delta) + o(\delta) \tag{7.11}$$

$$P_k((r+1)\delta) \approx P_k(r\delta) + P_{k-1}(r\delta)\lambda f(r\delta) + P_{k+1}(r\delta)\mu - P_k(r\delta)\ (\lambda f(r\delta) + \mu) + o(\delta),\ 1 \le k \le K-1 \tag{7.12}$$

$$P_K((r+1)\delta) \approx 1 - \sum_{k=0}^{K-1} P_k((r+1)\delta) + o(\delta)\,, \tag{7.13}$$

which are the finite difference scheme applied to Equations (7.8) and (7.9). For convenience, we drop the subscription $F_e$, and let the expected value and dynamics be that under the Nash equilibrium by default. The expected cost

$$\mathbb{E}[C(r\delta)] \approx \begin{cases} \dfrac{(\alpha+\beta)\lambda p_e}{2\mu} & r = 0 \\ \dfrac{\alpha+\beta}{\mu} q(r\delta) + \beta r\delta & r > 0\,, \end{cases} \tag{7.14}$$

where $q(r\delta) \equiv \sum_{k=1}^{K} kP_k(r\delta)$ is the approximated expected queue size at slot $r$. Increasing $K$ or decreasing $\delta$ clearly improves the accuracy of the approximation, but this is at the expense of calculation speed. We set $K = \min\{m : \sum_{k=0}^{m} \lambda^k e^{-\lambda}/k! \ge 1 - 10^{-6}\}$, and $\delta = 0.001$ in the rest of the chapter.

The values of $t_a$ and $t_b$ are approximated by $r_a\delta$ and $r_b\delta$. In the following, we explain how to find $p_e$, $f_e$, $r_a$ and $r_b$ using the discrete approximation. When the value of $p_e$ is given, the expected cost $\mathbb{E}[C(0)]$ incurred by customers arriving at time zero can be calculated.

It follows from [42] that $F_e$ has a zero density along the interval $(0, t_a)$, which means $f(r\delta) = 0$ until $r \ge r_a$. For $r = 1, 2, \ldots, r_a$, since $f(r\delta) = 0$, the queue size dynamics at time $r\delta$ can be calculated using Equations (7.11)-(7.13), the expected cost to arrive at $r\delta$ can then be determined. From Equation (7.7), the expected cost incurred by customers arriving at anytime in $(0, r_a\delta)$ is not less than $\mathbb{E}[C(0)]$. Hence, to determine the value of $r_a$, we keep computing the queue dynamics, and then the expected cost for $t = r\delta$ from

$r = 1$ until $\mathbb{E}(C(t)) \leq \mathbb{E}(C(0))$, then $r_a = \inf\{r : \mathbb{E}(C(r\delta)) \leq \mathbb{E}(C(0)), r \geq 1\}$. In Haviv [42], the author calculated $r_a$ by working out the expression of the expected cost at time $r\delta$ when $f(r\delta) = 0$. Here we use an alternative way, and we put a more detailed explanation of the method in Haviv [42] and its comparison with our work in the remark below.

For $r \geq r_a$, the arrival density $f(r\delta)$ is defined by Equation (7.1), then the queue size dynamics can be obtained using Equations (7.11)-(7.13), until $f(r\delta) \leq 0$, and $r_b = \sup\{r : f(r\delta) \leq 0, r > r_a\}$. Thus, given the value of $p_e$, the values of $r_a$, $r_b$, and $f(r\delta)$ in $[r_a\delta, r_b\delta]$ can be determined. Another condition that $p_e$, $f_e$, $r_a$ and $r_b$ need to satisfy is

$$p_e + \int_{t=t_a}^{t_b} f_e(t)\, dt = 1\,. \tag{7.15}$$

Hence, we can initially guess a value for $p_e$, and then adjust it iteratively using the bisection method until Equation (7.15) is satisfied. Specifically, we start with $p_1 = 0, p_2 = 1$, and always set $p_e = \dfrac{p_1 + p_2}{2}$. At the end of each iteration, we set $p_2 = p_e$ if the total probability is greater than one, and $p_1 = p_e$ otherwise. This calculation process is summarized in Algorithm 6.

**Remark**: Note that in the case without early birds, there is an interval $[0, t_a]$ where the equilibrium density is 0. Thus given the atom size at time zero, the expected number of customers at any time $t \leq t_a$ has an analytic expression. This expression was derived in Haviv [42, Lemma 3.3], where the author proposed two methods to calculate its quantity. One method is computing it with the assistance of Bessel's functions, and the other method is estimating it using a Monte Carlo simulation procedure. The goal of working out the expression is to find a time at which if a customer arrives, her expected cost will be the same as the expected cost if she arrives at time 0. In this section, we do not adopt this expression for our numerical estimation process. Although our method to estimate $t_a$ seems cumbersome, it performs well to obtain $t_a$ compared with the method using the expression of the expected number of customers as in Haviv [42, Lemma 3.3]. Actually, in the numerical examples we tried, it calculated $t_a$ faster.

**Algorithm 6** Equilibrium arrival distribution for the model without early birds.

**Input:** $\lambda,\ \mu, \alpha,\ \beta,\ \delta$

**Output:** $p_e,\ f_e$

**procedure** $f_{\mathcal{W}/\mathcal{O}}(\lambda, \mu, \alpha, \beta, \delta)$

  **Initialization:** $p_1 = 0,\ p_2 = 1,\ p = \dfrac{p_1 + p_2}{2}$

  **while** $p_2 - p_1 > 10^{-6}$ **do**

    $r = 1,\ \ P_k(0) = \left( \dfrac{(\lambda p)^k \, e^{-\lambda p}}{k!} \right)_{k=0:K}$

    $c = \dfrac{(\alpha + \beta)\lambda p}{2\mu}$

    **do**

      calculate $P_k(r\delta)$ for $k = 0, 1, \ldots, K$ using Equations (7.11)-(7.13)

      **if** $\left( \beta\,(r\delta) + (\alpha + \beta)\, \sum_{k=0}^{K} \dfrac{k\, P_k(r\delta)}{\mu} \right) > c$ **then**

        $f_e(r\delta) = 0$

      **else**

        $f_e(r\delta) = \dfrac{(1 - P_0(r\delta))\mu}{\lambda} - \dfrac{\beta\mu}{(\alpha + \beta)\lambda}$

      $r = r + 1;$

      $\zeta = p_e + \delta\, f\boldsymbol{e}$, where $\boldsymbol{e}$ is a vector of 1's of the appropriate size.

    **while** $\zeta < 1$ and $\min f \geq 0$

    **if** $\zeta >= 1$, $\min f >= 0$ **then**

      $p_2 = p_e$

    **else**

      $p_1 = p_e$

    $p_e = \dfrac{p_1 + p_2}{2}$

## 7.3 Estimation of $\theta$

Assume that the system manager knows the value of $\mu$, but not $\theta$. It follows from Equation (7.1) $F_e$ is defined by $\theta$, $\lambda$, and $\mu$. Thus, the queue size distribution is affected by $\theta$ via $F_e$. The question is how to construct an estimator for $\theta$ from observations of queue sizes. To answer this question, we first write down the expression for the expected cost of any two points in $\{0\} \cup [t_a, t_b]$, and then derive $\theta$ in terms of the expected queue size at these two points. This is discussed in detail in the following. From the definition of the Nash equilibrium, when others arrive according to $F_e$, each customer is indifferent between arriving at any $t \in \{0\} \cup [t_a, t_b]$. Hence, in the model without early birds, Equation (7.6) and (7.10) imply that

$$\frac{(\alpha+\beta)}{\mu}\,\mathbb{E}[Q(t)]+\beta\,t \;=\; \frac{(\alpha+\beta)}{2\,\mu}\,\mathbb{E}[Q(0)] \;=\; c \qquad \forall\,t\in[t_a,t_b]\,, \tag{7.16}$$

thus

$$\theta=\frac{\beta}{\alpha+\beta}=-\frac{\mathbb{E}[Q(s)]-\mathbb{E}[Q(t)]}{\mu\,(s-t)}=-\frac{\mathbb{E}[Q(t)]-\mathbb{E}[Q(0)]/2}{\mu\,t} \qquad \forall\,s,t\in[t_a,t_b]\,. \tag{7.17}$$

That is, $\theta$ can be expressed as a function of the expected queue size at any two instants in $\{0\}\cup[t_a,t_b]$.

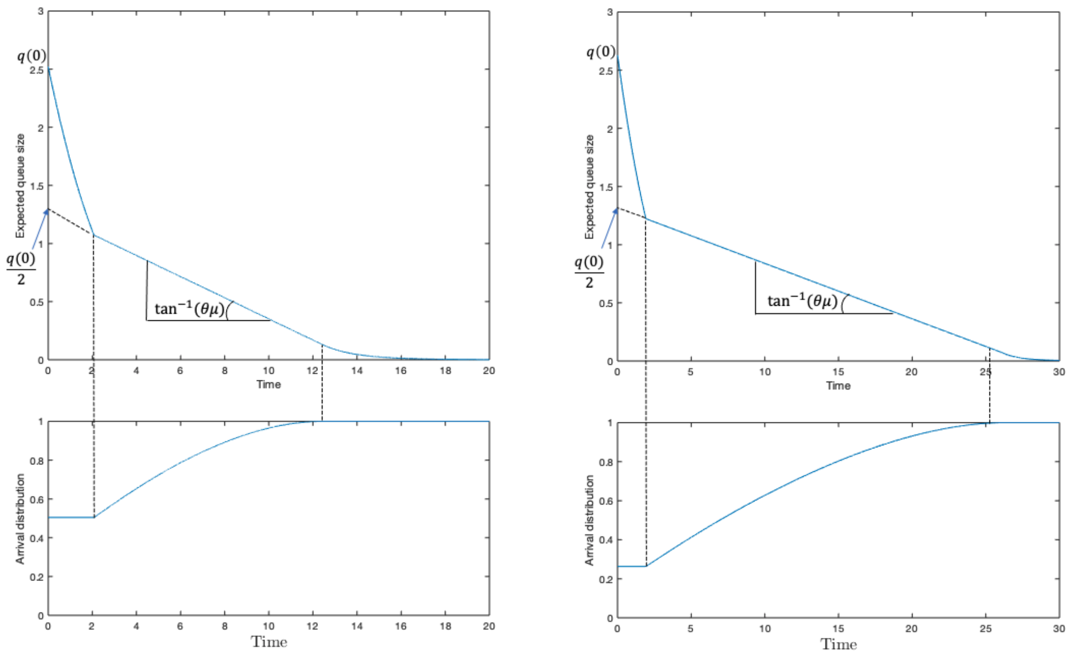
### 7.3.1 Estimation of the expected queue sizes

With Algorithm 7 and 6, we can calculate the approximate expected queue size $q(t)$ when customers use the Nash equilibrium. Figure 7.1 depicts $q(t)$ and two observations can be made. First, $q(t)$ on $[r_a\delta, r_b\delta]$ forms a straight line, whose slope is $-\theta\mu$. Second, if we extend the line to $t=0$, then the point of intersection is $q(0)/2$, which can also be inferred from Equation (7.17).

**Algorithm 7** Expected queue size calculation

**Input:** $\lambda,\ \mu, \alpha,\ \beta,\ \delta,\ T$
**Output:** $q(t)$
$K=\min\{m:\sum_{k=0}^{m}\lambda^k e^{-\lambda}/k!\ge 1-10^{-6}\}$
**switch** model **do**
    **case** without early birds
        $[p_e,f_e]=f_{\mathcal{W}/\mathcal{O}}(\lambda,\mu,\alpha,\beta,\delta)$
    **case** with early birds
        $[w,f_e]=f_{\mathcal{W}}(\lambda,\mu,\alpha,\beta,\delta)$
    **case** with service closing time
        $[p_e,f_e]=f_{\mathcal{CT}}(\lambda,\mu,\alpha,\beta,\delta)$
**procedure** QUEUE SIZE DISTRIBUTION UNDER EQUILIBRIUM
    $r=0$
    **while** $P_0(r\delta)<1$ **do**
        calculate $P_k(r\delta)$ for $k=0,1,\ldots,K$ using $f_e$ and Equations (7.11)-(7.13)
        $r=r+1$
$q(r\delta)=\sum_{k=0}^{K}P_k(r\delta)$ for $r=0,1,2,\ldots$.

Equation (7.17) implies that two observation points in the support each day are enough to estimate $\theta$. However, observe that to use (7.17), we first need to estimate $[t_a,t_b]$, and to




(a) (b)

Figure 7.1: The Nash equilibrium arrival distribution and the expected queue size under it.

achieve this we need more than two observations as will be shown below. Suppose we choose a collection of time instants to observe the queue size every day. We assume that the observations on each day are obtained at the same time slots, so we drop the index $l$ on $t_i$ and write the samples on the $l$th day as $\{t_i,\, \Xi_i^{(l)}\}_{i=1}^{m}$. It follows from our assumption of statistical independence between days that $\{t_i,\, \Xi_i^{(l)}\}_{i=1}^{m}$ are from independent realisations of the process. Since $\Xi_i^{(1)}, \Xi_i^{(2)}, \ldots, \Xi_i^{(n)}$ are independent and identically distributed as $Q(t_i)$, the expected queue size can be estimated by the sample mean $\hat{q}_n(t_i) \equiv \frac{1}{n}\sum_{l=1}^{n} \Xi_i^{(l)}$ at the $i$th sampling time instant. Figure 7.2 depicts $\hat{q}_n(t_i)$ for three simulations with $n = 1000$ and $\{t_i\}_{i=1}^{m} = \{0, \delta, 2\delta, \ldots\}$, which is the same grid that we used to solve the differential equations (7.11) -(7.13). Note that although $\{t_i\}_{i=1}^{m}$ can take any value along the positive real line, in the simulation, $\{t_i\}_{i=1}^{m} \subseteq \hat{\mathcal{T}}$. To make the comparison between the expected value and the mean, $q(t)$ is also plotted. We will show later that we do not need such a large value of $m$ to estimate $\theta$. We chose a fine grid here to display the estimation of the expected queue size along the interval $[0, T]$.

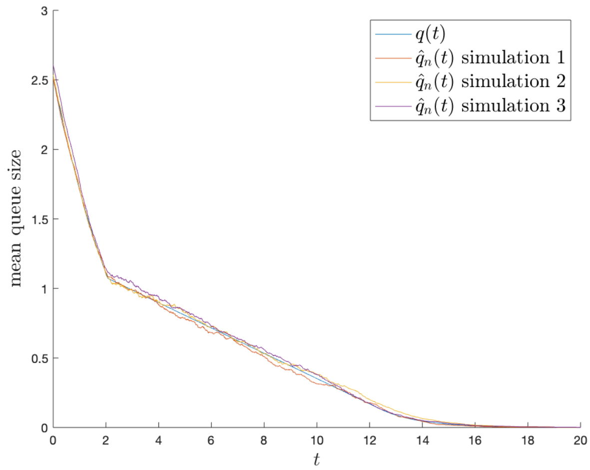



Figure 7.2: The sample mean of the queue sizes.

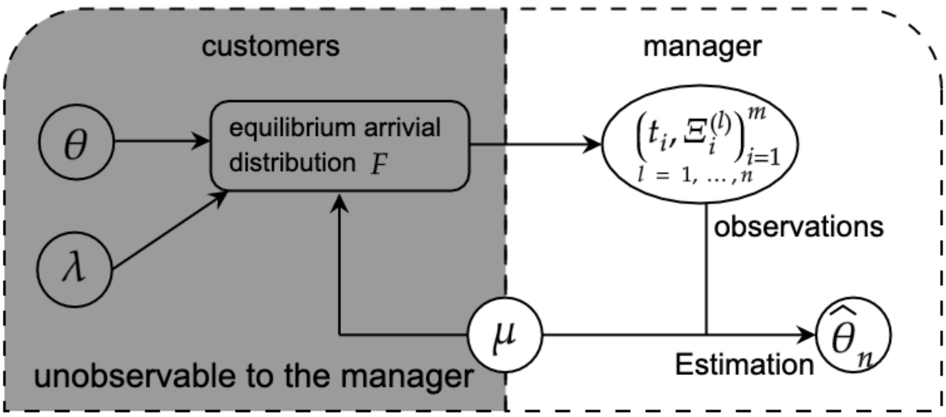



Figure 7.3: An overview of the observation and estimation process.

**Remark**: Parameters $\lambda$ and $\mu$ are easy to estimate even if they are unknown so we treat them as known throughout the chapter.

Figure 7.3 gives an overview of the observation and the estimation process, where $\hat{\theta}_n$ is the estimator derived from $\hat{\vartheta}_n$ which will be introduced in Section 7.3.4. The manager uses her observations and the value of $\mu$ to estimate $\theta$. We estimate the expected queue sizes from independent realizations of the queue. Now we move to the estimation of $\theta$ which requires the estimation of $t_a$ and $t_b$ first. Both estimations require comparing observations at different times either on average or for specific sample paths. Note that although the observations from different days are independent, the observations at different time for a single day are not. In the following, we first explain the method to estimate $t_a$ and $t_b$. We further need to carefully construct the pairing of observation instants in the construction of the sequence of estimators $\hat{\vartheta}_n(t_i, t_j)$ for $\theta$. To this end a heuristic rule of choosing the farthest away observation (in terms of time) is applied. The intuitive explanation for this rule is that the correlation between queue lengths at different sampling times decreases the further apart they are chosen. This intuition is verified by simulation experiments. Specifically, we randomly choose a point $t_r$ inside the estimated support, and estimate $\theta$ for every pair which includes $t_r$ and a point inside the estimated support except for $t_r$. The main observation from the simulation experiment is that indeed the farther away the two observation points are, the better the estimate is. This procedure is detailed in Section 7.3.3. Our proposed estimator in Section 7.3.4 is the result of taking an average of the estimators obtained by the optimal pairings described above.

### 7.3.2 Estimation of the support

To estimate $t_a$ and $t_b$, we first find the time instants in $\{t_i\}_{i=1}^m$ such that the queue size increases. By noticing that $[t_a, t_b]$ must include these times, we estimate the support boundary indices $\hat{a}_n$ and $\hat{b}_n$ by

$$\hat{a}_n = \min_{1\le l\le n}\{\inf_{2\le i\le m}\{i : \Xi_i^{(l)} - \Xi_{i-1}^{(l)} \ge 1\}\} \tag{7.18}$$

$$\hat{b}_n = \max_{1\le l\le n}\{\sup_{2\le i\le m}\{i : \Xi_i^{(l)} - \Xi_{i-1}^{(l)} \ge 1\}\}. \tag{7.19}$$

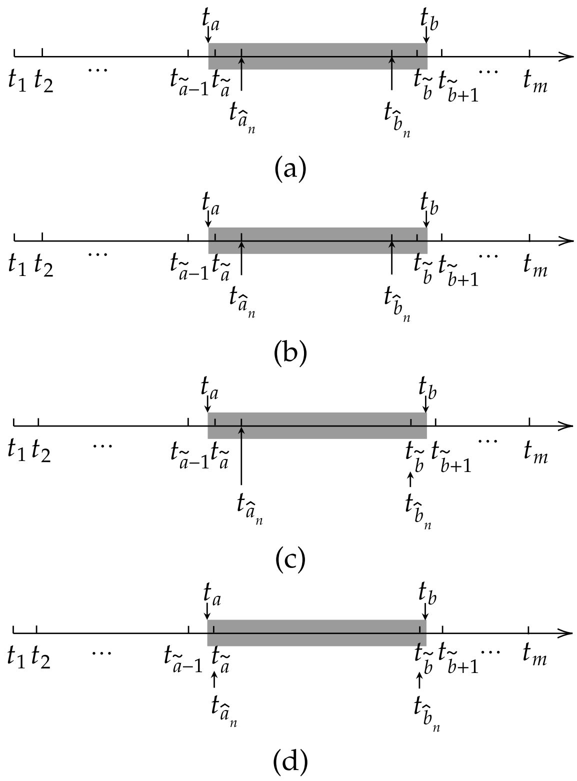


Figure 7.4: An illustration of $t_{\hat{a}_n}$, $t_{\hat{b}_n}$, $t_{\tilde{a}}$, and $t_{\tilde{b}}$.

Let

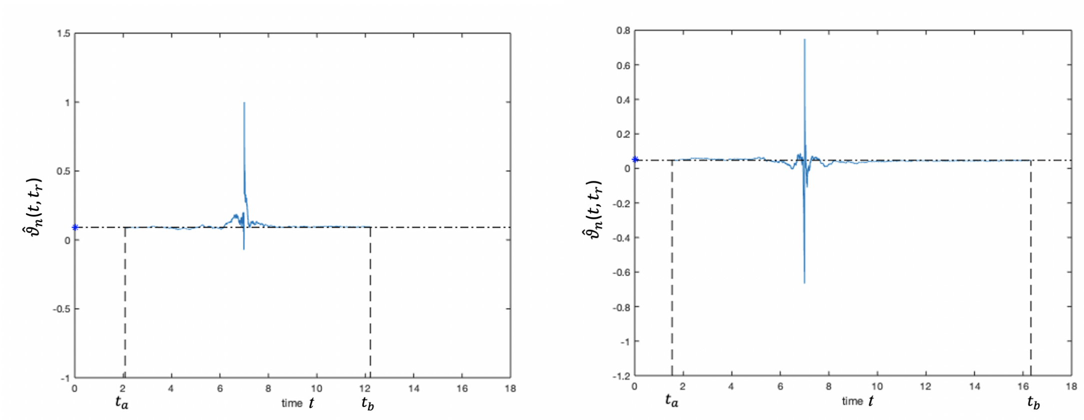
$$\tilde{a} \equiv \inf_{1\le i\le m} \{i : t_i \ge t_a\} \tag{7.20}$$

$$\tilde{b} \equiv \sup_{1\le i\le m} \{i : t_i \le t_b\}, \tag{7.21}$$

then the optimal estimates of $t_a$ and $t_b$ are $t_{\tilde{a}}$ and $t_{\tilde{b}}$, respectively. Figure 7.4 describes some examples of $t_{\hat{a}_n}$ and $t_{\hat{b}_n}$ with $t_a, t_b, t_{\tilde{a}}, t_{\tilde{b}}$ plotted as well, where the shaded interval is $[t_a, t_b]$. In Figure 7.4(a), the observation instants $\{t_i\}_{i=1}^m$ includes both $t_a$ and $t_b$, but this event happens with probability 0. In that case, $t_{\tilde{a}} = t_a$ and $t_{\tilde{b}} = t_b$. With probability 1, $\{t_i\}_{i=1}^m$ does not include $t_a$ or $t_b$, as in Figure 7.4(b)(c)(d). In any case, it follows from (7.18) and (7.19) that

$$t_a \le t_{\tilde{a}} \le t_{\hat{a}_n}, \qquad t_{\hat{b}_n} \le t_{\tilde{b}} \le t_b .$$

Note that the reason to estimate the support is to select time points inside $[t_a, t_b]$ to




(a) $\alpha = 2, \beta = 0.2, \lambda = 5, \mu = 1, t_r = 7$. (b) $\alpha = 2, \beta = 0.1, \lambda = 10, \mu = 1, t_r = 7$.

Figure 7.5: Estimates of $\theta$ using a reference instant.

estimate $\theta$. Even if the estimation for the support is biased, the estimation for $\theta$ is not affected by the bias as long as the chosen pair is inside $[t_a, t_b]$, which always holds since $[t_{\hat{a}_n}, t_{\hat{b}_n}] \subset [t_a, t_b]$.

### 7.3.3 Estimation of $\theta$ using a reference point

If the observation time slots in $\{0\} \cup [t_a, t_b]$ include slots $t_i$ and $t_j$, or slots $t_i$ and 0, then for any $i, j \in \{1, \ldots, m\}$ we can estimate $\theta$ by

$$\hat{\vartheta}_n(t_i, t_j) = -\frac{\hat{q}_n(t_i) - \hat{q}_n(t_j)}{\mu(t_i - t_j)} \quad \text{or} \quad \hat{\vartheta}_n(t_i, 0) = -\frac{\hat{q}_n(t_i) - \hat{q}_n(0)/2}{\mu t_i}, \tag{7.22}$$

To see the performance of the estimator in Equation (7.22), we first calculated $F_e$ for a given value of $\alpha$ and $\beta$ using Algorithm 6. Next we run the simulation $n$ times by letting customers arrive according to $F_e$, and recorded the queue size at each $t \in \mathcal{T}$. We estimated the boundary indices $\hat{a}_n$ and $\hat{b}_n$ using Equations (7.18) and (7.19). Then assumed that $\alpha$ and $\beta$ were unknown, let $\hat{\mathcal{T}} \equiv \left(\{0\} \cup [t_{\hat{a}_n}, t_{\hat{b}_n}]\right) \cap \{t_i\}_{i=1}^m$, and randomly selected a time point $t_r(> 0) \in \hat{\mathcal{T}}$ as a reference point. For each $t \in \hat{\mathcal{T}}/\{t_r\}$, calculated $\hat{\vartheta}_n(t, t_r)$.

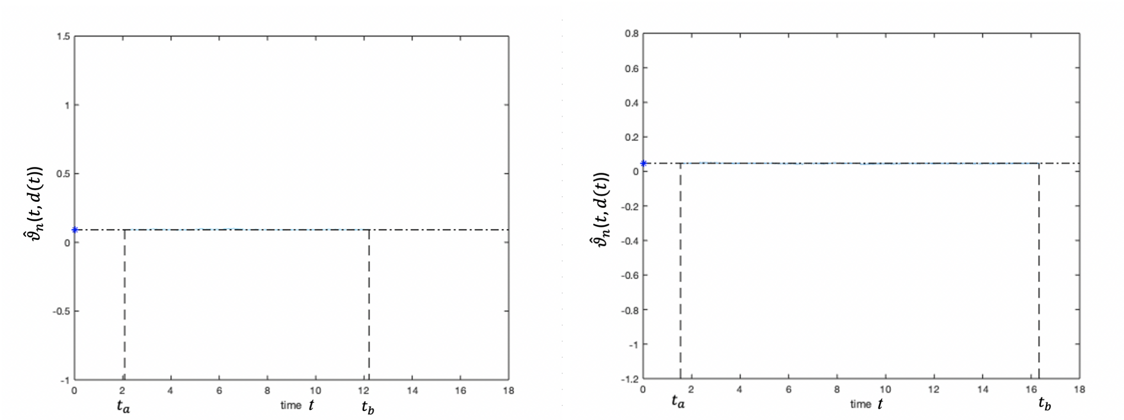
Two examples of $\hat{\vartheta}_n(t, t_r)$ for $t \in \hat{\mathcal{T}}/\{t_r\}$ where $t_r = 7$, under different parameter settings were depicted in Figure 7.5. In Figure 7.5(a), $t_a = 1.523, t_b = 16.594$, the estimates $t_{\hat{a}_n}$ and $t_{\hat{b}_n}$ were 1.538 and 16.324, respectively. In Figure 7.5(b), $t_a = 2.075, t_b = 12.415$, the estimates $t_{\hat{a}_n}$ and $t_{\hat{b}_n}$ were 2.076 and 12.210, respectively. The true $\theta$ value was plotted by the dotted line. It can be observed from both plots that the closer the two points are, the more biased the estimation is. The explanation is that the dependency between the queue sizes at two time slots increases when the interval between them decreases. In Section 7.4, we will see the estimator variance increases in quadratic rate with the distance.

**Remark**: It follows from Equation (7.1) and $f(t_b) = 0$ that $\theta$ can also be expressed as

$$\theta = 1 - P_0(t_b)\,. \tag{7.23}$$

This means if we can estimate $t_b$ by $t_{\hat{b}}$, we can estimate $P_0(t_b)$ by counting how many empty queues there are at $t_{\hat{b}}$ and divide it by the total sample size $n$, then $\theta$ can be estimated. However, this requires an accurate estimate of $t_b$ first, which requires both a large $m$ such that $\{t_i\}_{i=1}^{m}$ includes a point that is very close to it, and a large sample size $n$. Also, in the model with service closing time which we will introduce in Section 7.6, there doe not necessarily exist a time that satisfies $f(t_b) = 0$, thus no expression like (7.23).

### 7.3.4 The estimator




(a) $\alpha = 2, \beta = 0.2, \lambda = 5, \mu = 1$. (b) $\alpha = 2, \beta = 0.1, \lambda = 10, \mu = 1$.

Figure 7.6: Estimates of $\theta$ using the farthest instant.

With the aforementioned reasoning in mind, it is better to choose two observation points that are far away from each other. Denote the cardinal number of $\hat{\mathcal{T}}$ by $|\hat{\mathcal{T}}|$. For any $t \in \hat{\mathcal{T}}$, let $d(t) = \arg\max_{t_i \in \hat{\mathcal{T}}} |t_i - t|$ and we calculated $\hat{\vartheta}_n(t, d(t))$. The results are depicted in Figure 7.6. It is observed from the plots that $\hat{\vartheta}_n(t, d(t))$ is generally better than $\hat{\vartheta}_n(t, t_r)$, especially for $t$ which is close to $t_r$.

When $|\hat{\mathcal{T}}| > 2$, we propose an estimator

$$\hat{\theta}_n \equiv \frac{1}{|\hat{\mathcal{T}}|} \sum_{t \in \hat{\mathcal{T}}} \hat{\vartheta}_n(t, d(t)) . \tag{7.24}$$

by taking the mean of the estimates calculated by every pair in $\hat{\mathcal{T}}$.

We assume the sampling points does not include time 0 in the rest of the chapter. The analysis is similar if 0 is included.

## 7.4 Asymptotic analysis

We first state the main results of this section with the proofs detailed in the following subsections. In the following, $\mathcal{N}(\boldsymbol{\mu}, \Sigma)$ denotes a normally distributed random variable with mean vector $\boldsymbol{\mu}$ and covariance matrix $\Sigma$. An estimator is said to be strongly consistent if as the number of observation days increases, the resulting sequence of estimates converges almost surely to the true value. Theorem 16 proves the strong consistency of our estimator $\hat{\theta}_n$. We establish the asymptotic normality of our estimator $\hat{\theta}_n$, and prove in Theorem 17 that as $n \to \infty$, the estimation error scaled by $\sqrt{n}$ converges to a zero-mean normal random variable, whose variance can be numerically calculated. We explain in detail how the variance is approximated in Section 7.4.3. The proofs of Theorem 16 and 17 are provided in Section 7.4.1 and Section 7.4.2, respectively.

**Theorem 16.** *As $n \to \infty$, $\hat{\theta}_n \xrightarrow{a.s.} \theta$.*

**Theorem 17.** *Let $k_i \equiv |\hat{\mathcal{T}}| (t_i - d(t_i)) \mu$, $g_i \equiv \sum_{j \neq i} \frac{1}{k_j} \mathbb{1}_{\{d(t_j) = t_i\}} - \frac{1}{k_i}$, $v(t) \equiv \mathrm{Var}_{F_e}[Q(t)]$ and $\rho(s,t) \equiv \mathrm{Cov}_{F_e}[Q(s), Q(t)]$ for any $s, t \in [t_a, t_b]$. As $n \to \infty$,*

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, \sum_{i=\tilde{a}}^{\tilde{b}} g_i^2 v(t_i) + 2 \sum_{i=\tilde{a}}^{\tilde{b}} \sum_{j>i}^{\tilde{b}} g_i g_j \rho(t_i, t_j)\right) .$$

### 7.4.1 Strong consistency

The proof of Theorem 16 requires finite first and second moments of the queue size $Q(t)$. The total number of customers in the system is bounded by the a Poisson distributed random variables with rate $\lambda$, so as long as $0 < \lambda < \infty$, $v(t) < \infty$ and $|\rho(s,t)| < \infty$.

*Proof of Theorem 16.* The first and second moments of $Q(t)$ are bounded, it follows from Kolmogorov strong law of large numbers [59, p251] that

$$\hat{q}_n(t) \rightarrow_{a.s.} \mathbb{E}[Q(t)] \,. \tag{7.25}$$

Also, it follows from Equation (7.22) that both $\hat{\vartheta}_n(t, d(t))$ and $\hat{\theta}_n$ are linear functions of $\hat{q}_n(t)$, thus the continuous mapping theorem in van der Vaart [83, Theorem 2.3] implies

$$\hat{\vartheta}(t, d(t)) \rightarrow_{a.s.} -\frac{\mathbb{E}[Q(t)] - \mathbb{E}[Q(d(t))]}{(t - d(t))\mu} = \theta \,, \qquad \hat{\theta}_n \rightarrow_{a.s.} \frac{1}{m}\sum_{i=1}^{m} \theta = \theta \,. \tag{7.26}$$

□

We have explained in Section 7.3.2 that the reason to estimate $[t_a, t_b]$ is to choose two points inside it for the estimation, and our method assures that $[t_{\hat{a}}, t_{\hat{b}}] \subset [t_a, t_b]$. Moreover, the optimal estimates of $t_a$ and $t_b$, using our method, are $t_{\tilde{a}}$ and $t_{\tilde{b}}$ defined in Equations (7.20) and (7.21), respectively. Although it is not essential to have accurate estimates of $t_a$ and $t_b$, we prove in the following proposition that $t_{\hat{a}_n}$ and $t_{\hat{b}_n}$ converges to $t_{\tilde{a}}$ and $t_{\tilde{b}}$, respectively. Moreover, Proposition 2 is used in establishing the asymptotic distribution of the errors in the proof of Theorem 17.

**Proposition 2.** *If $t_{\hat{a}_n}$ and $t_{\hat{b}_n}$ exist, then as $n \to \infty$, $t_{\hat{a}_n} \xrightarrow{a.s.} t_{\tilde{a}}$, $t_{\hat{b}_n} \xrightarrow{a.s.} t_{\tilde{b}}$.*

*Proof.* We label the observation times $\{t_i\}_{i=1}^{m}$ in a way such that $t_{i+1} > t_i$ for $i = 1, \ldots, m - 1$. Thus $t_{\tilde{a}+1} > t_{\tilde{a}}$. For $t \geq t_a$, $F_e(t)$ is increasing, so $F_e(t_{\tilde{a}+1}) > F_e(t_{\tilde{a}})$. Our estimated $t_{\hat{a}_n}$ is greater than or equal to $t_{\tilde{a}}$ from the way we estimated it from the $n$ samples in Equations (7.18) and (7.19). Given that there are arrivals after time 0, the probability that the arrivals are all after time $t_{\tilde{a}+1}$ is $1 - (F_e(t_{\tilde{a}+1}) - F_e(t_{\tilde{a}}))/(1 - p_e)$. If the estimated $t_{\hat{a}_n}$ from $n$ samples is greater than $t_{\tilde{a}}$, then the arrivals after $t_{\tilde{a}}$ in the $n$ samples must all happen after time $t_{\tilde{a}+1}$. Also, as the observations of different days are independent, we have that

$$\mathbb{P}\left(t_{\hat{a}_n} > t_{\tilde{a}}\right) \leq \left(1 - \frac{F(t_{\tilde{a}+1}) - F(t_{\tilde{a}})}{1-p}\right)^n . \tag{7.27}$$

Therefore,

$$\lim_{n\to\infty} \mathbb{P}\left(|t_{\hat{a}_n} - t_{\tilde{a}}| > 0\right) = \lim_{n\to\infty} \mathbb{P}\left(t_{\hat{a}_n} - t_{\tilde{a}} > 0\right) = 0 . \tag{7.28}$$

Similarly, we have

$$\mathbb{P}\left(t_{\hat{b}_n} < t_{\tilde{b}}\right) \leq \left(1 - \frac{F(t_{\tilde{b}}) - F(t_{\tilde{b}-1})}{1-p}\right)^n , \tag{7.29}$$

and

$$\lim_{n\to\infty} \mathbb{P}\left(|t_{\hat{b}_n} - t_{\tilde{b}}| > 0\right) = \lim_{n\to\infty} \mathbb{P}\left(t_{\tilde{b}} - t_{\hat{b}_n} > 0\right) = 0 . \tag{7.30}$$

Moreover, convergence in probability of a monotone sequence implies the convergence with probability 1 [51, Lemma 3.2], so we conclude

$$t_{\hat{a}_n} \xrightarrow{a.s.} t_{\tilde{a}} \qquad t_{\hat{b}_n} \xrightarrow{a.s.} t_{\tilde{b}} . \tag{7.31}$$

□

### 7.4.2 Asymptotic distribution

*Proof of Theorem 17.* By Proposition 2 we know that $\{t_i\}_{i=\hat{a}_n}^{\hat{b}_n}$ converges almost surely to a fixed collection of times in $[t_a, t_b]$. That is, the vector

$$\hat{\boldsymbol{q}}_n = \left[\hat{q}_n(t_{\hat{a}_n}), \hat{q}_n(t_{\hat{a}_n+1}), \ldots, \hat{q}_n(t_{\hat{b}_n})\right]$$

converges almost surely to the same limit as

$$\left[\hat{q}_n(t_{\tilde{a}}), \hat{q}_n(t_{\tilde{a}+1}), \ldots, \hat{q}_n(t_{\tilde{b}})\right] .$$

Moreover, $\hat{q}_n(t)$ for any $t \in [t_{\tilde{a}}, t_{\tilde{b}}]$ satisfies the Central Limit Theorem because it is an average of independent and identically distributed observations with a known covariance matrix. Hence, letting

$$\boldsymbol{q} = \left[\mathbb{E}[Q(t_{\tilde{a}})], \mathbb{E}[Q(t_{\tilde{a}+1})], \ldots, \mathbb{E}[Q(t_{\tilde{b}})]\right] ,$$

we have

$$\sqrt{n}(\hat{\boldsymbol{q}}_n - \boldsymbol{q}) \xrightarrow{d} \mathcal{N}(0, \Sigma),$$

where a size of $(\tilde{b} - \tilde{a} + 1) \times (\tilde{b} - \tilde{a} + 1)$ matrix $\Sigma$ is given by $\Sigma_{ii} = v(t_{\tilde{a}+i-1})$ and $\Sigma_{ij} = \rho\left(t_{\tilde{a}+i-1}, t_{\tilde{a}+j-1}\right)$, $1 \le i, j \le \tilde{b} - \tilde{a} + 1$.

Next, by (7.17) the estimator can be written as the linear combination

$$\hat{\theta}_n = -\sum_{i=\hat{a}_n}^{\hat{b}_n} \frac{\hat{q}_n(t_i) - \hat{q}_n(d(t_i))}{k_i} = \sum_{i=\hat{a}_n}^{\hat{b}_n} g_i \hat{q}_n(t_i),$$

where $k_i$ and $g_i$ are as defined in Theorem 17. Again, Proposition 2 implies $\{t_i\}_{i=\hat{a}_n}^{\hat{b}_n}$ converges almost surely to $\{t_{\tilde{a}}, t_{\tilde{a}+1}, \ldots, t_{\tilde{b}}\}$. Then the delta method [83, Chapter 3] can be applied to conclude that

$$\sqrt{n}\left(\hat{\theta}_n - \theta\right) \xrightarrow{d} \mathcal{N}\left(0, \boldsymbol{g}\,\Sigma\,\boldsymbol{g}^T\right)$$

where $\boldsymbol{g} = [g_{\tilde{a}}, g_{\tilde{a}+1}, \ldots, g_{\tilde{b}}]$.

### 7.4.3 The variance computation

As $n \to \infty$, $\sqrt{n}\left(\hat{\theta}_n - \theta\right)$ converges to a zero-mean normal random variable with variance

$$\sum_{i=\tilde{a}}^{\tilde{b}} g_i^2 v(t_i) + 2 \sum_{i=\tilde{a}}^{\tilde{b}} \sum_{j>i}^{\tilde{b}} g_i g_j \rho(t_i, t_j). \tag{7.32}$$

The variance can be approximated numerically by a discrete approximation. Observe that the unknown components in (7.32) are $v(t_i)$ and $\rho(t_i, t_j)$. Following the scheme of Section 7.2.2, given $F_e$, we can calculate the queue size dynamics $P_k(r\delta)$, $0 \le k \le K$ for each time slot $r \in \{0, 1, 2, \ldots\}$. With $P_k(r\delta)$, $0 \le k \le K$, we can approximate both the first moment and the second moment of the queue size at $r\delta$ by $\sum_{k=0}^{K} k\, P_k(r\delta)$ and $\sum_{k=0}^{K} k^2\, P_k(r\delta)$, respectively. Then $v(t_i)$ can be obtained by $\sum_{k=0}^{K} k^2\, P_k(r\delta) - (\sum_{k=0}^{K} k\, P_k(r\delta))^2$.

The calculation of $\rho(t_i, t_j)$ is a bit more involved. From its definition,

$$\rho(t_i, t_j) = \mathbb{E}\left[Q(t_i)\, Q(t_j)\right] - \mathbb{E}\left[Q(t_i)\right] \mathbb{E}\left[Q(t_j)\right] \tag{7.33}$$

where the first term can be written as

$$\mathbb{E}\left[Q(t_i)\,Q(t_i)\right] \approx \sum_{1\le k,l\le K} k\,l\,\mathbb{P}\left[Q(t_j)=l\,Q(t_i)=k\right]$$
$$= \sum_{1\le k,l\le K} k\,l\,\mathbb{P}\left[Q(t_j)=l \mid Q(t_i)=k\right]\,\mathbb{P}\left[Q(t_i)=k\right]. \tag{7.34}$$

The reason to write $\mathbb{E}\left[Q(t_i)\,Q(t_i)\right]$ by its conditional probability is that the joint probability cannot be approximated directly. The conditional probability $\mathbb{P}\left[Q(t_j)=l \mid Q(t_i)=k\right]$ can be obtained by calculating $P_l(t_j)$, given $P_k(t_i)=1$ and $F_e$, which indicates that we also need to calculate $P_k(t_i+r\delta)$, $1\le k\le K$ for $r=1,2,\ldots$ until $r\delta > t_j$.

## 7.5 Numerical examples

In this section we show how the estimator performs with $\kappa = 20$ simulation examples. We denote by $\hat{\theta}_n^{(k)}$ the estimate of the $k$th simulation, and compare the estimates obtained with the same number of observation points $m$ but with different sample sizes $n$. We also compare the estimates with the same sample size but different number of observation points each day, and the results imply that the estimator is robust to the number of observation points.

The model we focus on is without early birds, whose service closing time $T=\infty$ theoretically. However, to make the simulations feasible, we cannot simulate the process tover an infinite time. Instead, we simulate the process until we think it is long enough, which means before the simulation stopping time, denoted by $T_s$, there should be a time interval that does not have any arrival for all the samples. Indeed, if $T$ is greater than $t_b$, then the strategic behavior of the customers is not affected by different values of $T$. The simulation time we use for the analysis is $T_s=20$, since we know from Figure 7.1 the system is empty at $t_b=20$. But in reality, we do not know the values of $\alpha$ and $\beta$, thus choosing the simulation time is a trial-and-error process.

Similar to the previous section, we run simulations from $F_e$ by discrete approximation. The sample is then $\{r_i\delta,\ \Xi_i^{(l)}\}_{i=1}^m$. We first run 20 simulations with $\alpha=2, \beta=0.2$, which means that $\theta\approx 0.091$, and $\lambda=5, \mu=1$ $m=T_s/\delta=20000$ for $n=1000,500,100,50$

Table 7.1: Estimation of $\theta(\approx 0.091)$ in 20 simulations for different $n$.

| Simulation $k$ | Estimation of $\theta$ for different sample sizes $n$ ($m = 20000$) | | | |
|---|---|---|---|---|
| | $n = 1000$ | $n = 500$ | $n = 100$ | $n = 50$ |
| 1 | 0.0911 | 0.0874 | 0.0840 | 0.0974 |
| 2 | 0.0885 | 0.0873 | 0.0672 | 0.0667 |
| 3 | 0.0942 | 0.0912 | 0.1124 | 0.0963 |
| 4 | 0.0892 | 0.0956 | 0.0852 | 0.1075 |
| 5 | 0.0930 | 0.0900 | 0.0934 | 0.1111 |
| 6 | 0.0880 | 0.0918 | 0.0880 | 0.1125 |
| 7 | 0.0917 | 0.0882 | 0.0896 | 0.1155 |
| 8 | 0.0912 | 0.0885 | 0.0967 | 0.0764 |
| 9 | 0.0882 | 0.0849 | 0.0898 | 0.0694 |
| 10 | 0.0926 | 0.0912 | 0.0761 | 0.0914 |
| 11 | 0.0860 | 0.0889 | 0.0930 | 0.0703 |
| 12 | 0.0951 | 0.0925 | 0.0755 | 0.0755 |
| 13 | 0.0916 | 0.0832 | 0.0787 | 0.1013 |
| 14 | 0.0928 | 0.0967 | 0.1051 | 0.0961 |
| 15 | 0.0861 | 0.0860 | 0.0824 | 0.1232 |
| 16 | 0.0958 | 0.0897 | 0.1064 | 0.0710 |
| 17 | 0.0895 | 0.0827 | 0.0856 | 0.0817 |
| 18 | 0.0956 | 0.0992 | 0.0913 | 0.1110 |
| 19 | 0.0999 | 0.0991 | 0.1112 | 0.1228 |
| 20 | 0.0952 | 0.0945 | 0.1241 | 0.1185 |
| AE | 0.0918 | 0.0904 | 0.0918 | 0.0958 |
| STD | 0.0036 | 0.0048 | 0.0141 | 0.0193 |
| MSE | $1.29\,10^{-5}$ | $2.20\,10^{-5}$ | $1.91\,10^{-4}$ | $3.79\,10^{-4}$ |

and present the estimates $\{\hat{\theta}_n^{(k)}\}_{k=1}^{20}$ in Table 7.1 for each $n$. In our performance analysis, we computed the average of the estimates (AE), the standard deviation of the estimates (STD), and the $\mathcal{L}_2$ distance (MSE) between $\theta$ and its estimate, which are defined as

$$AE = \frac{1}{\kappa}\sum_{k=1}^{\kappa}\hat{\theta}_n^{(k)}, \qquad STD = \sqrt{\frac{1}{\kappa-1}\sum_{k=1}^{\kappa}\left(\hat{\theta}_n^{(k)} - AE\right)^2}, \qquad MSE = \frac{1}{\kappa}\sum_{k=1}^{\kappa}\left(\hat{\theta}_n^{(k)} - \theta\right)^2.$$

It can be observed that although the estimator quality in terms of the mean and variance decreases with $n$, the mean does not differ too much, and it is close to the true $\theta(\approx 0.091)$, which is represented by the dotted line.

Next we assumed that the inter observation times were the same, and investigated how the value of $m$ affects the estimates. Let $\Delta$ represent the length of the inter observation interval, that is $\Delta := (r_{i+1} - r_i)\,\delta$ for any $i = 1, \ldots, m-1$. Note that when the value of $m$ or $n$ is too small, it is possible that there are less than two points in the estimated interval, and $\theta$ cannot be estimated then. Thus, we used $\eta$ to denote the number of sim-

Table 7.2: Estimation of $\theta(\approx 0.091)$ in 20 simulations for different $m$.

| | Estimation of $\theta$ for different intervals $m$ | | | |
|---|---|---|---|---|
| Simulation $k$ | $m = 20000$ ($\Delta = 0.001$) | $m = 40$ ($\Delta = 0.5$) | $m = 20$ ($\Delta = 1$) | $m = 4$ ($\Delta = 1$) |
| 1 | 0.0911 | 0.0919 | 0.0918 | 0.0976 |
| 2 | 0.0885 | 0.0876 | 0.0875 | 0.0916 |
| 3 | 0.0942 | 0.0944 | 0.0943 | 0.0958 |
| 4 | 0.0892 | 0.0894 | 0.0890 | 0.0786 |
| 5 | 0.0930 | 0.0926 | 0.0923 | 0.0934 |
| 6 | 0.0880 | 0.0883 | 0.0875 | 0.0822 |
| 7 | 0.0917 | 0.0929 | 0.0944 | 0.1022 |
| 8 | 0.0912 | 0.0901 | 0.0922 | 0.0974 |
| 9 | 0.0882 | 0.0884 | 0.0887 | 0.0906 |
| 10 | 0.0926 | 0.0935 | 0.0917 | 0.0894 |
| 11 | 0.0860 | 0.0850 | 0.0847 | 0.0864 |
| 12 | 0.0951 | 0.0950 | 0.0945 | 0.0878 |
| 13 | 0.0916 | 0.0915 | 0.0925 | 0.0890 |
| 14 | 0.0928 | 0.1002 | 0.0911 | N/A |
| 15 | 0.0861 | 0.0936 | 0.0856 | 0.0824 |
| 16 | 0.0958 | 0.0859 | 0.0958 | 0.0996 |
| 17 | 0.0895 | 0.0954 | 0.0891 | 0.0934 |
| 18 | 0.0956 | 0.0895 | 0.0949 | 0.1036 |
| 19 | 0.0999 | 0.0943 | 0.1029 | 0.1092 |
| 20 | 0.0952 | 0.0939 | 0.0940 | 0.0912 |
| AE | 0.0918 | 0.0917 | 0.0917 | 0.0927 |
| MSE | $1.29\,10^{-5}$ | $2.20\,10^{-5}$ | $1.91\,10^{-4}$ | $8.65\,10^{-3}$ |
| STD | 0.0036 | 0.0037 | 0.0041 | 0.0078 |

ulations for which $\theta$ was successfully estimated. this means that there are $20 - \eta$ out of the 20 simulations that could not give us an estimate of $\theta$. We present the results of 20 simulations with $\alpha = 2, \beta = 0.2, \lambda = 5, \mu = 1, n = 1000$ for $m = 2000, 40, 20, 4$, and show the estimates $\{\hat{\theta}_n^{(k)}\}_{k=1}^{20}$ in Table 7.2. It could be seen that only the 14th simulation was not effective, so $\eta = 19$.

In some cases sampling the queue many times during the day may be costly and it is therefore of interest to explore how many sampling instances are required to for a good estimator. The accuracy is obviously a function of the number of days sampled, and as we saw in the previous section accurate estimation is guaranteed as $n \to \infty$. To have an overview of estimates under different $(n, m)$ set, we calculated the estimates for $n = 1000, 500, 100, 50$, $m = 2000, 40, 20, 4$, and summarized the results as $AE(STD)$ in Table 7.3. In Table 7.3, $\eta$ is 20 by default, that is, if $\eta$ is not displayed, then all the 20 simulations were effective. For $m = 4$, the value of $\eta$ increased with $n$, and $\eta = 0$ when $n = 50$, which means $\theta$ could not be estimated even if we run the simulation for 20 times

Table 7.3: Estimation of $\theta(\approx 0.091)$ for different $(n, m)$ set, the results calculated from 20 simulations are summarized as $AE(STD)$.

| Sample size | $m = 2000\ (\Delta = 0.001)$ | $m = 40\ (\Delta = 0.5)$ | $m = 20\ (\Delta = 1)$ | $m = 4\ (\Delta = 5)$ |
|---|---|---|---|---|
| $n = 50$ | 0.0958<br>(0.0193) | 0.0932<br>(0.0212) | 0.0939<br>(0.0253) | N/A $\mid \eta = 0$<br>N/A |
| $n = 100$ | 0.0918<br>(0.0141) | 0.0924<br>(0.0161) | 0.0915<br>(0.0186) | 0.0980 $\mid \eta = 4$<br>(0.0067) |
| $n = 500$ | 0.0904<br>(0.0048) | 0.0902<br>(0.0053) | 0.0901<br>(0.0058) | 0.0892 $\mid \eta = 17$<br>(0.0131) |
| $n = 1000$ | 0.0918<br>(0.0036) | 0.0917<br>(0.0037) | 0.0917<br>(0.0041) | 0.0927 $\mid \eta = 19$<br>(0.0078) |

when $m = 4, n = 50$. Also note that although compared with the case $m = 4, n = 500$, the $STD$ when $m = 4, n = 100$ was smaller, the estimation accuracy was not satisfactory. The estimator quality was relatively robust to the number of observations each day. However, when the sample size was small it was better to sample the queue more often during the day to ensure an accurate estimator is obtained. The main conclusion from the simulation experiments is that in the case where we could estimate $\theta$, a few observations of the queue length every day are sufficient for efficient and accurate estimation of $\theta$.

In practice, we do not need to observe the queue size for $m$ times each day for every day. For example, we can observe it for several days with a large $m$ number of points, in order to estimate $t_a$ and $t_b$ first. Then, we only need to select more than one point (less than $m$) in $\{0\} \cup [t_{\hat{a}}, t_{\hat{b}}]$ to estimate $\theta$.

## 7.6 Extensions

The estimator in Section 7.3.4 can be easily modified to apply to the model with service closing time or with early birds. For the model with service closing time, we only discuss the model without early birds, but for other models with service closing time, the analysis is similar. When there is a service closing time $T$, the Nash equilibrium has three cases. If $T < t_a$, where $t_a$ is defined in (7.1) and (7.2), then the Nash equilibrium is to arrive at time zero with probability 1. If $t_a \leq T < t_b$, the equilibrium arrival distribution format is similar to the non closing service time case, but there does not exist a time $t_b$ that makes

$P_0(t_b) = \dfrac{\alpha}{\alpha+\beta}$. In particular, the values of $p_e$, $f_e(t)$, and $t_a$ satisfy

$$f_e(t) = \frac{\mu}{\lambda}\left(1 - P_0(t) - \theta\right) \qquad t \in [t_a, T] \tag{7.35}$$

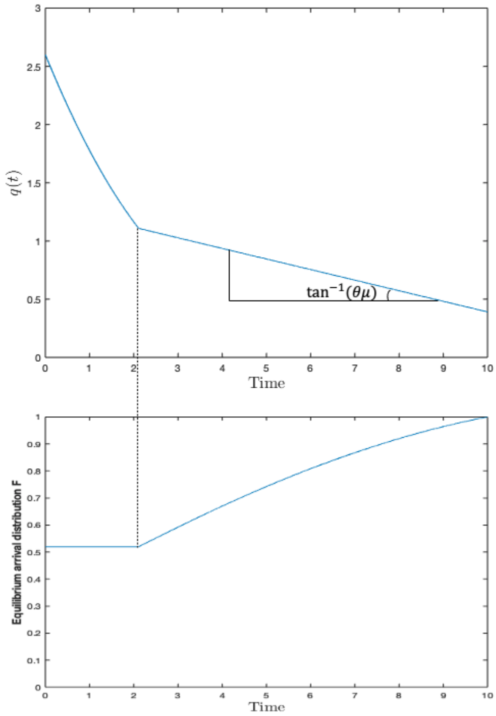
$$p_e = 1 - \int_{t=t_a}^{T} f_e(t)\, dt\,. \tag{7.36}$$

If $T \geq t_b$, the equilibrium arrival distribution is the same as the model without service closing time, so the estimator is the same. When $t_a \leq T < t_b$, although the equilibrium arrival distribution is different, the estimator is exactly the same as $\hat{\theta}_n$ proposed in Section 7.3. The only difference is we do not need to estimate $\hat{b}_n$ and the expected queue size will not drop to zero before $T$. The approximated expected queue size $q(t)$ and the Nash equilibrium $F_e$ when $t_a \leq T < t_b$, calculated using Algorithm 7 and 8, are plotted in Figure 7.7(a). It can be observed that although $q(t)$ does not drop to zero, the expected queue size is a straight line from $t_a$ to $T$. Hence, except for $T < t_a$, the estimator in (7.24) works for the model with service closing time.

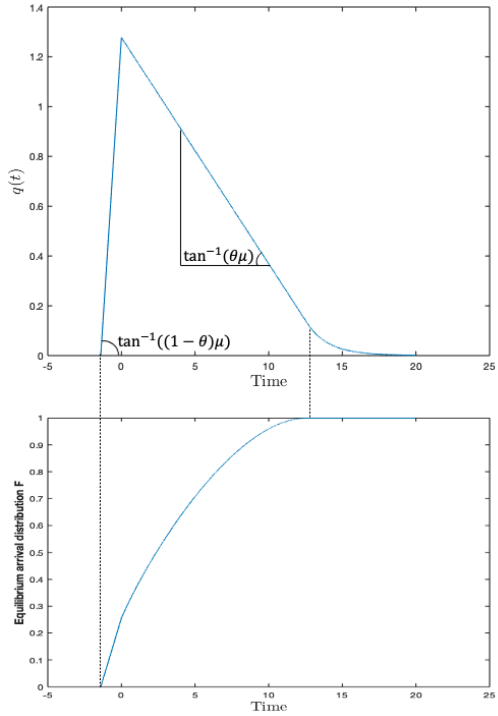
**Algorithm 8** Equilibrium arrival distribution for the model with service closing time.

**Input:** $\lambda$, $\mu$, $\alpha$, $\beta$, $\delta$, $T$
**Output:** $p_e$, $f_e$
**procedure** $f_{CT}(\lambda, \mu, \alpha, \beta, \delta, T)$
  **Initialization:** $p_1 = 0$, $p_2 = 1$, $p_e = \dfrac{p_1 + p_2}{2}$
  **while** $p_2 - p_1 > 10^{-6}$ **do**
    $P_k(0) = \left(\dfrac{(\lambda p)^k e^{-\lambda p}}{k!}\right)_{k=0:K}$
    $c = \dfrac{(\alpha+\beta)\lambda p}{2\mu}$
    **for** $r = 1 : \delta : \lceil \frac{T}{\delta} \rceil$ **do**
      calculate $P_k(r\delta)$ for $k = 0, 1, \ldots, K$ using Equations (7.11)-(7.13)
      **if** $\left(\beta\,(r\delta) + (\alpha+\beta)\sum_{k=0}^{K} \dfrac{k\,P_k(r\delta)}{\mu}\right) > c$ **then**
        $f_e(r\delta) = 0$
      **else**
        $f_e(r\delta) = \dfrac{(1 - P_0(r\delta))\mu}{\lambda} - \dfrac{\beta\mu}{(\alpha+\beta)\lambda}$
      **if** $f_e(r\delta) < 0$ **then**
        break
      $\zeta = p_e + \delta\, f\boldsymbol{e}$, where $\boldsymbol{e}$ is a vector of 1's of the appropriate size.
    **if** $\zeta >= 1$ **then**
      $p_2 = p_e$
    **else**
      $p_1 = p_e$
    $p_e = \dfrac{p_1 + p_2}{2}$

If the model is with early birds, The expected cost for the case without early birds can



(a) The model without early birds but with service closing time



(b) The model with early birds

Figure 7.7: The expected queue size when $\alpha = 2, \beta = 0.2, \lambda = 5, \mu = 1$.

then be written as

$$\mathbb{E}_F[C(t)] = \begin{cases} \dfrac{\alpha+\beta}{\mu}\mathbb{E}_F[Q(t)] - \alpha t & t < 0, \\ \dfrac{\alpha+\beta}{\mu}\mathbb{E}_F[Q(t)] + \beta t & t \geq 0\,. \end{cases} \tag{7.37}$$

Thus, the expected cost to arrive at any $t \in [-t_a, t_b]$ is the same. Hence,

$$-\alpha s + (\alpha+\beta)\frac{q(s)}{\mu} = \beta t + (\alpha+\beta)\frac{q(t)}{\mu} = c \qquad -t_a \leq s \leq 0 < t \leq t_b\,, \tag{7.38}$$

so

$$\theta = \begin{cases} 1 - \dfrac{q(s)-q(t)}{\mu(s-t)} & -t_a \leq s,\ t \leq 0 \\ \dfrac{s}{s-t} - \dfrac{q(s)-q(t)}{\mu(s-t)} & -t_a \leq s \leq 0 < t \leq t_b \\ -\dfrac{q(s)-q(t)}{\mu(s-t)} & 0 \leq s < t \leq t_b\,. \end{cases} \tag{7.39}$$

That is $\theta$ can be expressed as a function of the expected queue size in $[-t_a, t_b]$. Thus following the same estimation method as in Section 7.3, the estimator of $\theta$ can be constructed based on the sample mean of the observed queue lengths in an estimated support. The expected queue size $q(t)$ and the Nash equilibrium $F_e$ calculated using Algorithm 7 and 9, are plotted in Figure 7.7(b). It can be inferred from Equation (7.39) that the expected queue size when $t \in [-t_a, 0]$ and $t \in [0, t_b]$ should form straight lines with slope $(1-\theta)\mu$ and $-\theta\mu$, respectively.

**Algorithm 9** Equilibrium arrival distribution for the model with early birds.

**Input:** $\lambda$, $\mu, \alpha$, $\beta$, $\delta$, $T$
**Output:** $t_a$, $f_e$
**procedure** $f_{\mathcal{W}}(\lambda, \mu, \alpha, \beta, \delta)$
  **Initialization:** $t_1 = 0$, $t_2 = \dfrac{\lambda(\alpha+\beta)}{\mu\alpha}$, $t_a = \dfrac{t_1 + t_2}{2}$
  **while** $t_2 - t_1 > 10^{-6}$ **do**
    $r = 1$, $p_e = t_a \dfrac{\mu\,\alpha}{\lambda(\alpha+\beta)}$, $P_k(r\delta) = \left(\dfrac{(\lambda p)^k\, e^{-\lambda p}}{k!}\right)_{k=0:K}$
    $c = \alpha t_a$
    **do**
      calculate $P_k(r\delta)$ for $k = 0, 1, \ldots, K$ using Equations (7.11)-(7.13)
      $f_e(r\delta) = \dfrac{(1 - P_0(r\delta))\mu}{\lambda} - \dfrac{\beta\mu}{(\alpha+\beta)\lambda}$
      $r = r + 1$;
      $\zeta = p_e + \delta\, f\boldsymbol{e}$, where $\boldsymbol{e}$ is a vector of 1's of the appropriate size.
    **while** $\zeta < 1$ and $\min f \geq 0$
    **if** $F >= 1$, $\min f >= 0$ **then**
      $t_2 = t_a$
    **else**
      $t_1 = t_a$
    $t_a = \dfrac{t_1 + t_2}{2}$

Our methodology can be extended almost directly to other model variations, such as the model with order penalties in Ravner [74] and the model with earliness costs in Sherzer and Kerner [79]. We omit the details here. Furthermore, the method presented here has potential to be applied for systems with more elaborate dynamics. For example, different service regimes such as processor sharing, or non-Markovian systems such as a G/G/1 with a general distribution for the number of customers and service times. In such cases however, the queue length observations are not sufficient and one must be able to sample the virtual workload (or waiting times) at different time instants. If estimating the workload is possible then an estimation equation similar to ours can be constructed from the equilibrium condition that the expected cost is constant throughout the support of the equilibrium arrival distribution. Note that computing the equilibrium arrival distribution in elaborate systems is typically intractable, but nevertheless the cost parameters can be estimated as long as the components of the cost function can be observed.

# Chapter 8

# Conclusions

A centerpiece of the strategic behavior analysis is the interaction of actions, which can be either conflicting or in agreement. Deriving Nash equilibria calls for writing down a participant's payoff in terms of her and others' actions first and deriving her best response for every possible action taken by others. In the feedback queue model in Chapter 5 and 6, a customer's best response is decreasing when the others increase their threshold. This situation is referred to as 'avoid the crowd' (ATC). In the ATC case, if the Nash equilibrium exists, it is unique (see Hassin and Haviv [40, p7]).

On the other hand, there are situations where an individual's payoff is better if she follows others' behavior. This situation is referred to as 'follow the crowd' (FTC), and can result in multiple Nash equilibria (see Hassin and Haviv [40, p7]). Several possibilities can explain the story behind the FTC phenomenon. One is that customers' behavior is an indicator of the service quality. We tend to choose a popular restaurant even if we may need to queue. An alternative possibility is that the resource is limited, or everyone believes it is limited. The psychology of panic purchasing during the lockdown period lies in the fear of inadequate supply later. It is also possible that everyone's payoff is increased by a group action, and the group (collective) buying and the shuttle bus example in Hassin and Haviv [40, Chapter 1] fit in this explanation.

The FTC phenomenon can lead to some interesting research topics, since it can explain the cooperative behavior. Also, there may be more than one Nash equilibrium, and they lead to different values for social welfare. Coming up with a mechanism that leads the population to behave according to the Nash equilibrium with better social welfare is a direct follow-up study. What's more, counter-intuitive observations are thoughtprovoking. There are some situations where the individuals are not apparently following FTC behavior, and it is exciting to spot FTC (for some parameter settings) in a case where intuitively, customers might be expected are supposed to behave differently.

Another topic that particularly interests me is reneging behavior. Reneging is similar to balking in that it describes an abandonment before service. Nevertheless, the reneging of a customer will affect the payoff of customers joining after her, which may lower their desire to renege. Thus, to justify customers' incentive to renege when the customers in front of them choose to leave may be the first step of this research.

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