Author/s:
Singh, Chandan

Title:
Contributions to Grothendieck–Teichmüller theory and the genus zero Kashiwara–Vergne problem

Date:
2025

Persistent Link:
https://hdl.handle.net/11343/368394

Contributions to Grothendieck–Teichmüller theory and the genus zero Kashiwara–Vergne problem

by

Chandan Singh

ORCID: 0009-0007-7057-800X

A thesis submitted in total fulfillment for the
degree of Doctor of Philosophy

in the
School of Mathematics and Statistics
Faculty of Science
THE UNIVERSITY OF MELBOURNE

April, 2026

THE UNIVERSITY OF MELBOURNE

Abstract

School of Mathematics and Statistics
Faculty of Science

Doctor of Philosophy

by Chandan Singh
ORCID: 0009-0007-7057-800X

This thesis contributes to the study of genus zero Kashiwara–Vergne problem and Grothendieck–Teichmüller groups through operadic and topological lenses. First, we use the theory of moperads, Swiss-Cheese-type operads, to reformulate the Alekseev-Enriquez-Torossian construction of Kashiwara–Vergne solutions constructed from Drinfeld associators. This provides explicit operadic formulas to generate genus zero Kashiwara–Vergne solutions. More precisely, we define a correspondence between genus zero Kashiwara–Vergne solutions and isomorphisms between the parenthesized braid moperad and the parenthesized chord diagram moperad. Moreover, we show that this moperadic interpretation naturally gives rise to a new free and transitive action of a cyclotomic Grothendieck–Teichmüller group on genus zero KV solutions. Second, we provide a new operadic model for tangles and characterize the prounipotent Grothendieck–Teichmüller groups with certain automorphisms of completed cyclic operads. As a consequence, we provide a simple proof of the formality of the cyclic framed little disk operad. We establish the operadic action of the Grothendieck–Teichmüller group on the category of quantum tangles and chord diagrams with self-dual objects.

Declaration of Authorship

I, CHANDAN SINGH, declare that this thesis titled, “Contributions to Grothendieck-Teichmüller theory and the genus zero Kashiwara-Vergne problem” and the work presented in it are my own. I confirm that:

- The thesis comprises only my original work towards the DOCTOR OF PHILOSOPHY except where indicated in the preface;
- due acknowledgement has been made in the text to all other material used; and
- the thesis is fewer than the maximum word limit in length, exclusive of tables, maps, bibliographies and appendices as approved by the Research Higher Degrees Committee.

Signed: `Chandan Singh`

Date:

Preface

The thesis is divided into two parts, with each part representing collaborative work with other authors.

- **Part I** of the thesis is the article titled "*Genus zero Kashiwara–Vergne solutions from braids*" written in collaboration with Zsuzsanna Dancso, Iva Halacheva, Guillaume Laplante-Anfossi and Marcy Robertson [DHLA+25]. The thesis author contributed in the formulation of building a moperadic approach to the Kashiwara–Vergne solutions and contributed (more than 50%) in writing and mathematical content of Section 1, Section 3 and Section 4 of the article. This article is accessible through "arXiv: 2507.16243" and has not yet been submitted for publication.
- **Part II** of the thesis is an article in preparation titled "*Grothendieck-Teichmüller group through cyclic ribbon operad and its action on tangles*" written in collaboration with Marcy Robertson. The thesis author contributed (more othan 50%) to the proof of the main results and writing. The current part contains all unpublished original results. At the time of final thesis submission, the authors decided to split the article into in two. The first article appeared as "*Grothendieck–Teichmüller Symmetries of Cyclic Operads and Tangles*" joint with Marcy Robertson [RS25]. This article is accessible through "arXiv: 2511.05911" and submitted for publication. The second article titled "*Cyclic Symmetries of Chord Diagrams*" [Sin26], accessible through "arXiv: 2604.04688", is written independently by the thesis author.

Each part is self-contained and corresponds to independent research papers. There are minor notational inconsistencies between the parts. This choice ensures each part remains internally coherent.

The author acknowledges the following key support during his enrolment at the University of Melbourne.

- The Australian Government Research Training Program (RTP) Scholarship.
- The Australian Research Council Future Fellowship FT210100256.
- The University of Melbourne's Science Abroad Travelling Scholarship.

- Indian Institute of Science–IISc for a 6 month research visit while Australian borders were closed 2021. IISc is one of the Melbourne India Postgraduate Academy partner institutions. This arrangement was made possible by UoM Chancellery Global.

Acknowledgements

This journey has been long, challenging, filled with ups and downs, and deeply rewarding. I am grateful to many people who have made this thesis possible.

I am deeply grateful to my supervisor, Marcy Robertson, for her unwavering support, thoughtful guidance, deep insights, and remarkable patience throughout this journey. Her contribution extends far beyond the development of the thesis, shaping the way I approach and think about mathematics. I am especially thankful for her generosity with her time, her mentorship, and the amazing opportunities she has provided throughout my degree.

I would like to sincerely thank Diarmuid Crowley, Zsuzsanna Dancso, Christian Haesemeyer, Iva Halacheva, Arun Ram, and Sophie Raynor for their constructive feedback, comments, insightful discussions, and continued support over the years. Special thanks goes to my friends and colleagues Olivia Borghi, Tushar Joshi, Tamara Hogan, Guillaume Laplante-Anfossi, Kurt Stoeckl, and Jinwne XU.

My heartlfelt gratitude goes to Amanda Clegg, Manish Tiwari, and Shriya Agarwal, for believing in me, encouraging me, helping me stay grounded and sane throughtout this journey.

I am deeply indebted to Mainak Mukherjee, whose early encouragement inspired me to pursue advanced mathematics, and I also thank Muthupandian Ashokkumar, Robert Johanson, James McCluskey and Kiran Martin for their kind support.

Finally, I would like to thank my family for their invaluable support in helping me pursue my dream. I am especially grateful to my brother, Gunjan, for taking care of our parents at home. My sisters Priyanka, Pooja and Soni for their constant emotional support throughout.

Last but not least, I am grateful to God for their guidance, wisdom and blessings.

Contents

Abstract i

Declaration of Authorship ii

Preface iii

Acknowledgements v

1 Introduction 1
- 1.1 Motivation . . . 1
- 1.2 Summary of Part I . . . 5
- 1.3 Summary of Part II . . . 8

Part I: Genus zero Kashiwara–Vergne solutions from braids 12

2 Genus zero Kashiwara–Vergne solutions from braids 13
- 2.0 Introduction . . . 13
- 2.1 Preliminaries . . . 17
- 2.2 Genus Zero Kashiwara-Vergne Solutions . . . 38
- 2.3 Constructing KV Solutions . . . 48
- 2.4 Symmetric KV Solutions from Parenthesized Braids . . . 61
- Appendix A Operadic Structures and Kashiwara-Vergne Solutions . . . 66
- Appendix B Operadic Cohomology . . . 76
- References . . . 77

Part II: Grothendieck–Teichmüller Symmetries of Cyclic Operads 79

3 Grothendieck-Teichmüller group through Cyclic ribbon operad and its action on Tangles 80
- 3.0 Introduction . . . 80
- 3.1 Categorical Background . . . 84
- 3.2 Operads, Cyclic operads, Props . . . 87
- 3.3 Cyclic operad of ribbon braids . . . 93
- 3.4 Cyclic operad of chord diagrams . . . 98
- 3.5 Tangles and Chord diarams . . . 101

3.6 Drinfeld's Associators 107
3.7 Grothendieck-Teichmüller group 109
3.8 Graded Grothendieck-Teichmüller group 116
Appendix A Adjunctions between operads and cyclic operads 121
References 123

Bibliography **125**

Introduction

1.1 Motivation

The two parts of the thesis are studies of the Kashiwara-Vergne problem and Grothendieck-Teichmüller groups through a (m)operadic lens with an emphasis on their topological aspects.

Operads provide a systematic study of mathematical operations with multiple inputs and a single output. Operads originated in algebraic topology in the late 1960s, introduced by Boardman and Vogt [BV73] to study homotopy-invariant structures, and independently by J.P. May [May72] to characterize iterated loop spaces. These structures quickly found broad applicability beyond their original topological context, becoming central to algebra, category theory, and mathematical physics, see for reference [KSV95], [MSS02], [Lei04], [Fre09], [LV12], [Fre17a], [Fre17b]. Informally, an operad P is a collection of mathematical objects $\mathsf{P} = \{\mathsf{P}(n)\}_{n\geq 0}$ that comes equipped with a composition map $\circ_i : \mathsf{P}(m) \times \mathsf{P}(n) \to \mathsf{P}(m+n-1)$ that satisfies certain conditions of associativity and equivariance.

Similarly, Moperads are an operad-like structure which encode both monoid-type and module-type operations. The terminology was introduced by Willwacher [Wil16] in his study of unifying several approaches to study Kontsevich's formality morphism. Informally, given an operad P, a P-moperad P^1 is a sequence of objects $\mathsf{P}^1 = \{\mathsf{P}^1(n)\}_{n\geq 0}$ that comes with a monoid product $\circ_0 : \mathsf{P}^1(m) \times \mathsf{P}^1(n) \to \mathsf{P}^1(m+n)$ and a right P-module action $\circ_i : \mathsf{P}^1(m) \times \mathsf{P}(n) \to \mathsf{P}^1(m+n-1)$ that satisfies similar conditions of associativity and equivariance as the operad.

The following two (m)operads play a significant role in the present work.

- The *operad of little 2-disks* $\mathsf{D}_2 = \{\mathsf{D}_2(n)\}_{n\geq 0}$, is a topological operad whose n-ary operations $\mathsf{D}_2(n)$ consist of the space of all embeddings of k-disjoint disks into a standard unit disk via translation and dialation. The operadic composition is given by the composition of embeddings, visualised as the insertion of one disk into another. If we modify the definition of D_2 to also allow embeddings by rotation, we call the resulting operad the *operad of framed little disks* FD_2 [Get94]. This operad is central in string topology [CV05], Goodwillie–Weiss Embedding Calculus [LW23], recognition principle and Batalin-Vilkovisky (BV) algebras [SW03] to name a few.
- The *moperad of little 2-disks* $\mathsf{D}_2^1 = \{\mathsf{D}_2^1(n)\}_{n\geq 0}$ is a D_2-moperad, whose n-ary operation $\mathsf{D}_2^1(n)$ consists of all configurations of k-disjoint embedding of little disks into a unit puncured disk such that the embeddings are compositions of translation and dialation. The space $\mathsf{D}_2^1(n)$ is homotopically equivalent to a cylinder with n marked points. The monoid product in D_2^1 is given by stacking of cylinders, and the module action is similar to the operadic composition in D_2, as the puncture remains untouched. This moperad plays a key role in defining the rational models for configuration spaces of points on surfaces [CIW19] and variations of this moperad have been extensively used in the theory of Drinfeld's associators [CiL24].

The (m)operads D_2 and D_2^1 parameterize the configuration spaces $\mathrm{Conf}_n(\mathbb{C})$ and $\mathrm{Conf}_n(\mathbb{C}^\times)$ of points in the plane $\mathbb{C}$ and in the punctured plane $\mathbb{C}^\times$, respectively. By taking their induced operadic structure on the fundamental groupoids of these topological (m)operads, we get their groupoid models : the *operad of parenthesized braids* PaB and the *moperad of parenthesized braids* PaB^1 in groupoids as in [Fre17a] and [CG20].

In [AKKN17], Alekseev, Kawazumi, Kuno and Naef reformulated the Kashiwara–Vergne (KV) problem in terms of the topology of free loop spaces on surfaces. Specifically, they introduced a family of *higher genus* KV problems, defined as formality isomorphisms for certain Lie bialgebras associated with the free loop spaces on compact, oriented surfaces of genus g with $n+1$ boundary components. These formality isomorphisms are referred to as KV solutions of type $(g, n+1)$. In this framework, the classical KV problem studied in [KV78, AET10, AT12] corresponds to the case $(0,3)$, which is associated with the free loop space on the thrice-punctured sphere.

An important feature of this framework is that KV solutions of type $(0, n+1)$ can be constructed inductively from solutions of type $(0,3)$ via a gluing procedure that resembles operadic composition. This observation was emphasized in both [AKKN17] and [AKKN18], and suggests that the family of genus zero KV solutions admits a composition structure analogous to that of an operad. In Part I of the thesis, we examine this construction in detail and a group-coloured non-symmetric operad of genus zero KV solutions.

A Drinfeld associator is a group-like element of a formal non-commuting power series $\mathbb{K}\langle\langle X, Y\rangle\rangle$ in two variables X, Y subject to satisfying some algebraic equations [Dri90]. The set of such associators, denoted by $\mathsf{Assoc}(\mathbb{K})$, is shown to be in bijection with the set of object-fixing completed operad isomorphisms $\mathsf{PaB}_{\mathbb{K}} \to \mathsf{PaCD}_{\mathbb{K}}$ [BN98, Fre17a],

$$\mathsf{Assoc}(\mathbb{K}) \cong \mathrm{Iso}^{+}(\mathsf{PaB}_{\mathbb{K}} \longrightarrow \mathsf{PaCD}_{\mathbb{K}}).$$

The existence of the genus zero KV problem from [AT12] and its explicit solution constructed in [AET10] both relies on the existence of Drinfeld associators.

The approach to KV solutions through to configuration spaces was inspired by the connection between KV solutions and Drinfeld associators. These associators, appearing in the work of Drinfeld [Dri90], were constructed using Knizhnik–Zamolodchikov (KZ) equations and encode the monodromy of flat connections on configuration spaces of points in the complex plane that satisfy key equations called *pentagon* and *hexagons*. The KV solutions (of type $(0, 3)$) can be explicitly constructed from the Drinfeld associators [AET10]. In the other way around, every KV solution gives a *KV associator*, however, these are not the same: *KV associators* and *Drinfeld associators* live in different spaces. It is still unknown if the construction can be reversed, in other words, if Drinfeld associators can be constructed from KV solutions [AT12].

Additionally, the (prounipotent) Grothendieck-Teichmüller groups GT and GRT have free and transitive actions on $\mathsf{Assoc}(\mathbb{K})$ from the left and right, respectively. GT and GRT actions commute with each other; therefore, the triple $(\mathsf{GT}, \mathsf{Assoc}(\mathbb{K}), \mathsf{GRT})$ forms a bitorser.

Work of Bar-Natan [BN98] established a one-to-one correspondence between Drinfeld associators and *homomorphic expansions* (a type of universal knot invariant) of parenthesized braids. Building on this, Bar-Natan and Dancso introduced in [BND17] the theory of *welded tangled foams*, a class of 4-dimensional knotted objects generalizing virtual tangles. They identified the solutions to the KV problem with the *homomorphism expansions* of *welded tangled foams*. Their work relied on a presentation of welded tangeld foams as a *circuit algebra*, which is a different type of operad-like structure called a prop [DHR21]. This hinted at a potential operadic approach to the KV problem which was pursued in [DHR23] and [DHR22].

Contributing to the broader goal of developing a topological and operadic framework for the Kashiwara–Vergne (KV) problem, **Part I** of the thesis focuses on the role of the *braid group with a frozen strand* B_n^1—the fundamental group of the configuration space

$\mathrm{Conf}_n(\mathbb{C}^\times)$ of n ordered points in the punctured complex plane. This group appeared in the work of Alekseev, Enriquez, and Torossian [AET10], where it was used to express the compatibility conditions between Drinfeld associators and the tangential automorphisms underlying KV solutions.

The braid groups B_n^1 admit a natural description as the morphism sets of the moperad PaB^1, a module over the operad of parenthesized braids PaB. This (m)operadic viewpoint allows us to reinterpret the equations defining KV solutions as arising from the coherence relations in the moperad structure. In particular, it provides a starting point for studying *higher genus* KV problems by constructing configuration space models and moperadic structures that generalise PaB^1 to surfaces of arbitrary genus. This perspective forms the basis for the constructions and results developed throughout **Part I**.

Inspired by Grothendieck's *Esquisse d'un programme*, Drinfeld introduced in [Dri90] a profinite group now known as the Grothendieck–Teichmüller group, designed to act on a system of profinite braid groups in a way compatible with the operadic structure of genus zero configuration spaces. This construction was motivated by Grothendieck's idea of studying the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ via its actions on fundamental groupoids of moduli spaces, and particularly through the so-called *Teichmüller tower*. Drinfeld's Grothendieck–Teichmüller group $\widehat{\mathsf{GT}}$ encodes universal symmetries of these towers, especially those arising in genus zero.

Following this foundational work, the Grothendieck–Teichmüller group and its variants have been studied extensively and now appear across many areas of mathematics. In particular, $\widehat{\mathsf{GT}}$ acts on a range of algebraic and topological structures, including the (profinite or prounipotent completions of) classical braid groups, the genus zero mapping class groups, and the fundamental groups of the projective line minus a finite set of points. These actions extend in various ways to generalised braid groups and mapping class groups of higher-genus surfaces, as well as to combinatorial structures such as Dessins d'Enfants. Moreover, $\widehat{\mathsf{GT}}$ appears in the representation theory of braided and balanced tensor categories, and in constructions related to the universal Ptolemy–Teichmüller groupoid. For an overview of these developments and a survey of the broad influence of $\widehat{\mathsf{GT}}$ theory, see [Sch97].

In the context of operads, a canonical example of a $\widehat{\mathsf{GT}}$-action is its action on the operad of parenthesized braids PaB, as described in [Fre17a]. This action extends to the operad of parenthesized ribbon braids PaRB, as shown in [BdBHR19]. The operad PaRB serves as a groupoid model for the framed little disk operad FD_2, which is known to admit a cyclic structure [Bud08]. While a complete proof of the cyclic structure on PaRB has not yet appeared in the literature, Campos, Idrissi, and Willwacher provided a partial description in [CIW19], and the existence of such a structure was anticipated by Kontsevich in [Kon17]. **Part II** of this thesis arose from an effort to rigorously establish a cyclic structure on PaRB and to explore its consequences. These include a characterization of $\widehat{\mathsf{GT}}$ in terms of automorphisms of a cyclic operad and an operadic description of its action on tangles using homotopy-theoretic methods.

1.2 Summary of Part I

Background: The Kashiwara-Vergne (KV) conjecture, formulated in 1978 by Kashiwara and Vergne, arose from efforts to better understand the Baker-Campbell-Hausdorff (BCH) formula and its implications for the theory of invariant distributions on Lie groups [KV78]. In particular, for a free lie algebra $\mathfrak{lie}_2$ on two generators x_1 and x_2, the BCH formula is given by

$$\mathfrak{bch}(x_1, x_2) := x_1 + x_2 + \frac{1}{2}[x_1, x_2] + \frac{1}{12}[x_1, [x_1, x_2]] - \frac{1}{12}[x_2, [x_1, x_2]] + \cdots ,$$

where the Lie bracket is given by $[x, y] = xy - yx$. The series $\mathfrak{bch}(x_1, x_2)$ is not always convergent, it depends on the Lie algebra. The original question was about reformulating the BCH formula as a pair of two convergent power series.

The KV conjecture remained open for several decades until it was formally resolved by Alekseev and Meinrenken in [AM06] using methods from Poisson geometry and deformation quantization. Before [AM06], the conjecture had been solved for solvable Lie algebras, $\mathfrak{sl}_2$, and quadratic Lie algebras by Alekseev and Meinrenken [KV78], Rouvière [Rou81] and Vergne [Ver99], respectively.

The KV problem has several important consequences. First, the KV problem implies the Duflo theorem, a fundamental result in representation theory, the isomorphism between the center of the universal enveloping algebra $U\mathfrak{g}$ and the ring of invariant polynomials $(S\mathfrak{g})^{\mathfrak{g}}$ for a finite-dimensional Lie algebra $\mathfrak{g}$ [Duf77]. This implies a cohomology ring isomorphism $H(\mathfrak{g}, U\mathfrak{g}) \cong H(\mathfrak{g}, S\mathfrak{g})$, an instrumental isomorphism that simplifies computational aspects of deformation quantization [Sho00, PT04]. Second, the KV problem has a topological interpretation; in particular, it is an invariant of a four-dimensional knotted object called *welded foams* in [BND17], and the Kashiwara-Vergne symmetry groups KV and KRV are isomorphic to the automorphism group of completed *circuit algebras of welded foams* and its associated graded space of *arrow diagrams*, respectively [DHR23]. More recently, an algorithmic method to calculate the KV solution one degree at a time was formulated in [DHLAR23], highlighting its deep connections with Drinfeld's associators and its computational aspects.

Main results: The KV conjecture [KV78] was formulated by Alekseev and Torrosian in terms of basis-conjugating automorphisms of free lie algebras in [AT12] as follows.

A solution to the Kashiwara-Vergne conjecture, hereafter a *KV Solution*, is a tangential automorphism $F \in \mathrm{TAut}_2$ such that

$$F(e^{x_1}e^{x_2}) = e^{x_1+x_2} \tag{KV I}$$

and satisfies a non-commutative divergence condition

$$\exists h(z) \in \mathbb{K}[[z]],\ J(F) = \mathrm{tr}(h(x_1) + h(x_2) - h(x_1 + x_2)). \tag{KV II}$$

Based on Drinfeld's observation in [Dri90] that one can produce tangential derivations of free lie algebras from the elements of the Lie algebra of infinitesimal braids, Alekseev, Enriquez and Torossian in [AET10] provided explicit KV solutions constructed from Drinfeld associators.

In **Part I**, we use an operadic gadget called a *moperad*, defined as a Swiss-Cheese type of operad in [Wil16]. It is equivalently defined as monoids in the category of right modules over a given operad. The key contribution here is to observe the *moperadic structures* in the Alekseev, Enriquez and Torossian construction [AET10].

There are four key moperads that play a fundamental role in this work:

- The moperad of parenthesized braids PaB^1 apeared in [CG20], can be seen as a groupoid model for the configuration of points in a punctured plane. The moperad PaB^1 is a specific operadic module over the operad of parenthesized braids PaB, and it is a model of the braid group, $\mathsf{B}_n^1 \cong \pi_1(\mathrm{Conf}(\mathbb{C}^\times))$. The maps out of PaB^1 are universal among braided module categories.
- The moperad of shifted parenthesized braids PaB^+ is a groupoid model for the configuration of points in a plane. The moperad PaB^+ is a specific operadic module on PaB obtained by applying a *shift functor*, the functor that shifts the operadic arity by positive one to induce the module structure.
- The moperad of chord diagrams CD^+. It is seen as a module over the operad of chord diagrams CD, which models the completed Drinfeld-Kohno Lie algebras as its morphism set. The moperad CD^+ coincides with the moperad CD^1 that appears in [CG20].

- The moperad of tangential automorphisms TAut^1. This is a moperad in prounipotent groupoids and is seen as a module over the operad of special automorphisms SAut.

We first show that any isomorphism of the completed moperads $(\varphi^1, \varphi)\colon \mathsf{PaB}^1_{\mathbb{K}} \xrightarrow{\cong} \mathsf{CD}^+$ gives a KV solution. Reinterpreting the Alekseev–Enriquez–Torossian formula, this result reveals the existence of two additional relations, called *mixed pentagon* and *octagon* deeply embedded in the construction of KV solutions from [AET10]. These additional relations appear as a variant of the family of cyclotomomic symmetries, which comes from the monodromy of flat connections on configuration space of points in a punctured complex plane $\mathbb{C}^\times$. In particular, these extra relations mirror the mixed pentagon and octagon equations of a braided module category over a braided monoidal category, see for example [Bro13].

In the reverse direction, it is not clear if a KV solution gives such a moperad isomorphism, however, we showed the existence of a moperad isomorphism from a *symmetric KV solutions*–a KV solutions equipped with natural involution. In particular, it turns out that every symmetric KV solution gives an operad morphism $\varphi : \mathsf{PaB}_{\mathbb{K}} \longrightarrow \mathrm{SAut}$ and a moperad morphism $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \longrightarrow \mathrm{TAut}^1$.

A KV associator is a special automorphism in $G \in \mathrm{TAut}_3$ that satisfies certain equations called pentagon and hexagon in SAut_4 and SAut_3 respectively, which are reminiscent of the pentagon and hexagon equations that define Drinfeld associators. The set of KV associators is not empty, which follows from the existence of Drinfeld associators. These two associators are closely related. Drinfeld associators evaluate in the Drinfeld-Kuhno Lie algebra, whereas KV associators take values in certain automorphisms of (degree-completed) free Lie algebras. It is still unknown whether all KV associators are Drinfeld associators.

However, if the KV associator is a Drinfeld associator, then the moperad map $\varphi : \mathsf{PaB}^1_{\mathbb{K}} \longrightarrow \mathrm{TAut}^+$ factors through CD^+. In this case, the known actions of the acyclotomic Grothendieck-Teichmüller group GT_1 and its graded version GRT_1 on the set of Drinfeld associators are recovered (cf. [Dri90]).

Taking a module version of the classical Grothendieck-Teichmüller group GT (and GRT) one defines the group GT^1 (and GRT^1) which satisfies all the defining relations of GT (and GRT) together with a *mixed pentagon* and *octagon*. The (prounipotent) Grothendieck-Teichmüller groups GT^1 and GRT^1 are identified with the groups

$$\mathrm{Aut}^+(\mathsf{PaB}^1_{\mathbb{K}}) \quad \text{and} \quad \mathrm{Aut}^+(\mathsf{PaCD}^1_{\mathbb{K}})$$

of automorphisms of the completed moperad of parenthesized braids and chord diagrams that act as identities on objects, respectively (e.g. [CG20]). The groups GT^1 and GRT^1 are closely related to the cyclotomic variant of the Grothendieck–Teichmüller groups [Enr07]. It is already known from [AT12, AET10] that there are free and transitive actions of the Kashiwara-Vergne symmetric groups KV and KRV from left and right respectively, on the set SolKV of KV solutions that commute with each other, thus giving a bitorser triple $(\mathsf{KV}, \mathsf{SolKV}, \mathsf{KRV})$. Through the embeddings of acyclotomic groups GT_1 into KV and GRT_1 into KV, one obtain the injection of bitorsors map

$$(\mathsf{GT}_1, \mathsf{Assoc}_1(\mathbb{K}), \mathsf{GRT}_1) \to (\mathsf{KV}, \mathsf{SolKV}, \mathsf{KRV}).$$

Moreover, if the KV associators coincide with Drinfeld associators, then these actions coincides with the actions coming from the triple $(\mathsf{GT}_1, \mathsf{Assoc}_1(\mathbb{K}), \mathsf{GRT}_1)$. Lifting of these actions in the Grothendieck-Teichmüller modules GT^1 and GRT^1, we show that the triple $(\mathsf{GT}^1_{\lambda=1}, \mathsf{Assoc}^1(\mathbb{K}), \mathsf{GRT}^1_{\lambda=1})$ forms a bitorser, here $\mathsf{Assoc}^1(\mathbb{K})$ is the set of triples $(\mu, f, g) \in \mathbb{K} \times \exp(\mathfrak{t}_3) \times \exp(\mathfrak{t}_2)$, where the pair (μ, f) satisfies deining relations (the pentagon and hexagon equations) for Drinfeld associators and the pair g satisfies two additional relations called mixed pentagon and octagon equations. The actions of GT^1 and GRT^1 intertwine with the actions of $\mathsf{KV}(2)$ and $\mathsf{KRV}(2)$.

1.3 Summary of Part II

Background: Grothendieck in his *Esquisse d'un programme* [Gro97] envisioned an approach to understand the absolute Galois group of rationals $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ through its action on the tower of moduli spaces of algebraic curves known as the Teichmüller Tower $T_{g,n}$. This is the tower of the étale fundamental groupoid of the moduli spaces $\mathcal{M}_{g,n}$ of genus g curves with n marked points. The hope was to characterize $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with the group of automorphisms of the profinite tower $\widehat{T}_{g,n}$. He suggested the existence of a universal symmetric group which encodes the hidden symmetries of the Teichmüller tower and which contains or coincides with $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Following [Gro97], Drinfeld in his celebrated paper [Dri90] formulated such a universal symmetric group, now known as the Grothendieck-Teichmüller group, in the study of associators and deformations of certain Hopf algebras. These groups come in three flavours: prounipotent GT, pro-ℓ GT_ℓ and profinite $\widehat{\mathsf{GT}}$. Moreover, he indicated and Ihara [Iha94] constructed an injection $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \widehat{\mathsf{GT}}$, and later the existence of a morphism $\widehat{\mathsf{GT}} \longrightarrow \mathrm{Aut}(T_{0,n})$ such that $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho} \widehat{\mathsf{GT}} \longrightarrow \mathrm{Aut}(T_{0,n})$ was established in [BFL+99]. It is still an open problem to show whether $\widehat{\mathsf{GT}}$ is isomorphic to $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Building on Drinfeld's work on characterizing GT as a group automorphisms of prounipotent braids, Bar-Natan and Fresse in [BN98, Fre17a] identified GT and its graded version GRT with the groups

$$\mathrm{Aut}^+(\mathsf{PaB}_{\mathbb{K}}) \quad \text{and} \quad \mathrm{Aut}^+(\mathsf{PaCD}_{\mathbb{K}})$$

of automorphisms of the completed operad of parenthesized braids and chord diagrams which acts as identities on objects respectively. The first evidence of $\widehat{\mathsf{GT}}$ action on ribbon-braid-like objects was presented by Lochak and Schneps in [LS97], in particular $\widehat{\mathsf{GT}}$ action on the elements of the automorphism group of *universal Ptolemy- Teichmüller groupoids*, closely related to the Richard Thompson's group. The universal Ptolemy-Teichmüller groupoids are too large to be identified with $\widehat{\mathsf{GT}}$. After two long decades, Boavida, Horel, and Robertson in [BdBHR19], $\widehat{\mathsf{GT}}$ was identified with the group $\mathrm{Aut}^+(\widehat{\mathsf{PaRB}})$ based to homotopy theoretic methods and similarly in the prounipotent case $\mathsf{GT} \cong \mathrm{Aut}^+(\mathsf{PaRB}_{\mathbb{K}})$.

Actions of the Grothendieck-Teichmüller group have been studied in many seemingly unrelated areas of mathematics such as quantized deformation theory for Lie bialgebras, string topology, homological stability, and rational homotopy theory. Studying GT actions on mathematical objects of topological nature, for example, topological operads, moduli spaces, TQFT's, mapping class groups, braids, ribbon-braids, knots etc., are crucial to understanding arithmatic symmetries in the forms of combinatorial and geometric data, contributing directly to the central theme of Grothendieck's *Esquisse d'un programme.*

Main results: We show that the category of framed tangles can be described as the metric prop of a cyclic operad of parenthesized ribbon braids. This operadic interpretation allows us to explicitly describe an action of GT on the category of tangles. Moreover, we show that this approach justifies the Galois actions discussed in Appendix D of [KT98].

More precisely, the operad of parenthesized ribbon braids PaRB, defined in [Wah01], is a (cofibrant) groupoid model for the operad of framed little disks FD_2. Following Campos-Idrissi-Willwacher and Müller-Woike ([CIW19], [MW22]), we provide an explicit proof that PaRB is a cyclic operad. The cyclic operad PaRB^{cyc} provides an equivalent alternative description of the category of (parenthesized) tangles $\mathbf{T}'$ with self-dual objects. Using the metric prop construction of [HV02], we obtain a *tensor category*. In particular, we establish an isomorphism of prounipotent tensor categories

$$\Pi(\widehat{\mathsf{PaRB}_{\mathbb{K}}^{cyc}}) \cong \widehat{\mathbf{T}'}_{\mathbb{K}}$$

over any field $\mathbb{K}$ of characteristic 0. We show that the cyclic structure on PaRB is preserved under the operad isomorphism $\mathsf{PaRB} \longrightarrow \mathsf{PaRB}$. This implies that the action of the Grothendieck-Teichmüller group on ribbon braids extends to its cyclic version, in other words, we show that the prounipotent GT is isomorphic to the group of homotopy automorphisms of the prounipotent completion of the cyclic operad PaRB^{cyc},

$$\mathsf{GT} \cong \mathrm{HoAut}(\widehat{\mathsf{PaRB}}^{cyc}_{\mathbb{K}}).$$

This verifies Kontsevich's suggestion on the cyclicity of the framed Kimura-Stasheff-Voronov moduli space [Kon17, Section 2.4]. This theorem has several immediate consequences. For example, we exploit the GT action to provide a new proof of the formality of the cyclic framed little disks operad FD^{cyc} over rationals (e.g. [CIW19, Theorem 3.1], [GS12]). The action of GT on Turaev's category of tangles $\widehat{\mathbf{T}}_{\mathbb{K}}$ is then immediate after passing it through the metric prop of PaRB^{cyc}.

The above result holds over the field $\mathbb{Q}_\ell$ of ℓ-adic completion of $\mathbb{Q}$, for any prime ℓ. This implies an action of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ on the prounipotent completion of the category of tangles $\widehat{\mathbf{T}}_{\mathbb{Q}_\ell}$ that factors through $\mathsf{GT}(\mathbb{Q}_\ell)$ via Ihara's group morphism $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathsf{GT}(\mathbb{Q}_\ell)$ from [Iha94]. The Galois action obtained via Ihara's description can be shown to pass through our operadic machinery to the category of tangles. Moreover, this action agrees with the action of the absolute Galois group suggested by Kassel-Turaev in [KT98, Appendix D].

Similarly to the cylic characterization of GT, we show that its graded version GRT is isomorphic to the group $\mathrm{Aut}^+(\mathsf{PaRCD}^{cyc}_{\mathbb{K}})$ using the cyclic structure of the Drinfeld-Kuhno Lie algebra suggested in [CIW19, Wil24]. Drinfeld associators are universal solutions to the associativity constraints of braided monoidal categories. They are group-like elements of the (non-commuting) formal power series $\mathbb{K}\langle\langle X, Y\rangle\rangle$ in two variables X, Y subject to satisfying some algebraic equations called pentagon and hexagons [Dri90]. We establish a bijection between the set of Drinfeld associators and the set $\mathrm{Iso}^+(\mathsf{PaRB}^{cyc}_{\mathbb{K}} \longrightarrow \mathsf{PaRCD}^{cyc}_{\mathbb{K}})$ of object-fixing isomorphisms of completed cyclic operads. In summary, we show

$$\mathsf{GRT} \cong \mathrm{Aut}^+(\mathsf{PaRCD}^{cyc}_{\mathbb{K}}), \quad \text{and} \quad \mathsf{Assoc}(\mathbb{K}) \leftrightarrow \mathrm{Iso}^+(\mathsf{PaRB}^{cyc}_{\mathbb{K}} \longrightarrow \mathsf{PaRCD}^{cyc}_{\mathbb{K}}).$$

As a consequence, the triple

$$(\mathrm{Aut}^+(\mathsf{PaRB}^{cyc}_{\mathbb{K}}), \mathrm{Iso}^+(\mathsf{PaRB}^{cyc}_{\mathbb{K}} \longrightarrow \mathsf{PaRCD}^{cyc}_{\mathbb{K}}), \mathrm{Aut}^+(\mathsf{PaRCD}^{cyc}_{\mathbb{K}}))$$

forms a bitorser with a left $\mathrm{Aut}^+(\mathsf{PaRB}_{\mathbb{K}}^{cyc})$ action and a right $\mathrm{Aut}^+(\mathsf{PaRCD}_{\mathbb{K}}^{cyc})$ action. Moreover, the above identifications give a bijection of bitorsers

$$(\mathsf{GT}, \mathsf{Assoc}(\mathbb{K}), \mathsf{GRT}) \leftrightarrow (\mathrm{Aut}^+(\mathsf{PaRB}_{\mathbb{K}}^{cyc}), \mathrm{Iso}^+(\mathsf{PaRB}_{\mathbb{K}}^{cyc} \longrightarrow \mathsf{PaRCD}_{\mathbb{K}}^{cyc}), \mathrm{Aut}^+(\mathsf{PaRCD}_{\mathbb{K}}^{cyc})).$$

Inspired by [KT98], Furusho studied Galois actions on knots and tangles from a knot-theoretic lens, in particular through his ABC construction, in [Fur17, Fur20], however, this work builds an operadic machinery that gives an operadic approach of the Galois actions on tangles.

Part I

Genus zero Kashiwara–Vergne solutions from braids

GENUS ZERO KASHIWARA–VERGNE SOLUTIONS FROM BRAIDS

ZSUZSANNA DANCSO, IVA HALACHEVA, GUILLAUME LAPLANTE-ANFOSSI, MARCY ROBERTSON,
AND CHANDAN SINGH

Abstract. Using the language of moperads —monoids in the category of right modules over an operad— we reinterpret the Alekseev–Enriquez–Torossian construction of Kashiwara–Vergne (KV) solutions from associators. We show that any isomorphism between the moperad of parenthesized braids with a frozen strand and the moperad of chord diagrams gives rise to a family of genus zero KV solutions operadically generated by a single classical KV solution. We show that the Grothendieck—Teichmüller module groups act on the latter, intertwining the actions of the KV symmetry groups. In the other direction, we show that any symmetric KV solution gives rise to a module map from parenthesized braids with a frozen strand to tangential automorphisms of free Lie algebras. This map factors through the moperad of chord diagrams if and only if the associated KV associator is a Drinfeld associator.

Introduction

The Kashiwara–Vergne (KV) equations were posed in [KV78] as part of an effort to use the combinatorial and structural properties of the Baker–Campbell–Hausdorff (BCH) formula to give an algebraic proof of the Duflo isomorphism [Duf77]. Informally, a solution to the KV equations consists of a pair of Lie series satisfying compatibility and divergence conditions that, if fulfilled, yield the desired algebraic framework; the *KV conjecture* asserted that such solutions exist. A general solution to the equations was eventually constructed by Alekseev and Meinrenken [AM06], building on prior work by Torossian [Tor02] using configuration space integrals defining Kontsevich's formality morphism [Kon99, Kon03].

In [AT12], Alekseev and Torossian reformulated the KV equations in terms of *tangential automorphisms*, which are automorphisms of the prounipotent group $\exp(\mathfrak{lie}_n)$ associated to the completed free Lie algebra on n generators, $\mathfrak{lie}_n$. Tangential automorphisms act by conjugating each free generator by – potentially different – elements in $\exp(\mathfrak{lie}_n)$ (Definition 3.2). Tangential automorphisms of $\mathfrak{lie}_n$ form a group TAut_n. In this formulation, a solution to the KV equations is a tangential automorphism

$$F : \exp(\mathfrak{lie}_2) \to \exp(\mathfrak{lie}_2)$$

satisfying two identities which rectify the failure of associativity and commutativity of the exponential map at the Lie algebra level. Recall that the product of exponentials satisfies

$$e^{x_1}e^{x_2} = e^{\mathfrak{bch}(x_1,x_2)},$$

where $\mathfrak{bch}(x_1,x_2) := x_1+x_2+\frac{1}{2}[x_1,x_2]+\frac{1}{12}[x_1,[x_1,x_2]]-\frac{1}{12}[x_2,[x_1,x_2]]+\cdots$ is the Baker–Campbell–Hausdorff (BCH) series in $\mathfrak{lie}_2$. The first KV equation (SolKVI) requires the tangential automorphism $F \in \mathrm{TAut}_2$ to satisfy $F(e^{x_1}e^{x_2}) = e^{x_1+x_2}$. For more detail and the second KV equation (SolKVII) see Section 3.

The associativity property of the Baker–Campbell–Hausdorff series gives rise to coherence relations that constrain how KV solutions behave under nested compositions. In [AT12], Alekseev and Torossian introduce the class of *KV associators* – tangential automorphisms $G : \exp(\mathfrak{lie}_3) \to \exp(\mathfrak{lie}_3)$ of the form

$$G = F^{1,23}F^{2,3}(F^{12,3}F^{1,2})^{-1},$$

which encode the behavior of a KV solution F under nested applications of the BCH formula (Definition 3.33). The KV associators satisfy a pentagon identity that reflects the associativity of repeated applications of the

---

IH was supported by the National Science Foundation Grant No. DMS-2302664. GLA, MR and CS were supported by the Australian Research Council Future Fellowship FT210100256. GLA was supported by the Andrew Sisson Fund, Novo Nordisk Foundation grant NNF20OC0066298, Villum Fonden Villum Investigator grant 37814, and Danish National Research Foundation grant DNRF157. CS acknowledges the support of an Australian Government Research Training Program (RTP) Scholarship.

BCH formula, together with a pair of hexagon equations encoding symmetry under cyclic permutation of inputs. These identities mirror those satisfied by Drinfeld associators (Definition 1.21). Indeed, Alekseev and Torossian [AT12] showed that every Drinfeld associator gives rise to a KV solution and corresponding KV associator. Later, Alekseev, Enriquez and Torossian [AET10] gave a now-famous explicit formula constructing KV solutions from Drinfeld associators. It remains an open question – known as the Alekseev-Torossian conjecture [AT12] – whether every KV associator arises from a Drinfeld associator.

Expansions. Both Drinfeld associators and Kashiwara–Vergne solutions correspond to specific kind of topological invariants, called *homomorphic expansions*, for different knotted structures. Let us first give a loose introduction to homomorphic expansions.

Homomorphic expansions are particularly useful when defined on a finitely presented algebraic structure $\mathcal{T}$, typically of topological nature, such as the pure braid group on n strands, or the operad of parenthesized braids. We allow formal linear combinations over a field $\mathbb{K}$ of characteristic 0, and denote the resulting linear extension by $\mathbb{K}\mathcal{T}$. The *augmentation ideal* $\mathbb{K}\mathcal{T} \supseteq I = \{\sum c_i T_i : c_i \in \mathbb{K}, T_i \in \mathcal{T}, \sum c_i = 0\}$ gives rise to a decreasing filtration on $\mathbb{K}\mathcal{T}$, called the *I-adic* of *pro-unipotent* filtration:

$$\mathbb{K}\mathcal{T} = I^0 \supseteq I^1 \supseteq I^2 \supseteq \cdots$$

We denote by $\mathcal{A}$ the degree completed associated graded structure with respect to this filtration:

$$\mathcal{A} = \Pi_{n=0}^{\infty} I^n / I^{n+1}.$$

A *homomorphic expansion* is an isomorphism between the I-adic completion $\widehat{\mathbb{K}\mathcal{T}} = \mathcal{T}_{\mathbb{K}}$, and the associated graded structure $\mathcal{A}$ (see Section 1.3). We note that in [BND17] a homomorphic expansion is defined to be a filtered homomorphism $Z : \mathbb{K}\mathcal{T} \to \mathcal{A}$ with the property that the associated graded map of Z is the identity on $\mathcal{A}$. Such a map induces a uniue isomorphism on the I-adic completion.

The *expansion problem* for $\mathcal{T}$ is the problem of finding a homomorphic expansion for $\mathbb{K}\mathcal{T}$. The papers [AKKN18a, AKKN20] refer to expansion problems as *formality problems*, as homomorphic expansions are related to, although weaker than formality isomorphisms in rational homotopy theory. Expansion problems are often difficult, and homomorphic expansions do not always exist. For a finitely presented structure $\mathcal{T}$, the search for a homomorphic expansion comes down to solving a finite set of equations in $\mathcal{A}$, in order to determine the Z-values of the generators of $\mathcal{T}$. Often, these equations turn out to be independently interesting.

The prototype for a topological expansion problem is the expansion problem for the *operad of parenthesized braids*. The set of homomorphic expansions

$$\left\{ \varphi \colon \mathsf{PaB}_{\mathbb{K}} \xrightarrow{\cong} \mathsf{PaCD} \right\}$$

between the completed operad of parenthesized braids $\mathsf{PaB}_{\mathbb{K}}$, and its associated graded operad, the operad of parenthesized chord diagrams PaCD, is in one-to-one correspondence with *Drinfeld associators* [BN98, Fre17, CRiL25]. Drinfeld associators (Definition 1.21) are solutions of the pentagon and hexagon equations in the exponentiated Drinfeld–Kohno Lie algebras [BN98, Fre17, Tam03, CRiL25], and they are of independent interest in quantum algebra as they encode the coherence conditions for braided monoidal categories.

Solutions to the Kashiwara–Vergne equations also correspond to homomorphic expansions of multiple topological objects: a class of knotted surfaces in four-dimensions called *welded foams* [BND17], and the Goldman–Turaev Lie bialgebra of loops in a punctured disc [AKKN18b]. However, the key construction of KV solutions from Drinfeld associators [AET10] is not phrased in the language of expansions. In fact, it is an interesting question whether it constructs a homomorphic expansion, and if so, for what structure. The motivating goal of this paper is to answer this question by uncovering the relevant structure – the *moperad* of braids with a frozen strand – and reinterpreting the Alekseev–Enriquez–Torossian construction as a construction of a homomorphic expansion for this structure.

Main results. The Alekseev–Enriquez–Torossian construction builds on the correspondence between Drinfeld associators and operad homomorphic expansions $\varphi \colon \mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$, showing that the local behavior of such a homomorphic expansion φ on a family of elementary morphisms in $\mathsf{PaB}_{\mathbb{K}}$ can be used to produce a tangential automorphism F_φ, which satisfies the KV equations. We revisit this construction and show that it amounts to extending φ to a homomorphic expansion for the *moperad of parenthesized braids with a frozen strand*, PaB^1 (Definition 2.16).

Moperads, which are monoids in the category of right modules over an operad (Definition 2.4), extend the classical operadic framework by allowing for compositions with one distinguished input [Wil16, CG20, CIW24]). The moperad PaB^1 consists of groupoids $\mathsf{PaB}^1(n)$ which model the fundamental groupoids of ordered configuration spaces of n points in the punctured plane. Topologically, these configurations can be pictured as lying in a cylinder, where the puncture is a marked boundary component. The two defining operations on PaB^1 reflect this geometry :

1. a monoid structure given by composition along the frozen strand, corresponding to gluing the unmarked boundary of a cylinder into the marked boundary of the other;
2. a right PaB-module structure encoding the operadic composition of configurations away from the puncture, analogous to the classical structure encoded by the little discs operad.

A homomorphic expansion for $\mathsf{PaB}^1_{\mathbb{K}}$ is a pair (φ^1, φ), where φ is a homomorphic expansion identifying the operad $\mathsf{PaB}_{\mathbb{K}}$ with CD, and φ^1 is a φ-equivariant equivalence $\varphi^1 : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$.

It was shown by Calaque and Gonzalez [CG20, Theorem 3.4] that the moperad PaB^1 admits an explicit finite presentation. Therefore, homomorphic expansions for $\mathsf{PaB}^1_{\mathbb{K}}$ are determined by their values on the generators. It turns out that homomorphic expansions of $\mathsf{PaB}^1_{\mathbb{K}}$ are in bijection with tuples $(\mu, f, g) \in \mathbb{K} \times \exp(\mathfrak{t}_3) \times \exp(\mathfrak{t}_2^+)$, where $\mathfrak{t}_n$ denotes the Drinfeld–Kohno Lie algebra and the superscript $+$ denotes a shift, i.e. $\mathfrak{t}_n^+ \cong \mathfrak{t}_{n+1}$. The values an expansion takes on the generators of PaB^1 must satisfy the equations arising from the relations in the presentation of PaB^1: the pentagon and hexagon equations of the operad PaB ensure that the pair (μ, f) is a Drinfeld associator (Theorem 1.23), and the additional *mixed pentagon*, *right pentagon* and *octagon* equations which arise in the moperad structure of PaB^1 for the pair (f, g) (Theorem 2.17, Theorem 2.23).

Building on this characterisation, we show that any operad homomorphic expansion $\varphi : \mathsf{PaB} \to \mathsf{CD}$ has a one-parameter family of extensions to a moperad homomorphic expansion (φ^1, φ). We re-interpret the Alekseev–Enriquez–Torossian construction to show that such an extension (φ^1, φ) (with coupling constant 1) always restricts to a tangential automorphism $F_{\varphi^1} : \exp(\mathfrak{lie}_n) \to \exp(\mathfrak{lie}_n)$, which satisfies the KV equations and is symmetric, that is, invariant under the Alekseev–Torossian involution [AT12]:

Theorem (Theorem 4.11). *Any moperad homomorphic expansion $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ with $\mu = 1$ restricts to a tangential automorphism F_{φ^1} on the free group $F_2 \subseteq \mathsf{PaB}^1((0(12)), (0(12)))$, and F_{φ^1} is a symmetric KV solution.*

There is a natural shift functor $(-)^+$ which produces a moperad morphism from any operad morphism (Example 2.5). Applying this finctor to the homomorphic expansion corresponding to a Drinfeld associator $\varphi : \mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$, one obtains a homomorphic expansion of moperads $\varphi^+ : \mathsf{PaB}^+_{\mathbb{K}} \to \mathsf{CD}^+$. Precomposing φ^+ with the natural inclusion $\mathsf{PaB}^1_{\mathbb{K}} \hookrightarrow \mathsf{PaB}^+_{\mathbb{K}}$, one recovers the Alekseev–Enriquez–Torossian formula [AET10, Theorem 4] for F_{φ^1}. This highlights that incorporating the defining relations of PaB^1 as a PaB-moperad, the KV equations arise from the pentagon and hexagon equations as a natural consequence of extending an operad map to a moperad map.

The set of KV solutions admits commuting left and right actions of the Kashiwara–Vergne groups KV and KRV analogous to the commuting left and right actions of the Grothendieck–Teichmüller groups GT and GRT on the set of Drinfeld associators [Dri90]. It was shown in [AET10] that the Alekseev–Enriquez–Torossian construction intertwines these actions, in other words, constructs a map of torsors between Drinfeld associators and KV solutions. Indeed, the groups GT_1 and GRT_1 embed as subgroups into the KV symmetry groups KV and KRV, respectively. Moreover, when a KV solution F_Φ arises from an associator Φ, the actions of KV and KRV on F_Φ coincide with the actions of GT_1 and GRT_1 on Φ ([AT12, Proposition 9.13], [AET10, Theorem 9]).

In [CG20], Calaque and Gonzalez showed that the group of object-fixing automorphisms of the moperad $\mathsf{PaB}^1_{\mathbb{K}}$ is isomorphic to a cyclotomic variant of the Grothendieck–Teichmüller group, Enriquez's *Grothendieck–Teichmüller module group* denoted GTM. This group was introduced in the context of braided module categories [Enr07], which are module categories over braided monoidal categories ([Enr07], [BZBJ18], [Kol20]). It governs deformation symmetries of these module categories, analogous to how the Grothendieck–Teichmüller group GT governs deformations of braided monoidal categories themselves.

In Section 4.1 we show that suitable local data extracted from an automorphism of $\mathsf{PaB}^1_{\mathbb{K}}$ yields an element of the symmetry group KV. Concretely, this gives an injective group homomorphism

$$\mathsf{GTM}_{\lambda=1} \longrightarrow \mathsf{KV},$$

where $\mathsf{GTM}_{\lambda=1}$ is the subgroup of GTM with twist parameter $\lambda = 1$. In other words, the Grothendieck–Teichmüller module group $\mathsf{GTM}_{\lambda=1}$ acts as a group of KV symmetries. One can similarly define a right action of the graded version of the Grothendieck–Teichmüller module group $\mathsf{GRTM}_{\lambda=1}$ on KV solutions. In Section 4.1 we conclude the following.

Theorem (Theorems 4.19 and 4.21)**.** *Let* $\mathsf{GTM}_{\lambda=1}$ *and* $\mathsf{GRTM}_{\lambda=1}$ *denote the subgroups of* GTM *and* GRTM *with* $\lambda = 1$*. Then the assignment of a KV solution* F_{φ^1} *to a moperad isomorphism* $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ *with* $\mu = 1$ *intertwines the commuting left and right actions of* $\mathsf{GTM}_{\lambda=1}$ *and* $\mathsf{GRTM}_{\lambda=1}$ *with the left and right actions of the KV symmetry groups* KV *and* KRV*.*

In contrast to the case of Drinfeld associators, there are intrinsic obstacles to formulating an operadic theory of KV solutions. Nonetheless, there exist descriptions of KV-solutions as homomorphic expansions. In [BND17, DHR23] KV solutions are characterised as homomorphic expansions of a *circuit algebra* or *wheeled prop* [DHR21]. However, the underlying algebraic structures present fundamental limitations. In particular, while tangential automorphisms $F \in \mathrm{TAut}(n)$ of $\exp(\mathfrak{lie}_n)$ assemble into an operad in the category of sets (Proposition 3.12), the partial compositions are not group homomorphisms. Nevertheless, this minimal operadic structure on the collection of groups $\mathrm{TAut}(n)$ is sufficient to obtain a family of generalized KV solutions of type $(0, n+1)$, or *genus zero KV solutions*, as introduced in [AKKN18b]. This extension also provides another expansion description of KV solutions – up to some technical details – as homomorphic expansions of the Goldman-Turaev Lie bialgebras on genus zero surfaces.

The genus zero extension of the classical KV problem allows operations involving multiple inputs. In particular, a genus zero KV solution is a tangential automorphism $F\colon \exp(\mathfrak{lie}_n) \to \exp(\mathfrak{lie}_n)$ satisfying the condition

$$F(e^{x_1} \cdots e^{x_n}) = e^{x_1+\cdots+x_n},$$

as well as a generalized second KV equation. In [AKKN18a], Alekseev, Kawazumi, Kuno, and Naef construct such higher-arity solutions by inductively "gluing together" classical KV solutions. We use their construction to define a non-symmetric, group-coloured operad SolKV of genus zero KV solutions, coloured by Duflo functions (Theorem 3.38). Then, given a homomorphic expansion (φ^1, φ) of the moperad $\mathsf{PaB}^1_{\mathbb{K}}$, we identify a family of tangential automorphisms $\{F_{0w}\}$, indexed by parenthesizations w of the identity permutation $12\cdots n$. This coincides with the suboperad of SolKV generated by the binary solution F_{φ^1} (Theorem 4.14). As a result we obtain explicit formulas for genus zero KV solutions of all arities arising from a Drinfeld associator.

A class of better behaved automorphisms of the free group are the *special automorphisms*: those tangential automorphisms of $\exp(\mathfrak{lie}_n)$ which act trivially on the abelianization. Special automorphisms in particular form an operad in the category of pro-unipotent groups: that is, the restrictions of the partial compositions from TAut are group homomorphisms on SAut.

A source of special automorphisms comes from Kohno's homomorphic expansion

$$\widehat{\mathsf{PB}}_n \xrightarrow{\cong} \exp(\mathfrak{t}_n)$$

which identifies the prounipotent completion of the pure braid group with the prounipotent group corresponding to the Drinfeld–Kohno Lie algebra $\mathfrak{t}_n$ of infinitesimal braids ([Koh85]). The semi-direct product decomposition of the pure braid group $\mathsf{PB}_n \cong \mathsf{F}_{n-1} \rtimes \mathsf{PB}_{n-1}$ induces, after completion, an action of $\widehat{\mathsf{PB}}_{n-1}$

on the prounipotent free group $\widehat{\mathsf{F}}_{n-1} \cong \exp(\mathfrak{lie}_{n-1})$. This action defines an injective group homomorphism $\exp(\mathfrak{t}_{n-1}) \hookrightarrow \mathrm{SAut}_{n-1}$, as in [AET10, Section 4], [AT12, Proposition 3.11].

In Section 3.2, we define two SAut-moperads in pro-unipotent groups, TAut^1 and SAut^1, for tangential and special automorphisms, respectively. The moperad SAut^1 arises as a natural target for the embedding of the Drinfeld–Kohno operad in SAut (see Lemma 3.22). The moperad TAut^1 realises the strongest natural operadic structure on tangential automorphisms.

Finally, we study symmetric KV solutions, which are KV solutions invariant under the Alekseev–Torossian involution (Definition 3.42), and which are most closely related to Drinfeld associators. Each symmetric KV solution has a corresponding *KV associator*, which satisfies the pentagon and hexagon equations in SAut_3. If a symmetric KV solution arises from a Drinfeld associator, the corresponding KV associator lies in the image of the canonical inclusion $\exp(\mathfrak{t}_3) \hookrightarrow \mathrm{SAut}_3$, and thus recovers the original Drinfeld associator. In Section 5, we show the following.

Theorem (Theorem 5.6 and Corollary 5.7)**.** *Every symmetric KV solution F defines a map of* SAut*-modules*

$$(\varphi_F^1, \varphi_F) : \mathsf{PaB}^1_{\mathbb{K}} \longrightarrow \mathrm{TAut}^+ .$$

which factors through an equivalence $\mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ *if and only if the KV associator* $G = F^{1,23}F^{2,3}(F^{12,3}F^{1,2})^{-1}$ *is in the image of* $\exp(\mathfrak{t}_3)$.

Aknowledgements. We thank Anton Alekseev, Dror Bar-Natan, Benjamin Enriquez, Florian Naef, Muze Ren and Pavol Ševera for useful discussions. We would like to thank Najib Idrissi and Hidekazu Furusho for comments on a earlier draft. We are grateful to the Université de Genève and the SwissMap research station in Les Diablerets where part of this work was carried out. The second author is grateful to the Sydney Mathematical Research Institute for a two month research visit during which this project was initiated.

Plan of the paper. The structure of the paper is as follows. Section 1 reviews the well known example of Drinfeld associators realized as homomorphic expansions for the operad of parenthesized braids, and the Grothendieck–Teichmuller groups as symmetries of these homomorphic expansions. Section 2 introduces moperads, the moperad structure of parenthesized braids with a frozen strand, and the expansion problem for this structure. Section 3 introduces genus zero Kashiwara–Vergne solutions and the associated Lie algebras and pro-unipotent groups from an operadic perspective. In Section 4, we present the main construction of KV solutions from Drinfeld associators through the lens of moperad expansions. We establish the compatibility of these KV solutions with the actions of the Grothendieck-–Teichmüller groups. Section 5 constructs module maps on $\mathsf{PaB}^1_{\mathbb{K}}$ from KV associators, and shows that when a KV associator is in the image of $\exp(\mathfrak{t}_3)$, the corresponding module map factors through a moperad isomorphism $\mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$. Finally, two appendices provide supplementary background and technical details on operadic composition of tangential and special derivations, and cohomological computations.

Contents

Introduction 13

1. Operad expansions: the operad of parenthesised braids and Drinfeld associators 18
   1.1. Operads 18
   1.2. The operad of parenthesised braids 20
   1.3. Completion and homomorphic expansions 23
   1.4. Expansions in action: chord diagrams and Drinfeld associators 26
   1.5. The Grothendieck–Teichmüller groups 27
2. Moperad expansions: The moperad of parenthesised braids with a frozen strand 29
   2.1. Moperads 29
   2.2. Completion and homomorphic expansions for moperads 32

2.3. Parenthesized braids with a frozen strand 32
2.4. Associators and moperad expansions 38
2.5. A variation on the Grothendieck–Teichmüller groups 41
3. Genus Zero Kashiwara–Vergne Solutions 43
3.1. Operadic structures on $\mathfrak{lie}$, $\mathfrak{tDer}$, $\mathfrak{sDer}$, TAut, SAut 44
3.2. Moperad structures for $\mathfrak{sDer}$ and $\mathfrak{tDer}$ 46
3.3. Kashiwara–Vergne (KV) solutions, associators, and symmetry groups 51
4. Constructing KV Solutions 55
4.1. Actions of GTM on KV solutions 63
5. Symmetric KV Solutions from Parenthesized Braids 67
5.1. An involution on $\mathsf{SolKV}(2)$ 68
5.2. Operad and moperad maps from KV solutions 68
5.3. Actions of GTM on KV associators 71
Appendix A. Operadic Structures and Kashiwara–Vergne Solutions 73
A.1. Operadic structure on Lie algebras and $\mathfrak{tDer}$ 73
A.2. The operad of special derivations $\mathfrak{sDer}$ 74
A.3. Operadic composition of KV solutions 77
A.4. Operadic composition of the KV symmetry groups 79
Appendix B. Cosimplicial Structures and Operadic Cohomology 81
References 83

1. Operad expansions: the operad of parenthesised braids and Drinfeld associators

In this section we review the correspondence between Drinfeld associators and homomorphic expansions for the operad of parenthesised braids. In order to do so, we recall the necessary definitions of operads, the operad of parethesised braids, completions, chord diagrams, and associators. While much of this material is known in the literature, this section serves the dual purpose of setting the conceptual framework for the later sections of new content.

1.1. **Operads.**

Let $(\mathrm{C}, \otimes, I)$ be a cocomplete closed symmetric monoidal category such that the monoidal product $\otimes$ commutes with small colimits in each variable. The examples of such categories used in this paper are the categories of Lie algebras, vector spaces, Hopf algebras, (pro-unipotent) groups and groupoids, and sets.

Let $\Sigma_n := \mathrm{Aut}(\underline{n})$ denote the symmetric group on the set $\underline{n} := \{1, \ldots, n\}$. Given two permutations $\sigma \in \Sigma_m$ and $\tau \in \Sigma_n$ and $1 \leq i \leq m$, one can *insert* τ into the entry of σ labeled i (in one-line notation), and shift all the larger entries appropriately to maintain bijectivity. This defines a new permutation $\sigma \circ_i \tau \in \Sigma_{m+n-1}$. In formulas:

$$(\sigma \circ_i \tau)(k) := \begin{cases} \sigma(k) & \text{if } 1 \leqslant k < \sigma^{-1}(i),\ \sigma(k) < i, \\ \sigma(k) + n - 1 & \text{if } 1 \leqslant k < \sigma^{-1}(i),\ \sigma(k) > i, \\ \tau(k - \sigma^{-1}(i) + 1) + (i - 1) & \text{if } \sigma^{-1}(i) \leqslant k \leqslant \sigma^{-1}(i) + n - 1, \\ \sigma(k - n + 1) & \text{if } \sigma^{-1}(i) + n \leqslant k \leqslant m + n - 1,\ \sigma(k - n + 1) < i, \\ \sigma(k - n + 1) + n - 1 & \text{if } \sigma^{-1}(i) + n \leqslant k \leqslant m + n - 1,\ \sigma(k - n + 1) > i. \end{cases} \tag{1.1}$$

A *sequence* in symmetric monoidal category C is an $\mathbb{N}$-graded collection of objects in

$$\mathcal{P} = (\mathcal{P}(0), \mathcal{P}(1), \ldots, \mathcal{P}(n), \ldots).$$

A *symmetric sequence* in C is a sequence $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ in which each entry is equipped with a right action of the symmetric group

$$\mathcal{P}(n) \otimes \Sigma_n \longrightarrow \mathcal{P}(n).$$

Definition 1.1. A (symmetric) *operad* $\mathcal{P}$ in $\mathrm{C} = (\mathrm{C}, \otimes, I)$ consists of a (symmetric) sequence $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$, along with:

– A distinguished unit map $\mathrm{id} : I \to \mathcal{P}(1)$;

– A family of a partial composition maps

$$\circ_i : \mathcal{P}(n) \otimes \mathcal{P}(m) \longrightarrow \mathcal{P}(n+m-1),\ 1 \leq i \leq n.$$

These partial composition operations are required to satisfy

(i) unitality: $\mu \circ_i \mathrm{id} = \mathrm{id} \circ_1 \mu = \mu \quad$ for all $\mu \in \mathcal{P}(n)$,

(ii) associativity: For all $\lambda \in \mathcal{P}(\ell)$, $\mu \in \mathcal{P}(m)$, and $\nu \in \mathcal{P}(n)$, we have

$$(\lambda \circ_j \mu) \circ_i \nu = \begin{cases} (\lambda \circ_i \nu) \circ_{j+n-1} \mu, & \text{if } 1 \leq i \leq j-1, \\ \lambda \circ_j (\mu \circ_{i-j+1} \nu), & \text{if } j \leq i \leq j+m-1, \\ (\lambda \circ_{i-m+1} \nu) \circ_j \mu, & \text{if } j+m \leq i \leq \ell+m-1. \end{cases}$$

(iii) and, if $\mathcal{P}$ is symmetric, equivariance : $\mu^{\sigma} \circ_{\sigma^{-1}(i)} \nu^{\tau} = (\mu \circ_i \nu)^{\sigma \circ_i \tau}$ for all $\mu \in \mathcal{P}(m)$, $\nu \in \mathcal{P}(n)$, $\sigma \in \Sigma_m$ and $\tau \in \Sigma_n$.

In order to define the operadic composition on KV-solutions we require a group-coloured refinement of the usual notion of operad: operations now come equipped with input and output types, and composition is defined only when these types coincide. Let $\mathfrak{C}$ be a small groupoid (or group, or finite set), whose objects we refer to as *colours*. A *$\mathfrak{C}$-colored sequence* in a monoidal category C is a collection of objects in C, indexed by finite lists of colours $(c_1, \ldots, c_n; c_0)$ in $\mathfrak{C}$:

$$\mathcal{P} = \{\mathcal{P}(c_1, \ldots, c_n; c_0)\}_{(c_1,\ldots,c_n;c_0)\in \mathrm{Ob}(\mathfrak{C})^{n+1}}.$$

If $\mathfrak{C}$ is a groupoid containing non-trivial isomorphisms, we require contravariant functoriality in each input color and covariant functoriality in the output color. That is, for morphisms $f_i : c_i \to c_i'$ and $g : c_0 \to c_0'$ in $\mathfrak{C}$, there are structure maps

$$\mathcal{P}(c_1, \ldots, c_n; c_0) \longrightarrow \mathcal{P}(c_1', \ldots, c_n'; c_0'),$$

functorial in each variable. In the case where $\mathfrak{C}$ is a set, these functorialities are trivial. A $\mathfrak{C}$-colored *symmetric sequence* in C is a sequence $\mathcal{P}$ equipped, for each $n \geq 0$, with a right action of the symmetric group Σ_n,

$$\mathcal{P}(c_1, \ldots, c_n; c_0) \times \Sigma_n \longrightarrow \mathcal{P}(c_{\sigma(1)}, \ldots, c_{\sigma(n)}; c_0).$$

Definition 1.2. A (symmetric) *$\mathfrak{C}$-colored operad* $\mathcal{P}$ in $\mathrm{C} = (\mathrm{C}, \otimes, I)$ consists of a $\mathfrak{C}$-colored (symmetric) sequence $\mathcal{P} = \{\mathcal{P}(c_1, \ldots, c_n; c_0)\}$, along with:

– For each $c \in \mathrm{Ob}(\mathfrak{C})$, a unit map $\mathrm{id}_c : I \to \mathcal{P}(c; c)$;

– A family of a partial composition maps

$$\circ_i : \mathcal{P}(c_1, \ldots, c_n; c_0) \otimes \mathcal{P}(d_1, \ldots, d_m; c_i) \longrightarrow \mathcal{P}(c_1, \ldots, c_{i-1}, d_1, \ldots, d_m, c_{i+1}, \ldots, c_n; c_0),\ 1 \leq i \leq n,$$

which satisfy to $\mathfrak{C}$-coloured versions of the unitality, associativity and, if relevant, equivariance axioms in Definition 1.1 (cf. [BM07, Definition 1.1], [Pet13, Section 3])

Remark 1.3. In the case where the groupoid $\mathfrak{C}$ has a single object and no non-identity morphisms (i.e., $\mathfrak{C} = \{*\}$), a $\mathfrak{C}$-colored (symmetric) operad reduces to an ordinary (symmetric) operad.

A *morphism* $\varphi : \mathcal{P} \to \mathcal{Q}$ of $\mathfrak{C}$-colored (symmetric) operads in C is a morphism of $\mathfrak{C}$-colored (symmetric) sequences in C which preserves all unit and composition maps.

Example 1.4. Let G be a group, viewed as a groupoid with one object. Define a non-symmetric G-colored operad $\mathcal{P}$ in Set as follows. For a tuple $(g_1, \ldots, g_n; g_0)$ in G^{n+1}, let

$$\mathcal{P}(g_1, \ldots, g_n; g_0) := \begin{cases} \{*\}, & \text{if } g_1 \cdots g_n = g_0, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Composition is defined by substitution: if $\mu \in \mathcal{P}(g_1, \ldots, g_n; g_0)$ and $\nu \in \mathcal{P}(h_1, \ldots, h_m; g_i)$, then $\mu \circ_i \nu$ is defined and equal to the unique element of

$$\mathcal{P}(g_1, \ldots, g_{i-1}, h_1, \ldots, h_m, g_{i+1}, \ldots, g_n; g_0),$$

provided $h_1 \cdots h_m = g_i$. Unit elements are given by $\mathrm{id}_g \in \mathcal{P}(g; g)$ for all $g \in G$. This operad has no symmetric group actions, and is thus a non-symmetric operad colored by the group G.

1.2. **The operad of parenthesised braids.** In this section we recall the braid groups and their associated operadic structures. In the group setting we employ group-multiplication conventions—drawing braids bottom-to-top and writing braid words left-to-right. Upon passing to groupoids we instead use function-composition notation, with composition applied right-to-left; see Remark 1.6.

The *braid group* on n strands, denoted B_n, is the fundamental group of the space of unordered configurations of n points in the complex plane. Braids are usually drawn three-dimensional, with the vertical dimension representing the parametrization of the loop, as shown in Figure 1. The braid group has a useful finite presentation, known as the Artin presentation:

$$\mathsf{B}_n = \langle \beta_1, \ldots, \beta_{n-1} \mid \beta_i\beta_j = \beta_j\beta_i \text{ when } |i-j| \geq 2;\ \beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1} \rangle \,.$$

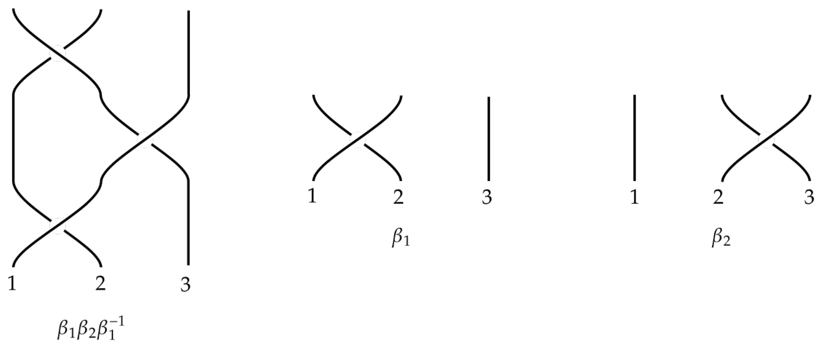


FIGURE 1. A braid in B_3, and the generators of B_3. The braid on the left is drawn bottom to top, and in terms of generators can be written as $\beta_1\beta_2\beta_1^{-1}$.

For $n \geq 1$, let $\Omega(n)$ denote the set of fully parenthesised permutations of the set $\{1, \ldots, n\}$. For example, $(1(32))$ and $((12)3)$ are elements of $\Omega(3)$. Alternatively, the set $\Omega(n)$ is in bijection with the set of planar binary rooted trees whose leaves are labeled with the set $\{1, \ldots, n\}$, see Figure 2 and [Fre17, Chapter 6.1.2].

For each $n \geq 1$, there is a natural action of the symmetric group Σ_n on $\Omega(n)$ given by the multiplication of permutations under the natural map

$$u : \Omega(n) \to \Sigma_n,$$

which forgets parentheses. For instance, $u\big((53)((42)1)\big) = (53421)$ – keeping in mind that permutations are written in one-line notation and the parentheses denote parenthetisations, not cycles. The symmetric sequence $\Omega = \{\Omega(n)\}_{n\geq 1}$ forms an operad in the category of sets with operadic composition

$$\Omega(n) \times \Omega(m) \xrightarrow{\circ_i} \Omega(n+m-1)$$

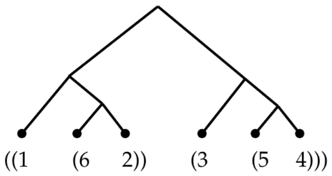


Figure 2. A parenthesised permutation represented by a planar binary rooted tree with labeled leaves.

given by inserting a parenthesised permutation $w' \in \Omega(m)$ into the entry labeled i in a parenthesised permutation $w \in \Omega(n)$. At the level of permutations insertion is defined as in (1.1), and the parenthetisation of w' is declared innermost. For example, we have $(1(32)) \circ_3 (21) = (1((43)2))$. See [Fre17, Chapter 6.1] for more details.

Definition 1.5. The *operad of parenthesised braids*, an operad in groupoids denoted PaB, is defined as follows. Let $\mathsf{PaB}(0) := \{*\}$ be the trivial groupoid. For each $n \geq 1$, the groupoid $\mathsf{PaB}(n)$ has objects $\mathrm{Ob}(\mathsf{PaB}(n)) := \Omega(n)$ and morphisms $\beta \in \mathrm{Hom}_{\mathsf{PaB}(n)}(w_1, w_2)$ given by elements of the braid group B_n whose underlying permutation is $u(w_2)^{-1}u(w_1)$. See Figure 3 for an example. The composition in the groupoid $\mathsf{PaB}(n)$ is given by the multiplication in the group B_n.

The symmetric group Σ_n acts on each groupoid $\mathsf{PaB}(n)$ via composition of permutations. The resulting symmetric sequence $\mathsf{PaB} = \{\mathsf{PaB}(n)\}_{n\geq 0}$ forms an operad in groupoids where the composition functors

$$\mathsf{PaB}(m) \times \mathsf{PaB}(n) \xrightarrow{\circ_i} \mathsf{PaB}(m+n-1)$$

are defined at the level of objects by the operadic composition of parenthesised permutations. The composition of morphisms is defined by inserting a parenthesised braid on n strands into the strand labeled i of a parenthesised braid on m strands, as shown in Figure 4. See [Fre17, Chapter 6] or [CRiL25] for more details.

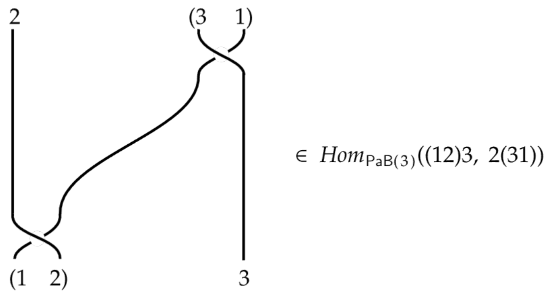


Figure 3. An example of a parenthesised braid.

Remark 1.6. A note on conventions: although composition in the groupoid PaB is defined via group multiplication, we follow the convention of writing compositions using function notation, from right to left. That is, given composable morphisms (braids) β_1 and β_2 in PaB, their composition $\beta_2 \circ \beta_1$ denotes "first apply β_1, then apply β_2," and in a diagram this would be drawn with β_1 at the bottom and β_2 at the top. In the group notation of the braid group the same composition would be written $\beta_1\beta_2$. This reversal of order is standard in categorical settings. Thus, technically, the automorphism group of a given object $w \in \Omega_n$ is PB_n^{op}.

The operad PaB admits a finite presentation with generators $R = R^{1,2} \in \mathrm{Hom}_{\mathsf{PaB}(2)}((12),(21))$ and $\Phi = \Phi^{1,2,3} \in \mathrm{Hom}_{\mathsf{PaB}(3)}((12)3, 1(23))$ shown in Figure 5. In other words, every morphism in the groupoid $\mathsf{PaB}(n)$ can be written as operadic and categorical compositions of the generating braid $R^{1,2}$, the associativity isomorphism $\Phi^{1,2,3}$, identity morphisms, and the permutations and inverses of these morphisms. Note that

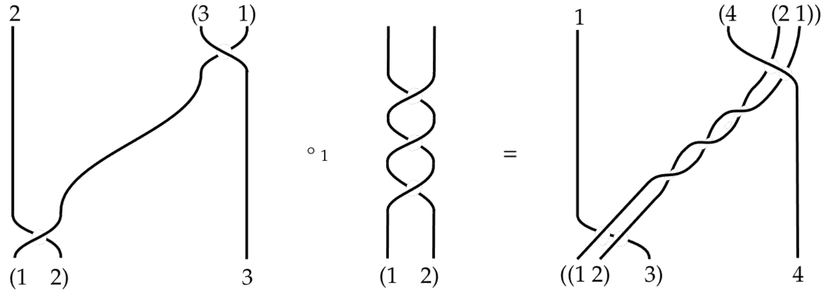


FIGURE 4. The operadic composition of parenthesised braids.

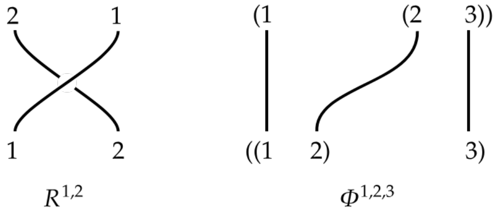


FIGURE 5. The generating morphisms of the operad PaB.

$R^{1,2}$ is the generating braid $\beta_1 \in \mathsf{B}_2$, viewed as a morphism in the groupoid $\mathsf{PaB}(2)$, and $\Phi^{1,2,3}$ is the identity braid viewed as a (non-identity) morphism $((12)3) \to (1(23))$.

Remark 1.7. Throughout, we use cosimplicial notation for the morphisms in PaB, to align with key sources and to help with the bookkeeping of permutations. In this notation, the superscript shows the indices of the strands on at the origin (domain) of a braid, omitting identity strands. In figures this means that the strands are numbered at the bottom, as shown in Figure 6. For example, the notation $R^{1,2}$ indicates, in particular, that $R^{1,2}$ is a morphism that starts from the parenthesised permutation (12); whereas $R^{2,1}$ starts from (21), and $R^{2,3} = \mathrm{id}_{(12)} \circ_2 R$ starts from 1(23). Double indices indicate doubled strands, for example, $R^{1,23} = R^{1,2} \circ_2 \mathrm{id}_{(12)}$, as in Figure 6. Including $\emptyset$ in the superscript, such as $\Phi^{\emptyset,1,2}$ denotes a composition with the trivial object $* \in \mathsf{PaB}(0)$, namely $(\Phi^{1,2,3} \circ_1 *)$. This has the effect of deleting a strand in the relevant braid. For more detail see the discussion of the unitary extension of PaB in [Fre17, Chapter 6].

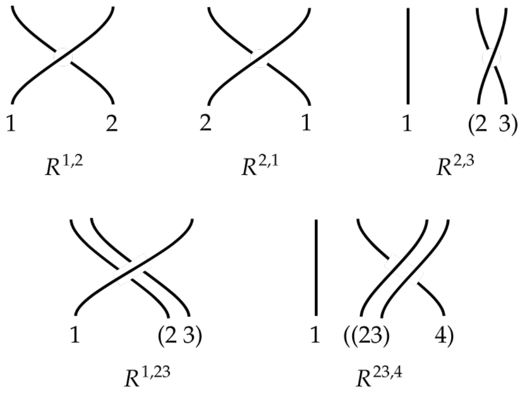


FIGURE 6. Examples of cosimplicial notation.

Now we are ready to state the finite presentation, or coherence, theorem for PaB:

Theorem 1.8 ([Fre17, Theorem 6.2.4])**.** *The parenthesised braids operad* PaB *is generated by the object* $(12) \in \mathsf{PaB}(2)$ *and the morphisms* $R := R^{1,2} \in \mathsf{PaB}(2)$ *and* $\Phi := \Phi^{1,2,3} \in \mathsf{PaB}(3)$*, subject to the following relations:*

$$\begin{aligned}
&\text{(U)} && \Phi^{\emptyset,1,2} = \Phi^{1,\emptyset,2} = \Phi^{1,2,\emptyset} = \mathrm{id}_{1,2} && \text{in } \mathrm{Hom}_{\mathsf{PaB}(2)}\big(12, 12\big)\,,\\
&\text{(P)} && \Phi^{1,2,34}\Phi^{12,3,4} = \Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3} && \text{in } \mathrm{Hom}_{\mathsf{PaB}(4)}\left(((12)3)4, 1(2(34))\right),\\
&\text{(H1)} && R^{1,3}\Phi^{2,1,3}R^{1,2} = \Phi^{2,3,1}R^{1,23}\Phi^{1,2,3} && \text{in } \mathrm{Hom}_{\mathsf{PaB}(3)}\left((12)3, 2(31)\right),\\
&\text{(H2)} && R^{1,3}(\Phi^{1,3,2})^{-1}R^{2,3} = (\Phi^{3,1,2})^{-1}R^{12,3}(\Phi^{1,2,3})^{-1} && \text{in } \mathrm{Hom}_{\mathsf{PaB}(3)}\left(1(23), (31)2\right).
\end{aligned}$$

1.3. **Completion and homomorphic expansions.** Let $\mathbb{K}$ be a field containing $\mathbb{Q}$. The *prounipotent completion* of a discrete group G, denoted $\mathsf{G}_{\mathbb{K}}$, is the universal prounipotent group over $\mathbb{K}$ equipped with a homomorphism from G. The construction of $\mathsf{G}_{\mathbb{K}}$ proceeds via the group algebra $\mathbb{K}[\mathsf{G}]$, which naturally carries the structure of a cocommutative Hopf algebra. Let $I \subset \mathbb{K}[\mathsf{G}]$ denote the augmentation ideal, that is, the kernel of the counit $\epsilon : \mathbb{K}[\mathsf{G}] \to \mathbb{K}$ given by $\epsilon(g) = 1$ for $g \in \mathsf{G}$. The *I-adic completion* of $\mathbb{K}[\mathsf{G}]$ is defined as the inverse limit

$$\mathbb{K}[\mathsf{G}]^{\wedge}_{I} := \varprojlim_{n} \mathbb{K}[\mathsf{G}]/I^{n}.$$

The prounipotent completion of G is then the prounipotent group of *group-like elements* in the completed Hopf algebra $\mathbb{K}[\mathsf{G}]^{\wedge}_{I}$:

$$\mathsf{G}_{\mathbb{K}} := \{g \in \mathbb{K}[\mathsf{G}]^{\wedge}_{I} \mid \Delta(g) = g \otimes g,\ \epsilon(g) = 1\}.$$

Every prounipotent group has an associated *Lie algebra*, in this case the lie algebra of primitive elements in $\mathbb{K}[\mathsf{G}]^{\wedge}_{I}$:

$$\mathfrak{g} := \mathrm{Prim}(\mathbb{K}[\mathsf{G}]^{\wedge}_{I}) = \{x \in \mathbb{K}[\mathsf{G}]^{\wedge}_{I} \mid \Delta(x) = x \otimes 1 + 1 \otimes x,\ \epsilon(x) = 0\}.$$

Since $\mathbb{K}[\mathsf{G}]^{\wedge}_{I}$ is complete and cocommutative, the exponential and logarithm series converge, giving mutually inverse bijections between group-like and primitive elements. Thus, every $g \in \mathsf{G}_{\mathbb{K}}$ can be uniquely written as an exponential $g = e^{x}$ for some $x \in \mathfrak{g}$.

Remark 1.9. There is an alternative, equivalent, way to define the Lie algebra over $\mathbb{K}$ associated to a group G, via limits of lower central series quotients $\mathsf{G}/\Gamma_n(\mathsf{G})$. The *lower central series* of G is defined inductively by $\Gamma_1(\mathsf{G}) := \mathsf{G}$ and $\Gamma_{n+1}(\mathsf{G}) := [\Gamma_n(\mathsf{G}), \mathsf{G}]$. Each quotient $\Gamma_n(\mathsf{G})/\Gamma_{n+1}(\mathsf{G})$ is abelian, and the associated graded object

$$\mathrm{gr}_{\Gamma}(\mathsf{G}) := \bigoplus_{n \geq 1} \Gamma_n(\mathsf{G})/\Gamma_{n+1}(\mathsf{G})$$

inherits a Lie bracket induced by the group commutator. If the group G is *formal*, one can identify this Lie algebra with the Lie algebra $\mathfrak{g} = \mathrm{Prim}(\mathbb{K}[\mathsf{G}]^{\wedge}_{I})$.

Example 1.10. Let F_n denote the free group on n generators. Its lower central series quotients $\Gamma_k(\mathsf{F}_n)/\Gamma_{k+1}(\mathsf{F}_n)$ are torsion-free, and the associated graded Lie algebra

$$\mathrm{gr}_{\Gamma}(\mathsf{F}_n) = \bigoplus_{k \geq 1} \Gamma_k(\mathsf{F}_n)/\Gamma_{k+1}(\mathsf{F}_n)$$

is canonically isomorphic to the free Lie algebra $\mathfrak{lie}_n$ on n generators, where all generators are of degree one. The prounipotent completion $(\mathsf{F}_n)_{\mathbb{K}}$ is naturally isomorphic to the prounipotent group associated to the degree-completed free Lie algebra:

$$(\mathsf{F}_n)_{\mathbb{K}} \cong \exp(\widehat{\mathfrak{lie}}_n).$$

Here, we have used the degree completion of $\mathfrak{lie}_n$, $\widehat{\mathfrak{lie}}_n$, so that the exponential map provides a bijection between $\widehat{\mathfrak{lie}}_n$ and the group-like elements of the completed Hopf algebra $\mathbb{K}[\mathsf{F}_n]^{\wedge}_{I}$. See, for example [Fre17, Proposition 8.4.1] for further details.

1.3.1. *Associated graded of a group.* Given a filtered algebra

$$A = \mathsf{F}_0 \supseteq F_1 \supseteq F_2 \supseteq \cdots ,$$

the associated graded algebra is defined as

$$\operatorname{gr} A := \prod_{n=0}^{\infty} F_n/F_{n+1},$$

with multiplication is defined by the Cauchy formula.

The powers I^n of the augmentation ideal define a complete decreasing filtration on $\mathbb{K}[\mathsf{G}]$, and the *completed associated graded algebra* is

$$\operatorname{gr}\big(\mathbb{K}[\mathsf{G}]\big) := \prod_{n\geq 0} I^n/I^{n+1},$$

with multiplication induced by the Cauchy product formula. This is a graded cocommutative Hopf algebra and we write $\mathcal{G}\big(\operatorname{gr}\mathbb{K}[\mathsf{G}]\big)$ for the associated group of group-like elements.

Definition 1.11. A *homomorphic expansion* for a group G is a filtered Hopf algebra isomorphism

$$\widehat{Z}\colon \mathbb{K}[\mathsf{G}]^{\wedge}_{I} \longrightarrow \operatorname{gr}\,\mathbb{K}[\mathsf{G}],$$

or equivalently an isomorphism of prounipotent groups

$$\widehat{Z}\colon \mathsf{G}_{\mathbb{K}} \xrightarrow{\cong} \mathcal{G}\big(\operatorname{gr}\mathbb{K}[\mathsf{G}]\big).$$

Informally, the existence of a homomorphic expansion means that the prounipotent completion of G is completely determined by the associated graded structure coming from the augmentation filtration.

Example 1.12. In Example 1.10 we described a bijection between $\widehat{\mathfrak{lie}}_n$ and the group-like elements of the completed Hopf algebra $\mathbb{K}[\mathsf{F}_n]^{\wedge}_{I}$. The graded Hopf algebra

$$\operatorname{gr}_I \mathbb{K}[\mathsf{F}_n] := \bigoplus_{k\geq 0} I^k/I^{k+1}$$

is naturally isomorphic to the universal enveloping algebra of the graded Lie algebra arising from the lower central series:

$$\operatorname{gr}_I \mathbb{K}[\mathsf{F}_n] \;\cong\; U(\operatorname{gr}_\Gamma(\mathsf{F}_n)\otimes\mathbb{K})\,.$$

Under this isomorphism, the degree 1 part I/I^2 is identified with the abelianisation $\mathsf{F}_n/\Gamma_2(\mathsf{F}_n)$, and the Lie bracket induced from commutators in F_n matches the bracket in $\operatorname{gr}_\Gamma(\mathsf{F}_n)$. In particular, the group-like elements of $\operatorname{gr}_I \mathbb{K}[F_n]$ are identified with the elements of the completed free Lie algebra via the exponential series. Combining this with the description of the prounipotent completion leads to an isomorphism

$$(\mathsf{F}_n)_{\mathbb{K}} \;\xrightarrow{\cong}\; \mathcal{G}(\operatorname{gr}_I \mathbb{K}[\mathsf{F}_n])\,,$$

showing that free groups admit homomorphic expansions in the sense of Definition 1.11.

1.3.2. *Prounipotent completion of groupoids and operads.* Prounipotent completion of groups extends naturally to groupoids. Given a groupoid G, its *prounipotent completion*,[1] $\mathrm{G}_{\mathbb{K}}$, is defined object-wise, in particular for each object $x \in \mathrm{Ob}(\mathrm{G})$ it recovers the prounipotent completion of the automorphism group of x:

$$\mathrm{Hom}_{\mathrm{G}_{\mathbb{K}}}(x,x) := \mathrm{Hom}_{\mathrm{G}}(x,x)_{\mathbb{K}}.$$

The construction of $\mathrm{G}_{\mathbb{K}}$ follows several steps. First, one constructs the $\mathbb{K}$-linear envelope $\mathbb{K}[\mathrm{G}]$ of the groupoid G. This is a $\mathbb{K}$-linear category whose morphism spaces $\mathrm{Hom}_{\mathbb{K}[\mathrm{G}]}(x,y)$ are the free $\mathbb{K}$-vector spaces generated by the sets $\mathrm{Hom}_{\mathrm{G}}(x,y)$. Note that $\mathrm{Hom}_{\mathbb{K}[\mathrm{G}]}(x,y)$ is moreover a coassociative coalgebra, by setting the morphisms in $\mathrm{Hom}_{\mathrm{G}}(x,y)$ to be group-like.

For any pair of objects (x,y) there is an "augmentation ideal" $I(x,y)$, namely, the kernel of the counit $\epsilon : \mathrm{Hom}_{\mathbb{K}[\mathrm{G}]}(x,y) \to \mathbb{K}$. We refer to the union of $I(x,y)$ over all pairs of objects as I. Define $I^n(x,y) \subseteq$

---

[1] The prounipotent completion of a groupoid G does not require G to have finitely many objects or morphisms. However, to ensure good algebraic behavior—such as convergence of the lower central series or well-behaved Lie algebras—it is common to assume that each automorphism group $\mathrm{Hom}_{\mathsf{G}}(x,x)$ is finitely generated and residually nilpotent.

$\mathrm{Hom}_{\mathbb{K}[\mathrm{G}]}(x,y)$ to be the subspace consisting of all morphisms which factor as a composition of n morphisms in I. The powers of I form nested linear subspaces within each morphism space:

$$\mathrm{Hom}_{\mathbb{K}[\mathrm{G}]}(x,y) \supseteq I(x,y) \supseteq I^2(x,y) \supseteq \ldots$$

The I-adic completion of each morphism space is defined as

$$\mathrm{Hom}_{\mathbb{K}[\mathrm{G}]}(x,y)^{\wedge}_{I} := \lim_{n} \big(\mathrm{Hom}_{\mathbb{K}[\mathrm{G}]}(x,y)\big)/I^n(x,y).$$

By taking the group-like elements in this I-adic completion, we obtain the pro-unipotent completion

$$\mathrm{Hom}_{\mathrm{G}_{\mathbb{K}}}(x,y) := \mathrm{Hom}_{\mathrm{G}}(x,y)_{\mathbb{K}}.$$

This procedure defines a functor

$$(-)_{\mathbb{K}} \colon \mathsf{Grp} \longrightarrow \mathsf{Grp}_{\mathbb{K}}$$

from the category of groupoids to the category of prounipotent groupoids over $\mathbb{K}$ (see [Fre17, §9.1]). In particular for each fixed object x, the completed endomorphism algebra $\mathrm{Hom}_{\mathrm{G}}(x,x)_{\mathbb{K}}$ is a prounipotent group over $\mathbb{K}$.

Proposition 1.13 ([Fre17, Proposition 9.2.2])**.** *The completion functor* $(-)_{\mathbb{K}} \colon \mathsf{Grp} \to \mathsf{Grp}_{\mathbb{K}}$ *is symmetric monoidal.*

Proposition 1.13 allows us to extend the completion functor to (coloured) operads in groupoids.

Definition 1.14. Let $\mathbb{K}$ be a field of characteristic zero, and let $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ be an operad in the category of groupoids. The *prounipotent completion* of $\mathcal{P}$ is the operad in prounipotent groupoids defined arity-wise by

$$\mathcal{P}_{\mathbb{K}} := \{\mathcal{P}(n)_{\mathbb{K}}\}_{n\geq 0}\,.$$

The composition maps $\circ_i \colon \mathcal{P}(n) \times \mathcal{P}(m) \longrightarrow \mathcal{P}(n+m-1)$ induce corresponding maps after completion:

$$\mathcal{P}(n)_{\mathbb{K}} \times \mathcal{P}(m)_{\mathbb{K}} \xrightarrow{\widehat{\circ}_i} \mathcal{P}(n+m-1)_{\mathbb{K}}$$

which define the operad structure on $\mathcal{P}_{\mathbb{K}}$.

1.3.3. *Associated graded structures for groupoids and operads.* As with prounipotent completion, the associated graded functor can be extended to groupoids. Given a groupoid G, the associated graded category of the $\mathbb{K}$-linear category $\mathbb{K}[\mathrm{G}]$ has the same objects as G, while morphism spaces are defined by

$$\mathrm{Hom}_{\mathrm{gr}\,\mathbb{K}[\mathrm{G}]}(x,y) = \prod_{n=0}^{\infty} I^n(x,y)/I^{n+1}(x,y).$$

We denote by $\mathcal{G}(\mathrm{gr}\,\mathbb{K}[\mathrm{G}])$ the restriction to group-like morphisms within each morphism space. A *homomorphic expansion* for a groupoid G is a filtered equivalence of $\mathbb{K}$-linear categories $\hat{Z} : \mathbb{K}[\mathrm{G}]^{\wedge}_{I} \to \mathrm{gr}\,\mathbb{K}[\mathrm{G}]$, or alternatively $\mathcal{Z} : \mathrm{G}_{\mathbb{K}} \to \mathcal{G}(\mathrm{gr}\,\mathbb{K}[\mathrm{G}])$. These notions naturally extend to (coloured) operads in groupoids.

Definition 1.15. For a field $\mathbb{K}$ of characteristic zero, and $\mathcal{P} = \{\mathcal{P}(n)\}_{n\geq 0}$ an operad in groupoids, the *associated graded* operad of $\mathcal{P}$ is the operad in Hopf groupoids defined arity-wise by

$$\mathrm{gr}\,\mathbb{K}[\mathcal{P}] := \{\mathrm{gr}\,\mathbb{K}[\mathcal{P}(n)]\}_{n\geq 0}\,.$$

The composition maps $\circ_i \colon \mathcal{P}(n) \times \mathcal{P}(m) \longrightarrow \mathcal{P}(n+m-1)$ induce corresponding maps on the associated graded linear categories:

$$\mathrm{gr}\,\mathbb{K}[\mathcal{P}(n)] \times \mathrm{gr}\,\mathbb{K}[\mathcal{P}(m)] \xrightarrow{\mathrm{gr}(\circ_i)} \mathrm{gr}\,\mathbb{K}[\mathcal{P}(n+m-1)]$$

which define the operad structure on $\mathrm{gr}[\mathcal{P}]$. The arity-wise restriction to group-like morphisms $\mathcal{G}(\mathrm{gr}\,\mathbb{K}\mathcal{P})$ is an operad in Hopf groupoids.

Definition 1.16. Given an (coloured) operad in groupoids $\mathcal{P}$ a *homomorphic expansion* for $\mathcal{P}$ is a filtered isomorphism of operads in $\mathbb{K}$-linear categories $\hat{Z} : \mathcal{P}^{\wedge}_{I} \to \mathrm{gr}\,\mathbb{K}[\mathcal{P}]$ or, equivalently, an isomorphism of operads in groupoids $\mathcal{Z} : \mathcal{P}_{\mathbb{K}} \to \mathcal{G}(\mathrm{gr}\,\mathbb{K}[\mathcal{P}])$.

1.4. **Expansions in action: chord diagrams and Drinfeld associators.** Kohno proved in [Koh85] that the Lie algebra associated with the prounipotent completion of the pure braid group PB_n is the Lie algebra now known as the *Drinfeld–Kohno* Lie algebra, $\mathfrak{t}_n$ (a.k.a. the Lie algebra of *infinitesimal braids*). Equivalently, the associated graded algebra of $\mathbb{K}[\mathsf{PB}_n]$ is the degree completed universal enveloping algbera of $\mathfrak{t}_n$. There is a finite presentation for $\mathfrak{t}_n$, namely, it is generated by the symbols

$$\{t_{ij} = t_{ji} \,|\, 1 \leq i, j \leq n, i \neq j\}$$

and subject to the relations $[t_{ij}, t_{ik} + t_{kj}] = 0$, and $[t_{ij}, t_{kl}] = 0$ whenever i, j, k, l are distinct indices. The grading on $\mathfrak{t}_n$ is given by assigning degree 1 to the generators. We work with the graded completion throughout the paper, and denote this by $\mathfrak{t}_n$ rather than $\hat{\mathfrak{t}}_n$ to simplify notation. In particular, there is a homomorphic expansion of prounipotent groups $(\mathsf{PB}_n)_{\mathbb{K}} \cong \exp(\mathfrak{t}_n)$ which identifies x_{ij} with $e^{t_{ij}}$ (cf. [Koh85] or the sketch of Theorem 10.0.7 in [Fre17]).

Definition 1.17. By convention, we set $\mathfrak{t}_0 = \mathfrak{t}_1 = \{0\}$. For $n \geq 2$, there is a natural right action of the symmetric group Σ_n on each Lie algebra $\mathfrak{t}_n$ which permutes the indices of the generators, that is, $(t_{ij})^\sigma := t_{\sigma^{-1}(i)\sigma^{-1}(j)}$, for some $\sigma \in \Sigma_n$. The symmetric sequence $\mathfrak{t} = \{\mathfrak{t}_n\}_{n\geq 0}$ forms and operad in the category of Lie algebras with partial composition maps

$$\mathfrak{t}_n \oplus \mathfrak{t}_m \xrightarrow{\circ_\alpha} \mathfrak{t}_{n+m-1},$$

$1 \leq \alpha \leq n$ defined on generators $t_{ij} \in \mathfrak{t}_n$ and $t_{kl} \in \mathfrak{t}_m$ as $t_{ij} \circ_\alpha t_{kl} := t_{ij} \circ_\alpha 0 + 0 \circ_\alpha t_{kl}$ with

$$t_{ij} \circ_\alpha 0 := \begin{cases} t_{(i+m-1)(j+m-1)} & \text{if } \alpha < i, \\ \sum_{\beta=1}^m t_{(i+\beta-1)(j+m-1)} & \text{if } \alpha = i, \\ t_{i(j+m-1)} & \text{if } i < \alpha < j, \\ \sum_{\beta=1}^m t_{i(j+\beta-1)} & \text{if } \alpha = j, \\ t_{ij} & \text{if } \alpha > j, \end{cases}$$

and $0 \circ_\alpha t_{kl} := t_{(k+\alpha-1)(l+\alpha-1)}$ for all α. Note that $0 \in \mathfrak{t}_1$ acts as an identity for the operadic $\circ_\alpha$ compositions.

Taking the universal enveloping algebra of a Lie algebra, and restricting to the group-like elements within it defines a functor from Lie algebras to prounipotent groups. In turn, this induces a functor from operads in Lie algebras to operads in prounipotent groups. In particular, the prounipotent groups $\exp(\mathfrak{t}_n)$, $n \geq 0$, assemble into an operad called the *operad of chord diagrams*. The name is due to a diagrammatic convention where elements of the universal enveloping algebra $U(\mathfrak{t}_n)$ are drawn as chord diagrams on n vertical strands, with t_{ij} represented by a horizontal *chord* connecting strand i and strand j, and multiplication given by vertical stacking.

Definition 1.18. For $n \geq 0$, let $\mathsf{CD}(n) := \exp(\mathfrak{t}_n)$ denote the prounipotent group of group-like elements[2] of $U(\mathfrak{t}_n)$. Each $\mathsf{CD}(n)$ inherits a symmetric group action from the Σ_n action on $\mathfrak{t}_n$ defined above. The symmetric sequence $\mathsf{CD} = \{\mathsf{CD}(n)\}_{n\geq 0}$ forms an operad in prounipotent groups in which the operadic composition is defined via the operadic composition of the operad of Drinfeld-Kohno Lie algebras (Definition 1.17),

$$e^x \circ_i e^y := e^{x \circ_i y}.$$

The associated graded operad of the operad PaB is the operad of *parenthesised* chord diagrams built on CD.

Definition 1.19. Let $\mathsf{PaCD}(0) = \{*\}$ be the trivial groupoid. For $n \geq 1$, define $\mathsf{PaCD}(n)$ as the prounipotent groupoid with $\mathrm{Ob}(\mathsf{PaCD}(n)) = \Omega(n)$ and morphisms

$$\mathrm{Hom}_{\mathsf{PaCD}(n)}(w_1, w_2) = \exp(\mathfrak{t}_n)$$

for any $w_1, w_2 \in \Omega(n)$. The symmetric group Σ_n acts on each groupoid $\mathsf{PaCD}(n)$ by permuting the objects, and the resulting symmetric sequence $\mathsf{PaCD} = \{\mathsf{PaCD}(n)\}_{n\geq 1}$ forms an operad in groupoids where the composition functors

$$\mathsf{PaCD}(n) \times \mathsf{PaCD}(m) \xrightarrow{\circ_i} \mathsf{PaCD}(n+m-1)$$

---

[2] Because we are working with the degree complete version of $\mathfrak{t}_n$ there is an isomorphism between the group-like elements of the universal enveloping algebra of $\mathfrak{t}_n$, $\mathcal{G}(U(\mathfrak{t}_n))$, and $\exp(\mathfrak{t}_n)$.

are defined at the level of objects by the operadic composition of parenthesised permutations. At the level of morphisms, the composition $e^x \circ_i e^y = e^{x \circ_i y}$ is defined as in the operad CD.

Remark 1.20. A homomorphic expansion for PaB is an isomorphism of operads $\mathsf{PaB}_{\mathbb{K}} \to \mathsf{PaCD}$. However, every operad equivalence $\varphi : \mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$ extends uniquely to an operad isomorphism $\varphi' : \mathsf{PaB}_{\mathbb{K}} \to \mathsf{PaCD}$ which acts as the identity on objects ([Fre17, Theorem 10.3.12]). In light of this correspondence and to simplify later discussions, we will study the equivalences $\mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$ in place of the expansions.

Definition 1.21. Let $\mathbb{K}$ be a field of characteristic zero. A *Drinfeld associator* is a pair $(\mu, f) \in \mathbb{K}^{\times} \times \exp(\mathfrak{lie}_2)$ that satisfies the following equations.

$$f(\xi_1, \xi_2) f(\xi_2, \xi_1) = 1, \tag{I}$$

$$e^{\mu\xi_1/2} f(\xi_3, \xi_1) e^{\mu\xi_3/2} f(\xi_2, \xi_3) e^{\mu\xi_2/2} f(\xi_1, \xi_2) = 1 \tag{H}$$

whenever $e^{\xi_1+\xi_2+\xi_3} = 1$ in $\exp(\mathfrak{t}_3)$, and

$$f(t_{23}, t_{34}) f(t_{12} + t_{13}, t_{24} + t_{34}) f(t_{12}, t_{23}) = f(t_{12}, t_{23} + t_{24}) f(t_{13} + t_{23}, t_{34}) \tag{P}$$

in $\exp(\mathfrak{t}_4)$. Equation (P) is called the *pentagon* equation and (H) is called the *hexagon* equation.

Remark 1.22. The hexagon equation is often written as

$$e^{\mu t_{12}/2} f(t_{13}, t_{12}) e^{\mu t_{13}/2} f(t_{23}, t_{13}) e^{\mu t_{23}/2} f(t_{12}, t_{23}) = 1 \tag{H'}$$

in $\exp(\mathfrak{t}_3)$, which is equivalent to (H). Since the element $t_{12} + t_{13} + t_{23}$ generates the center of $\mathfrak{t}_3$, further equivalent expressions of the hexagon (respectively, pentagon) equations can be obtained by adding any central element of $\mathfrak{t}_3$ (respectively, $\mathfrak{t}_4$) to any input of f within the equations. For example, a version of the hexagon equation using only the variables t_{12} and t_{13} is:

$$e^{\mu t_{12}/2} f(t_{13}, t_{12}) e^{\mu t_{13}/2} f(-t_{12} - t_{13}, t_{13}) e^{\mu(-t_{12}-t_{13})/2} f(t_{12}, -t_{12} - t_{13}) = 1.$$

The hexagon equation (H') is also equivalent to imposing the pair of equations

$$e^{(t_{12}+t_{13})/2} = f(t_{23}, t_{31})^{-1} e^{t_{13}/2} f(t_{21}, t_{13}) e^{t_{12}/2} f(t_{12}, t_{23})^{-1}, \tag{H1}$$

$$e^{(t_{13}+t_{23})/2} = f(t_{31}, t_{12}) e^{t_{13}/2} f(t_{13}, t_{32})^{-1} e^{t_{23}/2} f(t_{12}, t_{23}). \tag{H2}$$

provided (I) holds, and the equations (H1) and (H2) together imply the inversion relation (I) ([Dri90, §4]). So, one can equivalently define Drinfeld associators as pairs (μ, f) satisfying (H1), (H2), and (P), or (I), (H1), (H2), and (P).

The canonical isomorphism of prounipotent groups $(\mathsf{PB}_n)_{\mathbb{K}} \cong \exp(\mathfrak{t}_n)$ suggests that there exists an equivalence of operads in prounipotent groupoids $\mathsf{PaB}_{\mathbb{K}} \simeq \mathsf{CD}$. The following Theorem is proven in [BN98] (in a slightly different context), or [Fre17, Proposition 10.2.7]. In spirit, this theorem is a consequence of Theorem 1.8, as an operad equivalence is determined by its values on the generators of $\mathsf{PaB}_{\mathbb{K}}$.

Theorem 1.23. *There is a bijection between the set of operad equivalences $\varphi : \mathsf{PaB}_{\mathbb{K}} \longrightarrow \mathsf{CD}$ and the set of Drinfeld associators. The correspondence is given by the values*

$$\varphi(R^{1,2}) = e^{\frac{\mu t_{12}}{2}} \in \exp(\mathfrak{t}_2) \quad \text{and} \quad \varphi(\Phi^{1,2,3}) = f(t_{12}, t_{23}) \in \exp(\mathfrak{t}_3)$$

which satisfy the equations in Definition 1.21. In particular, for any operad equivalence φ, the pair $(\mu, f(t_{12}, t_{23}))$ is a Drinfeld associator, and vice versa.

1.5. **The Grothendieck–Teichmüller groups.** The set of Drinfeld associators is a bi-torsor under the actions of two prounipotent groups – the Grothendieck–Teichmüller group, GT, and the graded Grothendieck–Teichmüller group, GRT. We follow the conventions as in [Fre17, Chapter 10 and Chapter 11], and we will interchangeably refer to $(\mathsf{F}_2)_{\mathbb{K}}$ and $\exp(\mathfrak{lie}_2)$, under the isomorphism $(\mathsf{F}_2)_{\mathbb{K}} \cong \exp(\mathfrak{lie}_2)$ defined by $x_1 \mapsto e^{\xi_1}$ and $x_2 \mapsto e^{\xi_2}$.

Definition 1.24. The $\mathbb{K}$-prounipotent *Grothendieck–Teichmüller group*, denoted GT, is defined by the set of all pairs $(\lambda, f) \in \mathbb{K} \times (\mathsf{F}_2)_{\mathbb{K}}$ which satisfy

$$f(x_1,x_2)f(x_2,x_1)=1, \tag{1.2}$$

$$f(x_1,x_2)x_1^{\mu}f(x_3,x_1)x_3^{\mu}f(x_2,x_3)x_2^{\mu}=1 \quad \text{in} \quad (\mathsf{F}_2)_{\mathbb{K}} \quad \text{with} \quad x_3x_2x_1=1 \text{ and } \lambda=2\mu+1. \tag{1.3}$$

$$f(x_{23},x_{34})f(x_{13}x_{12},x_{34}x_{24})f(x_{12},x_{23})=f(x_{12},x_{24}x_{23})f(x_{23}x_{13},x_{34}) \quad \text{in} \quad (\mathsf{PB}_4)_{\mathbb{K}}. \tag{1.4}$$

The group multiplication is given by, for any two pairs $(\lambda_1, f_1(x_1,x_2))$ and $(\lambda_2, f_2(x_1,x_2))$, by

$$(\lambda_1,f_1)*(\lambda_2,f_2)=(\lambda_1\lambda_2, f_1(x_1,x_2)f_2(x_1^{\lambda_1}, f_1^{-1}x_2^{\lambda_1}f_1)).$$

The subgroup of GT consisting of all pairs (λ, f) with $\lambda = 1$ is denoted by GT_1. This subgroup is often referred to as the *acyclotomic* Grothendieck–Teichmüller group, as it coincides with the kernel of the cyclotomic character of the profinite version of the Grothendieck-Teichmüller group ([LS15], [Dri90]).

The group GT acts freely and transitively on the set of Drinfeld associators as follows [Fre17, Proposition 11.2.1]. Given a Drinfeld associator $(\mu, f(\xi_1,\xi_2)) \in \mathbb{K}^{\times} \times \exp(\mathfrak{lie}_2)$ and a GT element $(\lambda, g(x_1,x_2)) \in \mathbb{K} \times (\mathsf{F}_2)_{\mathbb{K}}$, there is an action

$$(\mu,f)*(\lambda,g)=(\mu\lambda, f(\xi_1,\xi_2)g(e^{\mu\xi_1}, f^{-1}e^{\lambda\xi_2}f)). \tag{1.5}$$

Note that the GT action is a left group action, but we use function composition notation above, hence the GT element (λ, g) is written on the right.

Following from the identification of operad equivalences $\varphi : \mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$ with Drinfeld associators, one can identify the group GT with the automorphisms of the operad $\mathsf{PaB}_{\mathbb{K}}$.

Theorem 1.25 ([BN98], [Fre17, Theorem 11.1.7])**.** *Let* $\mathrm{Aut}_0(\mathsf{PaB}_{\mathbb{K}})$ *denote the set of automorphisms* $\vartheta : \mathsf{PaB}_{\mathbb{K}} \to \mathsf{PaB}_{\mathbb{K}}$ *that act as the identity on objects. There is an isomorphism of prounipotent groups*

$$\mathrm{Aut}_0(\mathsf{PaB}_{\mathbb{K}}) \cong \mathsf{GT}.$$

The pair $(\lambda, f) \in \mathsf{GT}$ uniquely determines the automorphism $\vartheta \in \mathrm{Aut}_0(\mathsf{PaB}_{\mathbb{K}})$ which acts on generators as

$$\vartheta(R^{1,2}) = R^{1,2} \cdot (R^{2,1}R^{1,2})^{\nu}$$
$$\vartheta(\Phi^{1,2,3}) = \Phi^{1,2,3} \cdot f(x_{12},x_{23}),$$

where $\nu = (\lambda-1)/2$, and any such assignment defines an automorphism of $\mathsf{PaB}_{\mathbb{K}}$.

The *graded Grothendieck–Teichmüller group*, denoted GRT, acts on the right on the set of Drinfeld associators.

Definition 1.26. The $\mathbb{K}$-prounipotent *graded Grothendieck–Teichmüller group*, GRT, is defined as the semidirect product $\mathbb{K}^{\times} \ltimes \mathsf{GRT}_1$ consisting of the set of all pairs (λ, g), where $\lambda \in \mathbb{K}^{\times}$, and $g(\xi_1,\xi_2) \in \exp(\mathfrak{lie}_2) \subset \exp(\mathfrak{t}_3)$, satisfying the following equations:

$$g(\xi_1,\xi_2)=g(\xi_2,\xi_1)^{-1}, \tag{1.6}$$

$$g(\xi_3,\xi_1)\cdot g(\xi_2,\xi_3)\cdot g(\xi_1,\xi_2)=1 \quad \text{in } \exp(\mathfrak{t}_3) \quad \text{with} \quad \xi_1+\xi_2+\xi_3=0, \tag{1.7}$$

$$g(t_{12},t_{23}+t_{24})\cdot g(t_{13}+t_{23},t_{34})=g(t_{23},t_{34})\cdot g(t_{12}+t_{13},t_{24}+t_{34})\cdot g(t_{12},t_{23}) \quad \text{in } \exp(\mathfrak{t}_4). \tag{1.8}$$

The action of $\mathbb{K}^{\times}$ on GRT_1 is given by $\lambda \cdot \Phi(\xi_1,\xi_2) = \Phi(\lambda\xi_1,\lambda\xi_2)$ for $\lambda \in \mathbb{K}^{\times}$. The group multiplication in $\mathsf{GRT} = \mathbb{K}^{\times} \ltimes \mathsf{GRT}_1$ is defined as follows for any two pairs $(\lambda_1, g_1(\xi_1,\xi_2))$ and $(\lambda_2, g_2(\xi_1,\xi_2))$,

$$(\lambda_1, g_1(\xi_1,\xi_2)) * (\lambda_2, g_2(\xi_1,\xi_2)) = (\lambda_1\lambda_2, g_1(\xi_1,\xi_2)g_2(\lambda_1\xi_1, g_1^{-1}\cdot\lambda_1\xi_2\cdot g_1)).$$

The group GRT also acts freely and transitively on Drinfeld associators. Given a Drinfeld associator (μ, f) and an element (λ, g) in GRT, we have an action

$$(\lambda,g)*(\mu,f)=(\lambda\mu, g(\xi_1,\xi_2)f(\lambda\xi_1, g^{-1}\cdot\lambda\xi_2\cdot g)). \tag{1.9}$$

The following Theorem characterizes GRT as the automorphisms of the operad of *parenthesised* chord diagrams.

Theorem 1.27 ([BN98], [Fre17, Theorem 10.3.10])**.** *Let* $\mathrm{Aut}_0(\mathsf{PaCD})$ *be the set of operad isomorphisms* $\vartheta : \mathsf{PaCD} \to \mathsf{PaCD}$ *which act as the identity on objects. Then, there is an isomorphism of prounipotent groups*

$$\mathrm{Aut}_0(\mathsf{PaCD}) \cong \mathsf{GRT}.$$

2. Moperad expansions: The moperad of parenthesised braids with a frozen strand

Expansions of the moperad of parenthesised braids with a frozen strand were described [CG20] in the study of an operadic interpretation of cyclotomic associators. In the present paper, we use a closely related variant of this moperad, adapted for our study of Kashiwara–Vergne solutions. The precise relationship with the formulation of [CG20] is explained in Remark 2.19 later in this section.

2.1. **Moperads.** In this section we recall some preliminaries on moperads. For further details, see for instance [Wil16].

Definition 2.1. Let $\mathcal{P}$ be a one-colored symmetric operad in a symmetric monoidal category C. A *right $\mathcal{P}$-module* consists of a symmetric sequence $\mathcal{M} = \{\mathcal{M}(n)\}_{n\geq 0}$ in C together with structure maps

$$\circ_i : \mathcal{M}(k) \otimes \mathcal{P}(m) \longrightarrow \mathcal{M}(k+m-1), \quad \text{for } 1 \leq i \leq k,$$

called *partial right actions*, which satisfy the following axioms:

(i) For all $\mu \in \mathcal{M}(k)$, we have $\mu \circ_i \mathrm{id} = \mu$.

(ii) For all $\lambda \in \mathcal{M}(\ell)$, $\mu \in \mathcal{P}(m)$, and $\nu \in \mathcal{P}(n)$, we have

$$(\lambda \circ_j \mu) \circ_i \nu = \begin{cases} (\lambda \circ_i \nu) \circ_{j+n-1} \mu, & \text{if } 1 \leq i \leq j-1, \\ \lambda \circ_j (\mu \circ_{i-j+1} \nu), & \text{if } j \leq i \leq j+m-1, \\ (\lambda \circ_{i-m+1} \nu) \circ_j \mu, & \text{if } j+n \leq i \leq \ell+m-1, \end{cases}$$

(iii) For all $\mu \in \mathcal{M}(k)$, $\nu \in \mathcal{P}(m)$, and $\sigma \in \Sigma_k$, $\tau \in \Sigma_m$, we have

$$\mu^\sigma \circ_{\sigma^{-1}(i)} \nu^\tau = (\mu \circ_i \nu)^{\sigma \circ_i \tau}$$

where the permutation $\sigma \circ_i \tau$ refers to an action by the block permutation.

Example 2.2. Let $\mathcal{P}$ be an operad in the category C. Then $\mathcal{P}$ can be regarded as a right $\mathcal{P}$-module via its own operadic composition. Explicitly, we define the structure maps

$$\circ_i : \mathcal{P}(k) \otimes \mathcal{P}(m) \longrightarrow \mathcal{P}(k+m-1),$$

for $1 \leq i \leq k$, using the operad composition and note that this action satisfies all the axioms required of a right $\mathcal{P}$-module.

Definition 2.3. A *morphism of right $\mathcal{P}$-modules* $\varphi : \mathcal{M} \to \mathcal{N}$ is a morphism of symmetric sequences such that the diagram

$$\begin{tikzcd}
\mathcal{M}(k) \otimes \mathcal{P}(m) \arrow[r, "\circ_i"] \arrow[d, "\varphi(k)\otimes \mathrm{id}"'] & \mathcal{M}(k+m-1) \arrow[d, "\varphi(k+m-1)"] \\
\mathcal{N}(k) \otimes \mathcal{P}(m) \arrow[r, "\circ_i"] & \mathcal{N}(k+m-1)
\end{tikzcd}$$

commutes for all $k, m \geq 0$ and $1 \leq i \leq k$.

If $\mathcal{P}$ and $\mathcal{Q}$ are two operads in C, a map between a right $\mathcal{P}$-module $\mathcal{M}$ and a right $\mathcal{Q}$-module $\mathcal{N}$ is a pair $(\varphi, \phi) : \mathcal{M} \to \mathcal{N}$, where $\phi : \mathcal{P} \to \mathcal{Q}$ is an operad morphism and φ is a map of symmetric sequences compatible with the module structures via the operad map ϕ. That is, for all $k, m \geq 0$ and $1 \leq i \leq k$, the diagram

$$\begin{tikzcd}
\mathcal{M}(k) \otimes \mathcal{P}(m) \arrow[r, "\circ_i"] \arrow[d, "\varphi(k)\otimes \phi(m)"'] & \mathcal{M}(k+m-1) \arrow[d, "\varphi(k+m-1)"] \\
\mathcal{N}(k) \otimes \mathcal{Q}(m) \arrow[r, "\circ_i"] & \mathcal{N}(k+m-1)
\end{tikzcd}$$

commutes. That is, φ intertwines the right $\mathcal{P}$-action on $\mathcal{M}$ with the right $\mathcal{Q}$-action on $\mathcal{N}$ via the operad map ϕ. For an extensive treatment of right modules over operads and their morphisms, see [Fre09].

Definition 2.4. Let $\mathcal{P}$ be a one-coloured symmetric operad in C. A $\mathcal{P}$-*moperad* is a monoid $\mathcal{M}$ in the category of right $\mathcal{P}$-modules. That is, a symmetric sequence $\mathcal{M} = \{\mathcal{M}(n)\}$ together with two types of partial composition maps:

(1) An associative, unital monoid composition

$$\circ_0 : \mathcal{M}(k) \otimes \mathcal{M}(m) \longrightarrow \mathcal{M}(k+m)$$

which makes the symmetric sequence $\mathcal{M}$ into a monoid in the category of symmetric sequences.

(2) An operadic module composition

$$\circ_i : \mathcal{M}(k) \otimes \mathcal{P}(m) \longrightarrow \mathcal{M}(k+m-1)$$

for $1 \leq i \leq k$ which makes $\mathcal{M}$ into a right module over the operad $\mathcal{P}$.

These two operations are compatible in the following way

(3) For any $\lambda \in \mathcal{M}(k), \mu \in \mathcal{M}(p), \nu \in \mathcal{P}(n)$, we have

$$(\lambda \circ_0 \mu) \circ_i \nu = \begin{cases} \lambda \circ_0 (\mu \circ_i \nu), & \text{if } 1 \leq i \leq p, \\ (\lambda \circ_{i-p} \nu) \circ_0 \mu, & \text{if } p+1 \leq i \leq p+k. \end{cases}$$

See [Wil16, Definition 9] for more details.

Example 2.5. Just as every operad can be considered a right module over itself, every operad $\mathcal{P}$ can itself be considered as a $\mathcal{P}$-moperad. Given an operad $\mathcal{P}$ we may build a $\mathcal{P}$-moperad $\mathcal{P}^+$ by "shifting" the operad structure. More precisely, set

$$\mathcal{P}^+(k) := \mathcal{P}(k+1),$$

for $k \geq 0$, where we visualize the operations in $\mathcal{P}(k+1)$ as rooted trees with leaves labeled $\{0, 1, \ldots, k\}$ and define the Σ_k action on $\mathcal{P}^+(k)$ by restricting the Σ_{k+1}-action on $\mathcal{P}(k+1)$ to those permutations that fix 0. The monoid composition is the operadic composition on the first input. That is to say $\circ_0 : \mathcal{P}^+(k) \otimes \mathcal{P}^+(n) \to \mathcal{P}^+(k+n)$ is then the operadic composition $\mathcal{P}(k+1) \circ_0 \mathcal{P}(n+1) \to \mathcal{P}(k+n+1)$. The right $\mathcal{P}$-module action maps $\circ_i : \mathcal{P}^+(k) \otimes \mathcal{P}(n) \to \mathcal{P}^+(k+n-1)$ are given by operadic compositions $\mathcal{P}(k+1) \circ_i \mathcal{P}(n) \to \mathcal{P}(k+n)$, for $1 \leq i \leq k$.

In a moperad $\mathcal{M}$, an 'operation' in $\mathcal{M}(k)$ can be visualized as a rooted tree T whose set of leaves, $\mathrm{L}(T)$, is equipped with a (bijective) labeling $\{0, 1, \ldots, k\} \to \mathrm{L}(T)$, such that the leftmost leaf is always labeled by 0 (i.e. fixed) and the other leaves are labeled by elements of $\{1, \ldots, k\}$. The $\circ_0$-compositions are described by inserting the operations of $\mathcal{M}$ into the 0-th leaf. The $\circ_i$-compositions are given by inserting operations of $\mathcal{P}$ into the leaf labeled i, for $1 \leq i \leq k$.

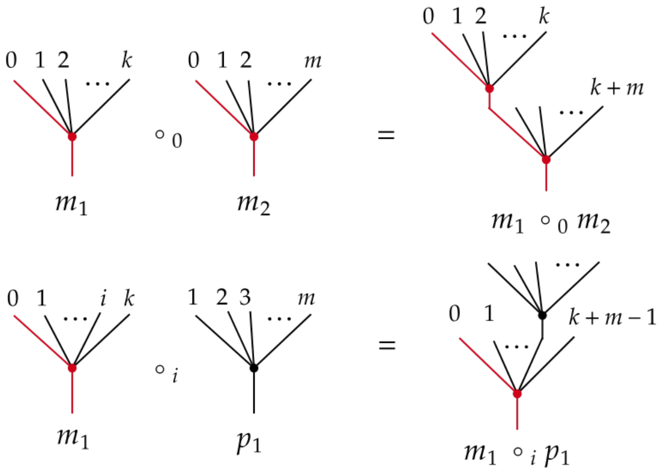


FIGURE 7. Moperad composition illustrated on rooted trees: here m_1 an operation in $\mathcal{M}(k)$, m_2 an operation in $\mathcal{M}(m)$ and p_1 an operation in $\mathcal{P}(m)$.

Remark 2.6. The depiction of moperad composition in Figure 7 suggests an alternative description of a $\mathcal{P}$-moperad $\mathcal{M}$ as a two-colored operad in which one color is "frozen" or "non-symmetric". Let $\mathfrak{C} = \{a, m\}$ be a discrete set of two colours. We can define a $\mathfrak{C}$-colored symmetric sequence $\mathcal{M} = \mathcal{M}(c_1, \dots, c_n; c_0)$ in which:

- $\mathcal{M}(c_1, \dots, c_n; c_0)$ is set to be an object of C called $\mathcal{P}(n)$ if $(c_1, \dots, c_n; c_0) = (m, \dots, m; m)$;
- $\mathcal{M}(c_1, \dots, c_n; c_0)$ is set to be an object of C called $\mathcal{M}(n)$ if $(c_1, \dots, c_n; c_0) = (a, m, \dots, m; a)$;
- $\mathcal{M}(c_1, \dots, c_n; c_0) = \emptyset$, the initial object of C, otherwise.

The symmetric group Σ_n acts on an object $\mathcal{P}(n)$ by permuting all of the inputs, i.e.

$$\mathcal{M}(c_1, \dots, c_n; c_0) \otimes \Sigma_n \xrightarrow{\sigma^*} \mathcal{M}(c_{\sigma(1)}, \dots, c_{\sigma(n)}; c_0).$$

The symmetric group Σ_n restricts to a trivial action on an object $\mathcal{M}(n)$ unless σ is a permutation which fixes 1, i.e.

$$\mathcal{M}(c_1, \dots, c_n; c_0) \otimes \Sigma_n \mapsto \begin{cases} \mathcal{M}(c_1, c_{\sigma(2)} \dots, c_{\sigma(n)}; c_0) \text{ if } \sigma(1) = 1; \\ \mathcal{M}(c_1, \dots, c_n; c_0) \text{ otherwise.} \end{cases}$$

In other words, $\mathcal{M}$ has two types of operations: algebra operations in color m, governed by the operad $\mathcal{P}$, and module operations with one distinguished input and output of color a, and any number of additional inputs of color m. Moreover, $\mathcal{M}$ is non-symmetric in the color a—that is, no permutations are allowed among inputs of color a ([Wil16, Definition 8]). Composition is defined as in any standard $\mathfrak{C}$-colored operad, respecting the symmetry and color constraints above.

Definition 2.7. A *map of $\mathcal{P}$-moperads* $\varphi : \mathcal{M} \to \mathcal{N}$ is a map of right $\mathcal{P}$-modules which is compatible with the monoid structures, that is, $\varphi(\mathcal{M}(k) \circ_0 \mathcal{M}(m)) = \varphi(\mathcal{M}(k)) \circ_0 \varphi(\mathcal{M}(m))$. We write $\mathrm{MOp}_{\mathcal{P}}(\mathrm{C})$ for the category of $\mathcal{P}$-moperads in C.

Similarly, given two distinct operads $\mathcal{P}$ and $\mathcal{Q}$, a map $(\varphi, \phi) : \mathcal{M} \to \mathcal{N}$ between a $\mathcal{P}$-moperad $\mathcal{M}$ and a $\mathcal{Q}$-moperad $\mathcal{N}$ is a pair (φ, ϕ) where $\phi : \mathcal{P} \to \mathcal{Q}$ is a map of operads and $\varphi : \mathcal{M} \to \mathcal{N}$ is a map of symmetric sequences such that (φ, ϕ) is a map of right modules and φ is compatible with the respective monoid structures. We write MOp(C) for the category of moperads in C, where the operad is allowed to vary.

Remark 2.8. The shift construction (Example 2.5) induces a functor $(-)^+ : \mathrm{Op}(\mathrm{C}) \to \mathrm{MOp}(\mathrm{C})$ which sends the operad $\mathcal{P}$ to the $\mathcal{P}$-moperad $\mathcal{P}^+$. This functor is faithful, meaning that every map of operads $\varphi : \mathcal{P} \to \mathcal{Q}$ gives rise to a map of moperads $\varphi^+ : \mathcal{P}^+ \to \mathcal{Q}^+$ and this assignment is injective. Not every moperad map

$\varepsilon : \mathcal{P}^+ \to \mathcal{Q}^+$ need come from shifting an operad map $\varphi : \mathcal{P} \to \mathcal{Q}$. This is reminiscent of the fact that, given two rings R and S, a module map $f : \mathsf{R} \to \mathsf{S}$ need not correspond to a ring homomorphism unless it preserves the full multiplicative structure.

2.2. **Completion and homomorphic expansions for moperads.** Let $\mathcal{P}$ be an operad and let $\mathcal{M} = \{\mathcal{M}(n)\}_{n\geq 0}$ be a $\mathcal{P}$-moperad. The *prounipotent completion* of $\mathcal{M}$ is the $\mathcal{P}_{\mathbb{K}}$-moperad $\mathcal{M}_{\mathbb{K}}$ defined as follows.

As a symmetric sequence, $\mathcal{M}_{\mathbb{K}}$ is given by the entrywise prounipotent completion

$$\mathcal{M}_{\mathbb{K}} = \{\mathcal{M}(n)_{\mathbb{K}}\}_{n\geq 0}.$$

The structure maps of $\mathcal{M}_{\mathbb{K}}$ are obtained by applying the prounipotent completion functor to the structure maps of $\mathcal{M}$: the monoid composition induces an associative and unital map

$$\circ_0 : \mathcal{M}(k)_{\mathbb{K}} \otimes \mathcal{M}(m)_{\mathbb{K}} \longrightarrow \mathcal{M}(k+m)_{\mathbb{K}},$$

and the right $\mathcal{P}$-module structure induces, for each $1 \leq i \leq k$, operadic module compositions

$$\circ_i : \mathcal{M}(k)_{\mathbb{K}} \otimes \mathcal{P}(m)_{\mathbb{K}} \longrightarrow \mathcal{M}(k+m-1)_{\mathbb{K}}.$$

We can similarly define the associated graded of a $\mathcal{P}$-moperad $\mathcal{M}$.

Definition 2.9. Let $\mathbb{K}$ be a field of characteristic zero, and $\mathcal{M} = \{\mathcal{M}(n)\}_{n\geq 0}$ a moperad in groupoids over the operad $\mathcal{P}$. The *associated graded* moperad of $\mathcal{M}$ is the moperad in Hopf groupoids over $\operatorname{gr}\mathcal{P}$, defined arity-wise by

$$\operatorname{gr}\mathbb{K}[\mathcal{M}] := \{\operatorname{gr}\mathbb{K}[\mathcal{M}(n)]\}_{n\geq 0}.$$

The action maps $\circ_i \colon \mathcal{M}(n) \times \mathcal{P}(m) \longrightarrow \mathcal{P}(n+m-1)$ induce corresponding maps on the graded linear categories:

$$\operatorname{gr}\mathbb{K}[\mathcal{M}(n)] \times \operatorname{gr}\mathbb{K}[\mathcal{P}(m)] \xrightarrow{\operatorname{gr}(\circ_i)} \operatorname{gr}\mathbb{K}[\mathcal{M}(n+m-1)] \ .$$

The same is true for the monoid multiplication $\circ_0 \colon \mathcal{M}(n) \times \mathcal{M}(m) \longrightarrow \mathcal{M}(n+m)$:

$$\operatorname{gr}\mathbb{K}[\mathcal{M}(n)] \times \operatorname{gr}\mathbb{K}[\mathcal{M}(m)] \xrightarrow{\operatorname{gr}(\circ_0)} \operatorname{gr}\mathbb{K}[\mathcal{M}(n+m)] \ .$$

This defines the moperad structure on $\operatorname{gr}[\mathcal{M}]$. The arity-wise restriction to group-like elements $\mathcal{G}(\operatorname{gr}\mathbb{K}\mathcal{P})$ is a moperad in graded Hopf groupoids.

Definition 2.10. Given a moperad in groupoids $\mathcal{M}$ over an operad $\mathcal{P}$, a *homomorphic expansion* for $\mathcal{M}$ is a pair $(\hat{Z}_{\mathcal{M}}, \hat{Z}_{\mathcal{P}})$, where $\hat{Z}_{\mathcal{P}}$ is a homomorphic expansion for $\mathcal{P}$, and $\hat{Z}_{\mathcal{M}}$, is a filtered isomorphism of mopeards in groupoids over $\hat{Z}_{\mathcal{P}}$:

$$\hat{Z}_{\mathcal{M}} : \mathcal{M}_I^{\wedge} \to \operatorname{gr}\mathbb{K}[\mathcal{M}], \quad \text{or} \quad \hat{Z}_{\mathcal{M}} : \mathcal{M}_{\mathbb{K}} \to \mathcal{G}(\operatorname{gr}\mathbb{K}[\mathcal{M}]).$$

2.3. **Parenthesized braids with a frozen strand.** The *braid group with a frozen strand*, denoted B_n^1, is the fundamental group of the space of unordered configurations of n points in the punctured plane $\mathbb{C} - \{0\}$. The group B_n^1 admits a finite presentation as in [AET10, § 3].

Proposition 2.11. *The group B_n^1 is finitely presented on generators $X_1, X_2, \ldots, X_n$, and $\beta_1, \ldots, \beta_{n-1}$, subject to the relations*

$$\beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}, \tag{2.1}$$

$$\beta_i\beta_j = \beta_j\beta_i \quad \textit{when} \quad |i-j| \geq 2, \tag{2.2}$$

$$\beta_i X_i \beta_i^{-1} = X_{i+1}, \tag{2.3}$$

$$\beta_i X_{i+1} \beta_i^{-1} = X_{i+1}^{-1} X_i X_{i+1}, \tag{2.4}$$

$$\beta_i X_j \beta_i^{-1} = X_j \quad \textit{when} \quad j \neq i, i+1. \tag{2.5}$$

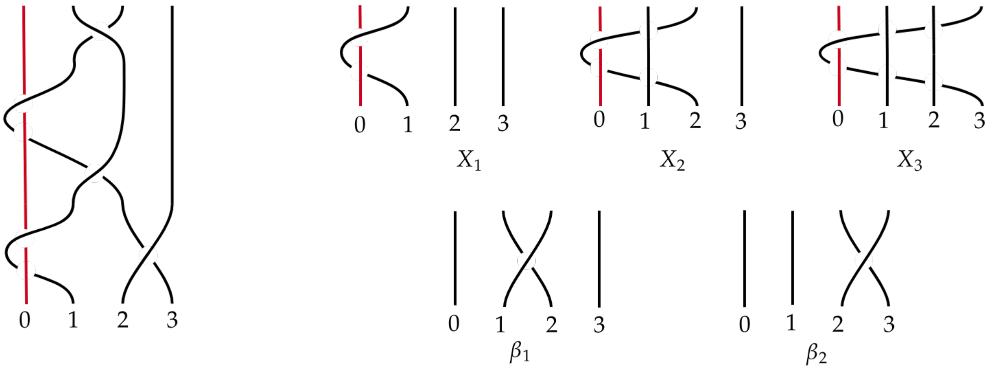


FIGURE 8. A braid with a frozen strand in B_3^1, and the generators of B_3^1. The frozen strand (puncture) is drawn leftmost, and shown in red. The generator X_1 represents the loop where the first point circles around the puncture in the positive direction, and $X_k = \beta_{k-1}X_{k-1}\beta_{k-1}^{-1}$ for $k \geq 2$. In terms of the generators, the braid on the left is (non-uniquely) expressed as $\beta_2 X_1 X_2$.

We can define a map $\iota : \mathsf{B}_n^1 \to \mathsf{B}_{n+1}$ by re-indexing the generators of the braid group B_{n+1}, as follows. Consider the presentation of B_{n+1} given by generators $\{\beta_0, \beta_1, ..., \beta_{n-1}\}$ and the Artin relations for the indices $\{0, ..., n-1\}$. Define $\iota(\beta_i) := \beta_i$ for $1 \leq i \leq n-1$, and $\iota(X_i) := \beta_{i-1}\beta_{i-2}\cdots\beta_1\beta_0^2\beta_1^{-1}\cdots\beta_{i-2}^{-1}\beta_{i-1}^{-1}$ for $1 \leq i \leq n$. See Fig. 8 for the intuition. We leave it to the reader to check that these images satisfy the relations of B_n^1 and define an injective homomorphism. On the level of fundamental groups of configuration spaces, the idea is to consider the puncture in the plane as an additional point.

Proposition 2.12. *The map* $\iota : \mathsf{B}_n^1 \hookrightarrow \mathsf{B}_{n+1}$ *defined above is an injective group homomorphism.* ☐

Remark 2.13. Inspecting the presentation of B_n^1 in Proposition 2.11, relation (2.3) makes clear that it is enough to include X_1 as a generator, and along with the β_i it generates the rest of the X_i's. Doing so leads to a new presentation on the generators $\{X_1, \beta_1, ..., \beta_{n-1}\}$, where relations (2.1) and (2.2) stay the same, (2.3) is omitted, and (2.4) and (2.5) are replaced by

$$\beta_1 X_1 \beta_1^{-1} = X_1^{-1}\beta_1^{-1}X_1\beta_1X_1, \tag{2.6}$$

$$\beta_i X_1 \beta_i^{-1} = X_1 \quad \text{when} \quad i \neq 1. \tag{2.7}$$

Notice that in the presentation of Proposition 2.11, relations (2.1) and (2.2) are the standard braid relations, and relations (2.3)–(2.5) describe the interactions of the (pure braid) generators X_i with the crossings β_i. Note in particular that there are no relations between the X_i generators. This is an important fact, as the $\{X_1, ..., X_n\}$ generate a normal subgroup of B_n^1 isomorphic to the free group F_n. Furthermore, there is a trivially intersecting copy of B_n in B_n^1, generated by the crossings $\{\beta_1, ..., \beta_n\}$. This gives an isomorphism

$$\mathsf{B}_n^1 \cong \mathsf{F}_n \rtimes \mathsf{B}_n, \tag{2.8}$$

which is a classical result, also discussed in [AET10, Proposition 15], and is central to some of our later constructions. The defining action of the braid group B_n on the free group F_n is reminiscent of the perhaps better-known action of the pure braid group PB_n on the free group F_n, which comes from the Fadell–Neuwirth fibration of ordered configuration spaces [FN62], which we now recall.

For any $n \geq 1$, there is a short exact sequence

$$1 \longrightarrow \mathsf{PB}_n \longrightarrow \mathsf{B}_n \xrightarrow{\pi} \Sigma_n \longrightarrow 1$$

where the map π sends a braid generator β_i to the transposition $(i\; i{+}1)$ in Σ_n. The kernel of π, denoted PB_n, is called the *pure braid group* on n strands.

Pure braid groups are generated by the elements

$$x_{ij} = \beta_{j-1}\dots\beta_{i+1}\beta_i^2\beta_{i+1}^{-1}\dots\beta_{j-1}^{-1}, \text{ for } 1 \leq i < j \leq n, \tag{2.9}$$

as shown in Figure 9.

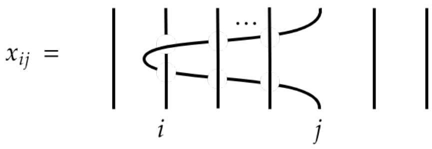


FIGURE 9. A pure braid generator.

Alternatively, the pure braid groups are the fundamental groups of the spaces of ordered configurations of n points in the complex plane at a chosen basepoint $p = (z_1, \dots, z_n)$,

$$\mathsf{PB}_n = \pi_1(\mathrm{Conf}_n(\mathbb{C}), p).$$

Using this identification, one can show that the long exact sequence in homotopy induced by the locally trivial fiber bundle $\mathrm{Conf}_{n+1}(\mathbb{C}) \to \mathrm{Conf}_n(\mathbb{C})$ given by forgetting one point $(z_1, \dots, z_n, z_{n+1}) \mapsto (z_1, \dots, z_n)$ reduces to a split exact sequence of groups:

$$1 \longrightarrow \mathsf{F}_n \longrightarrow \mathsf{PB}_{n+1} \longrightarrow \mathsf{PB}_n \longrightarrow 1 \ .$$

This produces an isomorphism $\mathsf{PB}_{n+1} \cong \mathsf{F}_n \rtimes \mathsf{PB}_n$ analogous to (2.8). In fact, the latter isomorphism $\mathsf{B}_n^1 \cong \mathsf{F}_n \rtimes \mathsf{B}_n$ restricts to an isomorphism

$$\mathsf{PB}_n^1 \cong \mathsf{F}_n \rtimes \mathsf{PB}_n, \tag{2.10}$$

where PB_n^1 is the *pure braid group with a frozen strand*, that is, the fundamental group of the space of ordered configurations of n points in the punctured plane.

Lemma 2.14. *There is a natural isomorphism of groups* $\mathsf{PB}_n^1 \cong \mathsf{PB}_{n+1}$.

Proof. Considering the puncture as an additional point produces an inclusion $\mathsf{PB}_n^1 \hookrightarrow \mathsf{PB}_{n+1}$. To see that this is surjective, note that for a pure braid it is always possible via homotopy to ensure that one given point stays fixed throughout a loop in $\mathrm{Conf}_n(\mathbb{C})$. ☐

Let Σ_n^+ denote the group of automorphisms of the set $\{0, 1, \dots, n\}$. There is an embedding $\Sigma_n \hookrightarrow \Sigma_n^+$ as the stabilizer subgroup of 0; call this subgroup Σ_n^1. Given $\sigma \in \Sigma_m^1$ and $\tau \in \Sigma_n$ we can *insert* τ into the position labeled i of σ, for $1 \leq i \leq m$, via the formulas (1.1). If $\sigma \in \Sigma_m^1$ and $\tau \in \Sigma_n^1$, then τ can be inserted into the $i = 0$ of σ:

$$(\sigma \circ_0 \tau)(k) := \begin{cases} 0 & \text{if } k = 0, \\ \tau(k) & \text{if } 1 \leq k \leq n, \\ \sigma(k-n) + n & \text{if } n+1 \leq k \leq n+m. \end{cases} \tag{2.11}$$

The result is a permutation on $\{0, 1, \dots, m+n\}$ that again fixes 0, thus, an element of Σ_{n+m}^1. Let $\Omega^1(n)$ denote the set of (maximally) parenthesized permutations in Σ_n^1. For example, $0((23)1)$, $(01)(32)$ and $(0(12))3$ are elements in $\Omega^1(3)$.

Definition 2.15. The symmetric sequence $\Omega^1 = \{\Omega^1(n)\}_{n\geq 0}$ forms an Ω-moperad in sets, called the *moperad of parenthesized permutations*. The monoid structure

$$\Omega^1(m) \times \Omega^1(n) \xrightarrow{\circ_0} \Omega^1(m+n)$$

is given by $(0p) \circ_0 (0q) := ((0q)\overline{p})$, for any element $p \in \Omega(m)$ and $q \in \Omega(n)$, where $\overline{p}$ denotes p with each entry shifted by m. The right Ω-module structure

$$\Omega^1(m) \times \Omega(n) \xrightarrow{\circ_i} \Omega^1(m+n-1),$$

for $1 \leq i \leq m$, is given by replacing entry labelled i in a parenthesized permutation in $\Omega^1(m)$ by a parenthesized permutation in $\Omega(n)$, and relabelling appropriately. The moperad Ω^1 comes with a map of symmetric sequences $\Omega(n) \longrightarrow \Omega^1(n) \xrightarrow{u} \Sigma^1_n$, where the first map adds 0 before an element in Ω and the second map forgets the parenthesization.

Next, we define the moperad of parenthesized braids, PaB^1, one of our central objects of study.

Definition 2.16. The *moperad of parenthesized braids*, a PaB-moperad in groupoids denoted PaB^1, is defined as follows. Let $\mathsf{PaB}^1(0) := \{*\}$ be the trivial groupoid. For each $n \geq 1$, let $\mathsf{PaB}^1(n)$ denote the groupoid whose objects $\mathrm{Ob}(\mathsf{PaB}^1(n)) := \Omega^1(n)$ are parenthesized permutations with 0 fixed; and whose morphisms $\beta \in \mathrm{Hom}_{\mathsf{PaB}^1(n)}(p, q)$ are elements of the braid group B^1_n with underlying permutation $u(q)^{-1}u(p)$. The composition in $\mathsf{PaB}^1(n)$ is given by multiplication in B^1_n (with multiplication written right to left, see Remark 1.6).

The symmetric group Σ_n acts on $\mathsf{PaB}^1(n)$ by permuting the labels. The symmetric sequence $\mathsf{PaB}^1 = \{\mathsf{PaB}^1(n)\}_{n\geq 0}$ forms a PaB-moperad in groupoids as follows.

(1) The monoid structure of PaB^1,

$$\mathsf{PaB}^1(n) \times \mathsf{PaB}^1(m) \xrightarrow{\circ_0} \mathsf{PaB}^1(n+m),$$

is intuitively given by inserting the second braid into the "frozen" 0-th strand of the first. At the level of objects, the functor $\circ_0$ acts as the monoid composition of parenthesized permutations. At the level of morphisms, viewing morphisms between arity n permutations as braids on $n+1$ strands, then $\circ_0$ is the operad insertion into the 0-th strand. See Figure 10 for an example of this composition.

(2) The right-module structure of PaB^1 over PaB is defined by the functors

$$\mathsf{PaB}^1(n) \times \mathsf{PaB}(m) \xrightarrow{\circ_i} \mathsf{PaB}^1(n+m-1),$$

for $1 \leq i \leq n$, as follows. On objects, $\circ_i$ is the right action of Ω on Ω^1 as in Definition 2.15. On morphisms, $\circ_i$ is defined by considering a morphism in arity n as a braid on $n+1$ strands indexed by 0 to n, and using the operad insertion into its i-indexed strand.

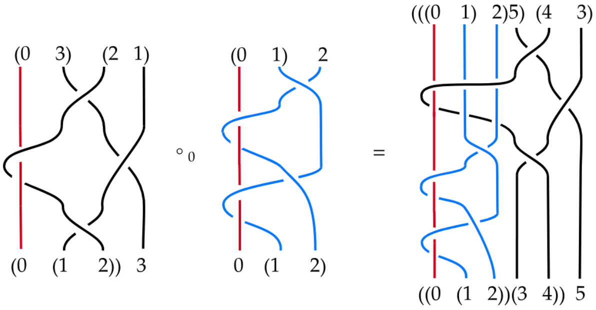


Figure 10. The monoid structure of PaB^1.

Like the operad PaB, the PaB-moperad PaB^1 admits a finite presentation [CG20, Theorem 3.4]. In other words, every morphism in PaB^1 can be obtained from a finite number of generating morphisms via operadic, monoidal, and categorical compositions, permutations, and PaB-actions; subject to finitely many relations.

There is a natural inclusion of the symmetric sequence $\mathsf{PaB} \hookrightarrow \mathsf{PaB}^1$ given by $R^{1,2} \mapsto \mathrm{id}_{(01)} \circ_1 R^{1,2}$. Visually, this amounts to adding a frozen strand indexed by zero to the left of $R^{1,2}$, outermost in the parenthesization. The isomorphic image of PaB under this inclusion, and in particular the generators $R^{1,2}$ and $\Phi^{1,2,3}$ are present in any finite presentation of PaB^1 and will be denoted the same.

The two generating morphisms of PaB^1 discussed below enable the "frozen strand" to interact with the other strands. The groupoid $\mathsf{PaB}^1(1)$ has one object (01) and the morphisms $\mathrm{Hom}_{\mathsf{PaB}^1(1)}((01),(01)) \cong \mathsf{B}_1^1 \cong \mathbb{Z}$ are freely generated by a single automorphism $E^{0,1}$, shown in Figure 11. The other generator is the associativity isomorphism in the groupoid $\mathsf{PaB}^1(2)$, denoted $\Psi^{0,1,2} \in \mathrm{Hom}_{\mathsf{PaB}^1(2)}((01)2, 0(12))$ shown in Figure 11.

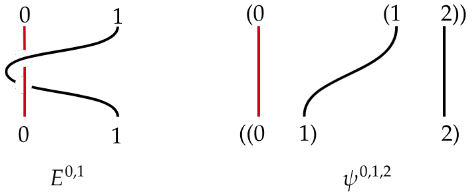


FIGURE 11. The generators of PaB^1 as a PaB-moperad.

Theorem 2.17 ([CG20, Theorem 3.4]). *As a* PaB*-moperad,* PaB^1 *is generated at the object level by* (01)*, and at the level of morphisms by the morphisms* $E^{0,1} \in \mathrm{Hom}_{\mathsf{PaB}^1(1)}((01),(01))$ *and* $\Psi^{0,1,2} \in \mathrm{Hom}_{\mathsf{PaB}^1(2)}((01)2, 0(12))$*, which are subject to the following relations:*

$$\Psi^{0,\emptyset,2} = \Psi^{0,1,\emptyset} = \mathrm{id}_{01} \quad \text{in } \mathrm{Hom}_{\mathsf{PaB}^1(1)}\big(01, 01\big)\,, \tag{U}$$

$$\Psi^{0,1,23}\Psi^{01,2,3} = \Phi^{1,2,3}\Psi^{0,12,3}\Psi^{0,1,2} \quad \text{in } \mathrm{Hom}_{\mathsf{PaB}^1(3)}\big(((01)2)3, 0(1(23))\big)\,, \tag{MP}$$

$$(\Psi^{0,1,2})^{-1}R^{2,1}E^{0,21}R^{1,2}\Psi^{0,1,2} = E^{01,2}E^{0,1} \quad \text{in } \mathrm{Hom}_{\mathsf{PaB}^1(2)}\big((01)2, (01)2\big)\,, \tag{RP}$$

$$E^{01,2} = (\Psi^{0,1,2})^{-1}R^{2,1}\Psi^{0,2,1}E^{0,2}(\Psi^{0,2,1})^{-1}R^{1,2}\Psi^{0,1,2} \quad \text{in } \mathrm{Hom}_{\mathsf{PaB}^1(2)}\big((01)2, (01)2\big)\,. \tag{O}$$

Remark 2.18. Here, again, we use cosimplicial notation to describe monoid and operadic compositions more concisely. For example, $E^{01,2} = E^{01} \circ_0 \mathrm{id}_{(01)}$. The $\emptyset$ notation in relation (U) describes the composition of a braid with the arity zero operation $* \in \mathsf{PaB}(0)$, which has the effect of deleting a strand in the braid.

Remark 2.19. There is a minor difference between our presentation of PaB^1 and that given in Theorem 3.4 of [CG20]. This is because in this paper we work with the *blackboard framing* when defining $E^{0,21}$, and in general when inserting into strands with non-zero winding number around the puncture. The paper [CG20] uses the annular framing, which adds a full twist to $E^{0,21}$ compared to the blackboard framing, see Figure 12. The definition of the operadic composition on the associated graded side is correspondingly slightly different. See Remarks 3.1, 3.6, and 5.1 of [CG20]. As a result, our relation (RP) includes additional half-twists compared to [CG20]. Both versions of the theorem are equivalent. The annular framing is more natural from the perspective of cyclotomic actions; the blackboard framing has easier compatibility with the KV problem.

At each parenthesized permutation $w \in \mathrm{Ob}(\mathsf{PaB}^1(n))$ there an isomorphism

$$\mathrm{Aut}_{\mathsf{PaB}^1(n)}(w) \cong \mathsf{PB}_n^1 \cong \mathsf{F}_n \rtimes \mathsf{PB}_n.$$

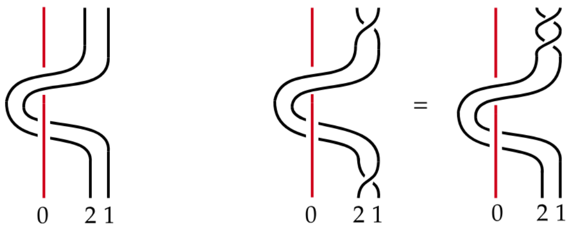


FIGURE 12. On the left, $E^{0,21}$ using the blackboard framing convention. On the right $E^{0,21}$ in the annular framing convention of [CG20].

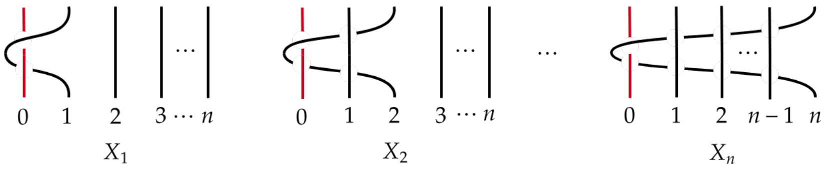


Figure 13. The free group generators $X_1, \ldots, X_n$ in PB^1_n.

Elements of the group PB^1_n, viewed as morphisms, can be expressed in terms of the moperad generators of PaB^1 – of course these expressions are only unique up to the relations in Theorem 2.17. Recall that the free group F_n embeds in PB_n, generated by the elements $X_1, \ldots, X_n$, as in Figure 13. These elements will play an important role, and the following lemma describes how to express them in terms of the generators of PaB^1.

Notation 2.20. For a permutation $\sigma \in \Sigma_n$, let $r_n(\sigma)$ denote the rightmost parenthesization of σ, that is, for $\sigma = (i_1, \ldots, i_n)$, we have $r_n(\sigma) := (i_1(i_2 \ldots (i_{n-1}i_n)))$. Let $r^1_n(\sigma)$ denote $\mathrm{id}_{(01)} \circ_1 r_n(\sigma) \in \mathrm{Ob}(\mathsf{PaB}^1(n))$. Similarly, let $l_n(\sigma)$ denote the leftmost parenthesization of σ: $l_n(\sigma) := (((i_1i_2)i_3 \ldots)i_n)$. Let $l^1_n(\sigma) = \mathrm{id}_{(01)} \circ_1 l_n(\sigma) = (0\,((i_1i_2)i_3 \ldots)\,i_n)$. Finally, let $l^0_n(\sigma)$ denote the altogether leftmost parenthesization of the concatenation 0σ. When σ is the identity permutation, we omit it from the notation, for example, $r^1_n = r^1_n(\mathrm{id})$.

Lemma 2.21. *The free group generators $X_1, \ldots, X_n$, viewed as automorphisms in $\mathrm{Hom}_{\mathsf{PaB}^1(n)}(r^1_n, r^1_n)$, can be expressed in terms of the moperad generators of PaB^1 as*

$$X_k = Q_k P_k(\mathrm{id}_{l^0_n(1\,2 \ldots k-1\,k+1 \ldots n)} \circ_0 E^{0,k}) P_k^{-1} Q_k^{-1},$$

where P_k is the re-association from $l^0_n(k\,1\,2 \ldots k-1\,k+1 \ldots n)$ to $l^1_n(k\,1\,2 \ldots k-1\,k+1 \ldots n)$

$$P_k := \Psi^{0,l_{n-1}(k\,1\,2 \ldots\, k-1\,k+1 \ldots n-1),n} \cdots \Psi^{0,((k\,1)\,2),3} \Psi^{0,(k\,1),2} \Psi^{0,k,1}, \tag{2.12}$$

and Q_k re-associates $l^1_n(k\,1\,2 \ldots\, k-1\,k+1\, \ldots\, n)$ to r^1_n, while crossing strand k under strands 1 through $k-1$:

$$\begin{aligned} Q_k := {} & \Phi^{n-2,n-1,n} \cdots (\Phi^{k-1,k,k+1} \cdots \Phi^{2,(3(4\ldots(k-1\ k))),k+1} \Phi^{1,(2(3\ldots(k-1\ k))),k+1}) \cdot \\ & (R^{k-1,k})^{-1} \cdots ((R^{4,k})^{-1} \Phi^{3,k,4} \Phi^{2,(3\ k),4} \Phi^{1,2(3\ k),4})((R^{3,k})^{-1} \Phi^{2,k,3} \Phi^{1,(2\ k),3})((R^{2,k})^{-1} \Phi^{1,k,2})(R^{1,k})^{-1} \end{aligned} \tag{2.13}$$

Proof. The equality $X_k = Q_k P_k(\mathrm{id}_{l^0_n(1\,2 \ldots k-1\,k+1 \ldots n)} \circ_0 E^{0,k}) P_k^{-1} Q_k^{-1}$ is between elements of $\mathrm{Hom}_{\mathsf{PaB}^1(n)}(r^1_n, r^1_n) \cong \mathsf{PB}^1_n$. In other words, one needs to check two statements:

(1) the morphisms on the right hand side, as given by (2.12) and (2.13), form a valid composition in PaB^1, and

(2) the two expressions map to the same (non-parenthesized) braids in PB^1_n via the isomorphism $\mathrm{Hom}_{\mathsf{PaB}^1(n)}(r^1_n, r^1_n) \cong \mathsf{PB}^1_n$.

Statement (1) is a straightforward combinatorial verification, illustrated in Figure 14.

To verify (2), we use that associators are trivial braids, so the equatlity simplifies as follows:

$$X_k = (R^{k-1,k})^{-1} (R^{k-2,k})^{-1} \cdots (R^{1,k})^{-1} E^{0,k} R^{1,k} \cdots R^{k-2,k} R^{k-1,k}.$$

Indeed, this is the definition of X_k in B^1_n, noticing that $E^{0,k}$ maps to X_1 under the isomorphism with the pure braid group. □

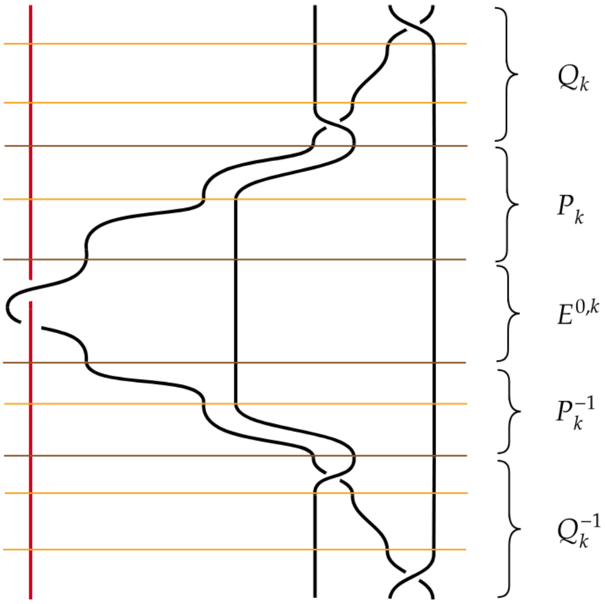


FIGURE 14. The expression of X_3 in PaB^1_3 in terms of the generators.

2.4. **Associators and moperad expansions.** The isomorphism $\mathsf{PB}^1_n \cong \mathsf{PB}_{n+1}$ is mirrored, on the graded side, by the isomorphism of Lie algebras $\mathfrak{t}^1_n \cong \mathfrak{t}_{n+1}$ which is defined by reindexing the generators of the Drinfeld–Kohno Lie algebra (discussed in Section 1.4). That is, $\mathfrak{t}^1_n$ is the Lie algebra generated by $\{t_{ij} \,|\, 0 \leq i, j \leq n, i \neq j\}$. In fact, at the level of Lie algebras, we have $\mathfrak{t}^+_n = \mathfrak{t}^1_n$, using the shift notation introduced in Example 2.5.

Definition 2.22. Recall that we write Σ^+_n for the group of automorphisms of the set $\{0, 1, \ldots, n\}$ and $\Sigma_n \subset \Sigma^+_n$ is the subgroup of those automorphisms $\sigma \in \Sigma^+_n$ with $\sigma(0) = 0$. The symmetric group Σ_n acts on the Lie algebra $\mathfrak{t}^+_n$ via the formula $(t_{ij})^\sigma := t_{\sigma^{-1}(i)\sigma^{-1}(j)}$, for $\sigma \in \Sigma_n \subset \Sigma^+_n$. The resulting symmetric sequence $\mathfrak{t}^+ = \{\mathfrak{t}^+_n\}_{n \geq 0}$ forms a $\mathfrak{t}$-moperad in (degree completed) Lie algebras in which the monoidal product $\mathfrak{t}^+_n \circ_0 \mathfrak{t}^+_m$ and the right $\mathfrak{t}$-action, $\mathfrak{t}^+_n \circ_i \mathfrak{t}_m$, $1 \leq i \leq m$, are given by the operadic composition in $\mathfrak{t}$, as in Definition 1.17.

Similarly, if we write $\mathsf{CD}^+(n) = \exp(\mathfrak{t}^+_n) \cong \exp(\mathfrak{t}_{n+1})$, we can define the CD-*moperad of chord diagrams* $\mathsf{CD}^+ := \{\mathsf{CD}^+(n)\}$ as the CD-moperad in prounipotent groups in which the monoidal product and the right CD-action are defined via the shifted $\mathfrak{t}$-moperad $\mathfrak{t}^+$,

$$e^x \circ_0 e^y := e^{x \circ_0 y} \quad \text{and} \quad e^x \circ_i e^y := e^{x \circ_i y}.$$

As in the operadic description of associators (Theorem 1.23), we use the presentation of PaB^1 as a PaB-moperad to classify moperad equivalences $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \longrightarrow \mathsf{CD}^+$. Such equivalences induce unique homomorphic expansions valued in PaCD^+, this is similar to the argument in [Fre17, Theorem 10.3.12]). The following is a simplified version of Theorem 5.5 in [CG20] where the authors provide a moperadic description of Enriquez's cyclotomic associators.

Theorem 2.23. *A moperad equivalence*

$$(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \longrightarrow \mathsf{CD}^+$$

is uniquely determined by the values

$$\varphi(R^{1,2}) = e^{\mu t_{12}/2} \in \exp(\mathfrak{t}_2) \quad \textit{and} \quad \varphi(\Phi^{1,2,3}) = f(t_{12}, t_{23}) \in \exp(\mathfrak{t}_3)$$

which define an operad equivalence $\mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$ *(and a Drinfeld associator), together with the values*

$$\varphi^1(E^{0,1}) = e^{\mu t_{01}} \in \exp(\mathfrak{t}^+_1), \quad \textit{and} \quad \varphi^1(\Psi^{0,1,2}) = g(t_{01}, t_{12}) \in \exp(\mathfrak{t}^+_2)$$

satisfying the equations:

$$\text{(MP)}\qquad g(t_{01},t_{12}+t_{13})\cdot g(t_{02}+t_{12},t_{23})=f(t_{12},t_{23})\cdot g(t_{01}+t_{02},t_{13}+t_{23})\cdot g(t_{01},t_{12})\qquad \textit{in } \exp(\mathfrak{t}_3^+),$$

$$\text{(O)}\qquad g(t_{01},t_{12})^{-1}\cdot e^{\mu t_{12}/2}\cdot g(t_{02},t_{12})\cdot e^{\mu t_{02}}\cdot g(t_{02},t_{12})^{-1}\cdot e^{\frac{\mu t_{12}}{2}}\cdot g(t_{01},t_{12})\cdot e^{\mu t_{01}}=1\qquad \textit{in } \exp(\mathfrak{t}_2^+).$$

Every moperad equivalence $(\varphi^1,\varphi):\mathsf{PaB}^1_{\mathbb{K}}\longrightarrow\mathsf{CD}^+$ is relative to an equivalence of operads $\varphi:\mathsf{PaB}_{\mathbb{K}}\to\mathsf{CD}$, which is indexed by a Drinfeld associator. Conversely, we now show that any equivalence of operads $\varphi:\mathsf{PaB}_{\mathbb{K}}\to\mathsf{CD}$ gives rise to a one-parameter family of equivalences of moperads $(\varphi^1,\varphi):\mathsf{PaB}^1_{\mathbb{K}}\longrightarrow\mathsf{CD}^+$. The first ingredient is the following moperad inclusion.

Lemma 2.24. *The group inclusion* $\mathsf{B}^1_n\hookrightarrow\mathsf{B}_{n+1}$ *extends to an injective map of* PaB*-moperads*

$$(\rho^1,\mathrm{id}):\mathsf{PaB}^1\hookrightarrow\mathsf{PaB}^+.$$

Proof. We define the map of PaB-moperads $(\rho^1,\mathrm{id}):\mathsf{PaB}^1\to\mathsf{PaB}^+$ as the identity on objects. Using the presentation of the PaB-moperad PaB^1 in Theorem 2.17 and the description of the subgroup inclusion $\mathsf{B}^1_n\hookrightarrow\mathsf{B}_{n+1}$ from Proposition 2.12 we define

$$\rho^1(E^{0,1}):=R^{1,0}R^{0,1}\quad\text{and}\quad\rho^1(\Psi^{0,1,2}):=\Phi^{0,1,2}$$

in $\mathrm{Hom}_{\mathsf{PaB}^+(1)}((01),(01))$ and $\mathrm{Hom}_{\mathsf{PaB}^+(2)}((01)2,0(12))$, respectively. The morphism $R^{0,1}$ in $\mathsf{PaB}^+(1)$ is identified with the morphism $R^{1,2}$ in $\mathsf{PaB}(2)$ and the morphism $\Phi^{0,1,2}$ in $\mathsf{PaB}^+(2)$ is identified with $\Phi^{1,2,3}$ in $\mathsf{PaB}(3)$.

To check that these assignments give a well-defined map of moperads , we note that the mixed pentagon relation (MP) translates to the pentagon equation (P) under the map ρ^1. Similarly, the equations (O) and (RP) translate to composites of the hexagon equations. More precisely, using (H1) and (H2) we have

$$\begin{aligned}\rho^1\left(E^{01,2}\right)=(R^{2,01})(R^{01,2})&=\left((\Phi^{0,1,2})^{-1}R^{2,1}\Phi^{0,2,1}R^{2,0}(\Phi^{2,0,1})^{-1}\right)\left(\Phi^{2,0,1}R^{0,2}(\Phi^{0,2,1})^{-1}R^{1,2}\Phi^{0,1,2}\right)\\&=(\Phi^{0,1,2})^{-1}R^{2,1}\Phi^{0,2,1}R^{2,0}R^{0,2}(\Phi^{0,2,1})^{-1}R^{1,2}\Phi^{0,1,2}\\&=\rho^1\left((\Psi^{0,1,2})^{-1}R^{2,1}\Psi^{0,2,1}E^{0,2}(\Psi^{0,2,1})^{-1}R^{1,2}\Psi^{0,1,2})\right),\end{aligned}$$

which shows that ρ^1 preserves the relation (O) from Theorem 2.17. To show that ρ^1 preserves equation (RP), using first the braid relations, we compute:

$$\begin{aligned}\rho^1\left(E^{01,2}E^{0,1}\right)&=R^{2,01}R^{01,2}R^{1,0}R^{0,1}=R^{2,01}R^{1,0}R^{0,1}R^{01,2}\\&=\underbrace{(\Phi^{0,1,2})^{-1}R^{2,1}\Phi^{0,2,1}R^{2,0}(\Phi^{2,0,1})^{-1}}_{\text{(H1)}}R^{1,0}R^{0,1}\underbrace{\Phi^{2,0,1}R^{0,2}(\Phi^{0,2,1})^{-1}R^{1,2}\Phi^{0,1,2}}_{\text{(H2)}}\\&=(\Phi^{0,1,2})^{-1}R^{2,1}\underbrace{\Phi^{0,2,1}R^{2,0}(\Phi^{2,0,1})^{-1}R^{1,0}\Phi^{2,1,0}}_{\text{(H2)}}\underbrace{(\Phi^{2,1,0})^{-1}R^{0,1}\Phi^{2,0,1}R^{0,2}(\Phi^{0,2,1})^{-1}}_{\text{(H1)}}R^{1,2}\Phi^{0,1,2}\\&=(\Phi^{0,1,2})^{-1}R^{2,1}R^{21,0}R^{0,21}R^{1,2}\Phi^{0,1,2}=\rho^1\left((\Psi^{0,1,2})^{-1}R^{2,1}E^{0,21}R^{1,2}\Psi^{0,1,2}\right).\end{aligned}$$

The lemma follows. □

Remark 2.25. We note that the moperad PaB^1 is not equivalent to the shifted moperad PaB^+. In particular, the morphism $R^{0,1}:(01)\to(10)$ does not exist in PaB^1, but is a morphism in PaB^+.

Proposition 2.26. *Every equivalence of operads* $\varphi:\mathsf{PaB}_{\mathbb{K}}\to\mathsf{CD}$ *extends to an equivalence of moperads* $(\varphi^1,\varphi):\mathsf{PaB}^1_{\mathbb{K}}\longrightarrow\mathsf{CD}^+$*.*

Proof. Given an equivalence of operads $\varphi:\mathsf{PaB}_{\mathbb{K}}\longrightarrow\mathsf{CD}$ we define an equivalence of moperads $(\varphi^+,\varphi):\mathsf{PaB}^+_{\mathbb{K}}\to\mathsf{CD}^+$ using the shift construction. Now, using the moperad inclusion from Lemma 2.24, we consider the composite

$$\begin{tikzcd}\mathsf{PaB}^1_{\mathbb{K}} \arrow[r, "\rho^1"] \arrow[rr, dashed, bend right, "\varphi^1"'] & \mathsf{PaB}^+_{\mathbb{K}} \arrow[r, "\varphi^+"] & \mathsf{CD}^+.\end{tikzcd}$$

We claim that $(\varphi^1, \varphi) = (\varphi^+, \varphi) \circ (\rho^1, \mathrm{id})$ defines an equivalence of moperads. Since, at the level of operads, we have $\varphi \circ \mathrm{id} = \varphi : \mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$, it remains to check that the values of $\varphi^1(E^{0,1})$ and $\varphi^1(\Psi^{0,1,2})$ satisfy the equations of Theorem 2.23. Since, the moperad map $(\rho^1, \mathrm{id}) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaB}^+_{\mathbb{K}}$ sends the associativity isomorphism $\Psi^{0,1,2}$ to the shift of the associativity isomorphism $\Phi^{1,2,3}$, we know that

$$\varphi^1(\Psi^{0,1,2}) = \varphi^+(\Phi^{0,1,2}) = f(t_{01}, t_{12}) \in \exp(\mathfrak{t}_2^+).$$

It follows immediately that $\varphi^1(\Psi^{0,1,2})$ satisfies (MP). Moreover, using the hexagon equation (H'), we have

$$\varphi^1(E^{0,1}) = \varphi^+(R^{1,0})\varphi^+(R^{0,1}) = e^{\frac{\mu t_{10}}{2}} e^{\frac{\mu t_{01}}{2}} = e^{\mu t_{01}} \in \exp(\mathfrak{t}_1^+).$$

This satisfies the equation

$$\begin{aligned} e^{\mu t_{01}} &= e^{\frac{\mu t_{01}}{2}} e^{\frac{\mu t_{10}}{2}} \\ &= \underbrace{f(t_{01}, t_{12})^{-1} e^{\frac{-\mu t_{12}}{2}} f(t_{12}, t_{02})^{-1} e^{\frac{-\mu t_{02}}{2}} f(t_{02}, t_{01})^{-1}}_{\text{(H')}} \cdot \underbrace{f(t_{01}, t_{02})^{-1} e^{\frac{-\mu t_{02}}{2}} f(t_{02}, t_{12})^{-1} e^{\frac{-\mu t_{12}}{2}} f(t_{12}, t_{01})^{-1}}_{\text{(H')}} \\ &= f(t_{01}, t_{12})^{-1} e^{\frac{-\mu t_{12}}{2}} f(t_{12}, t_{02})^{-1} e^{-\mu t_{02}} f(t_{02}, t_{12})^{-1} e^{\frac{-\mu t_{12}}{2}} f(t_{12}, t_{01})^{-1}. \end{aligned}$$

Replacing $f(t_{12}, t_{01})^{-1} = f(t_{01}, t_{12})$, and $f(t_{12}, t_{02})^{-1} = f(t_{02}, t_{12})$, we obtain that $\varphi^1(E^{0,1})$ satisfies the second defining equation (O) of a moperad equivalence. It follows that $(\varphi^1, \varphi) = (\varphi^+, \varphi) \circ (\rho^1, \mathrm{id})$ defines an equivalence of moperads. □

In fact, given an equivalence of operads $\varphi : \mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$, there is only a 1-parameter family of moperad equivalences $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ extending φ. Indeed, under the assumptions of Theorem 2.23, once $\varphi(\Phi^{1,2,3})$ has been fixed, there is only a 1-parameter family of choices for $\varphi^1(\Psi^{0,1,2})$, as we discuss below. This observation was made in [Enr07, Section 5] for a specialized setting in the context of cyclotomic associators, and communicated to the authors by Benjamin Enriquez.

Proposition 2.27. *If $f(x, y), g(x, y) \in \exp(\mathfrak{lie}_2)$ satisfy equation (MP) from Theorem 2.23, then $g(x, y) = e^{sy} f(x, y)$ for some $s \in \mathbb{K}$.*

Proof. Consider the projection $\pi_0 : \exp(\mathfrak{t}_3^+) \to \exp(\mathfrak{t}_3)$ defined by, for any pair $i < j$, if $i > 0$ then $t_{ij} \mapsto t_{ij}$, and if $i = 0$ then $t_{0j} \mapsto 0$. Applying π_0 to equation (MP), we get

$$g(0, t_{12} + t_{13}) \cdot g(t_{12}, t_{23}) = f(t_{12}, t_{23}) \cdot g(0, t_{13} + t_{23}) \cdot g(0, t_{12}).$$

Using the fact that $g(0, y) = e^{sy}$ for some $s \in \mathbb{K}$, we rewrite this equation as follows:

$$e^{s(t_{12}+t_{13})} \cdot g(t_{12}, t_{23}) = f(t_{12}, t_{23}) \cdot e^{s(t_{13}+t_{23})} \cdot e^{st_{12}}$$

Since $z = t_{12} + t_{13} + t_{23}$ is central in $\mathfrak{t}_3$, we have, $e^{s(z-t_{23})} \cdot g(t_{12}, t_{23}) = f(t_{12}, t_{23}) \cdot e^{s(z-t_{12})} \cdot e^{st_{12}}$ which can be simplified to

$$g(t_{12}, t_{23}) = e^{st_{23}} \cdot f(t_{12}, t_{23}),$$

completing the proof. □

As an immediate consequence, we have the following:

Corollary 2.28. *Let us denote by* Associators *the set of operad equivalences* $\mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$*, and by* Associators^1 *the set of moperad equivalences* $\mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$*. Then, we have an isomorphism of bitorsors*

$$\text{Associators}^1 \cong \text{Associators} \times \mathbb{K}.$$

The following lemma will help establish a crucial link to the Kashiwara–Vergne equations:

Lemma 2.29. *In* $\mathrm{Hom}_{\mathsf{PaB}^1(2)}\big(0(12), 0(12)\big)$*, we have the equality*

$$X_2 \circ X_1 = E^{0,12}. \tag{2.14}$$

Proof. Note that $\mathrm{Hom}_{\mathsf{PaB}^1(2)}\Big((0(12)),(0(12))\Big) \cong \mathsf{PB}_2^1 \subseteq \mathsf{B}_2^1$. Thus, to prove that two morphisms are equal in $\mathrm{Hom}_{\mathsf{PaB}^1(2)}\Big((0(12)),(0(12)\Big)$, it is enough to prove that they are equal as braids in B_2^1. For an intuitive proof, there is a homotopy of loops between the two sides, illustrated in Figure 15. For an algebraic proof, one can use the inclusion $\mathsf{PaB}^1 \hookrightarrow \mathsf{PaB}^+$ from Lemma 2.24 to apply a sequence of relations in B_3, with strands and generators re-indexed 0 to 2:

$$X_2X_1 = \beta_1^{-1}\beta_0\beta_0\beta_1\beta_0\beta_0 = \beta_1^{-1}\beta_0\beta_1\beta_0\beta_1\beta_0 = \beta_1^{-1}\beta_1\beta_0\beta_1\beta_1\beta_0 = \beta_0\beta_1^2\beta_0 = E^{0,12}$$

□

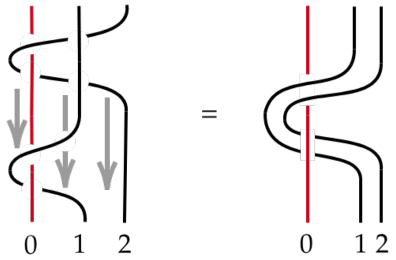


FIGURE 15. The loop homotopy showing that $X_2X_1 = E^{0,12}$.

2.5. **A variation on the Grothendieck–Teichmüller groups.** The prounipotent Grothendieck–Teichmüller group GT is defined by pairs (λ, f) satisfying Drinfeld's duality, hexagon, and pentagon relations (Definition 1.24). Such a pair induces an automorphism of the free group F_2 via

$$x \mapsto f(x,y)x^{\lambda}f(x,y)^{-1} \qquad \text{and} \qquad y \mapsto y^{\lambda}.$$

In the discrete setting, these relations are highly rigid: before completion, Drinfeld [Dri90, Proposition 4.1] shows that if $\lambda \in 2\mathbb{Z}+1$ and $f \in F_2$, then necessarily $(\lambda, f) = (1,1)$ or $(-1,1)$ and $\mathsf{GT} \cong \mathbb{Z}/2\mathbb{Z}$. After prounipotent completion, however, GT becomes the infinite dimensional prounipotent group from Definition 1.24.

Enriquez introduced a cyclotomic analogue of the Grothendieck–Teichmüller group by adjoining an additional module-type datum to the pair (λ, f). This Grothendieck–Teichmüller module group, GTM, is defined by quadruples (λ, μ, f, g) satisfying the usual Grothendieck–Teichmüller relations on (λ, f), together with an octagon relation and a mixed pentagon relation. As observed just before Proposition 5.3 of [Enr07], the octagon relation implies that (λ, μ, g) determines an automorphism of F_2 via

$$x \mapsto g(x,y)x^{\mu}g(x,y)^{-1} \qquad \text{and} \qquad y \mapsto y^{\lambda},$$

sending $x^{-1}y^{-1}$ to a conjugate of $(x^{-1}y^{-1})^{\mu}$. Furthermore, the scalar parameters are forced to agree, so that $\mu = \lambda$. One can therefore deduce the following definition from the prounipotent version of [Enr07, Proposition 5.4].

Definition 2.30. The group GTM consists of invertible elements $(\lambda, f, g) \in \mathbb{K}^{\times} \times (\mathsf{F}_2)_{\mathbb{K}} \times (\mathsf{F}_2)_{\mathbb{K}}$ such that $(\lambda, f) \in \mathsf{GT}$ and $g \in (\mathsf{F}_2)_{\mathbb{K}}$ satisfy the following equations:

$$\text{(MP)} \qquad g(x_{01}, x_{13}x_{12})g(x_{12}x_{02}, x_{23}) = f(x_{12}, x_{23})g(x_{02}x_{01}, x_{23}x_{13})g(x_{01}, x_{12}) \quad \text{in } (\mathrm{PB}_3^1)_{\mathbb{K}}.$$

$$\text{(O)} \qquad g(x,y)^{-1}y^{\frac{\lambda-1}{2}}g(z,y)z^{\nu}g(z,y)^{-1}y^{\frac{\lambda+1}{2}}g(x,y)x^{\nu} = 1 \quad \text{in } (\mathsf{F}_2)_{\mathbb{K}}, \quad \text{with } zyx = 1,\ \lambda = \nu + 1.$$

The product of two elements $(\lambda_1, f_1, g_1), (\lambda_2, f_2, g_2) \in \mathsf{GTM}$ is given by

$$(\lambda_1, f_1, g_1) * (\lambda_2, f_2, g_2) := (\lambda, f, g), \tag{2.15}$$

where $\lambda := \lambda_1\lambda_2$, $f := f_1 \cdot f_2(x_1^{\lambda_1}, f_1^{-1}x_2^{\lambda_1}f_1)$ and $g := g_1 \cdot g_2(x_1^{\nu_1}, g_1^{-1}x_2^{\lambda_1}g_1)$ with $\lambda_i = \nu_i + 1$ for $i = 1, 2$.

Enriquez further observes that the data of the tuple (λ, f, g) is essentially determined by the tuple (λ, f). Indeed, following Drinfeld, Enriquez observes in [Enr07, Proposition 5.3] that, in the discrete setting this additional datum is again rigid and forces $f = 1$, $\lambda = \pm 1$, and $g(x, y) = y^s$, for a unique integer s and so $\mathsf{GTM} \simeq \mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. The following is the prounipotent version of [Enr07, Proposition 5.4].

Proposition 2.31. *Every element* $(\lambda, f, g) \in \mathsf{GTM}$ *can be uniquely written in the form*

$$(\lambda, f, g) = (\lambda, f, y^s f)$$

for some $(\lambda, f) \in \mathsf{GT}$ *and some* $s \in \mathbb{K}$*. Equivalently, there is an isomorphism of prounipotent groups*

$$\mathsf{GTM} \cong \mathsf{GT} \ltimes \mathbb{K}.$$

Remark 2.32. The semidirect product description $\mathsf{GTM} \cong \mathsf{GT} \ltimes \mathbb{K}$ suggests that GTM should play a role analogous to that of a “module-valued” enhancement of GT. This invites comparison with the conjectural description of the symmetric Kashiwara–Vergne group as an extension of the Grothendieck–Teichmüller group by $\mathbb{K}$. We return to this perspective in a later section, where in Theorem 4.19 we construct an injective group homomorphism

$$\mathsf{GTM} \hookrightarrow \mathsf{KV}^{\mathrm{sym}}(2).$$

Every element $(\lambda, f, g) \in \mathsf{GTM}$ induces a moperad automorphism $(\vartheta^1, \vartheta) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaB}^1_{\mathbb{K}}$ acting trivially on objects. The operad component $\vartheta : \mathsf{PaB}_{\mathbb{K}} \to \mathsf{PaB}_{\mathbb{K}}$ is the automorphism corresponding to the Grothendieck–Teichmüller element $(\lambda, f) \in \mathsf{GT}$, determined by

$$\vartheta(R^{1,2}) = x_{12}^{\frac{\lambda-1}{2}} \cdot R^{1,2} \qquad \text{and} \qquad \vartheta(\Phi^{1,2,3}) = \Phi^{1,2,3} \cdot f(x_{12}, x_{23}),$$

and these values satisfy the pentagon and hexagon relations of Theorem 1.8. To extend ϑ to the moperad $\mathsf{PaB}^1_{\mathbb{K}}$, one sets

$$\vartheta^1(E^{0,1}) = (E^{0,1})^{\lambda} \qquad \text{and} \qquad \vartheta^1(\Psi^{0,1,2}) = \Psi^{0,1,2} \cdot g(x_{01}, x_{12}).$$

The mixed pentagon equation (MP) and octagon equation (O) are precisely the relations needed for these assignments to define a moperad automorphism; see [CG20, Proposition 4.8].

The following theorem is the special case of [CG20, Theorem 5.13] in which the group Γ is taken to be trivial. Its proof ultimately builds on [Fre17, Theorem 11.1.7].

Theorem 2.33. *Let* $\mathrm{Aut}_0(\mathsf{PaB}^1_{\mathbb{K}})$ *denote the group of moperad automorphisms*

$$(\vartheta^1, \vartheta) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaB}^1_{\mathbb{K}}$$

that act as the identity on objects. Then there is an isomorphism of prounipotent groups

$$\mathrm{Aut}_0(\mathsf{PaB}^1_{\mathbb{K}}) \cong \mathsf{GTM}.$$

Via this identification, the group GTM acts freely and transitively on the set of moperad equivalences:

$$(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \longrightarrow \mathsf{CD}^+,$$

by precomposition. Concretely, by Theorem 2.23, such a moperad equivalence is uniquely determined by a triple (μ, f, g), and the action of an element $(\lambda, f', g') \in \mathsf{GTM}$ is given by

$$(\mu, f, g) * (\lambda, f', g') := \big(\mu\lambda,\ f \cdot f'(e^{\mu\xi_1}, f^{-1}e^{\mu\xi_2} f),\ g \cdot g'(e^{\mu\xi_1}, g^{-1}e^{\lambda\xi_2} g)\big). \tag{2.16}$$

This picture admits a graded counterpart. Every moperad equivalence $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \longrightarrow \mathsf{CD}^+$ extends uniquely to a homomorphic expansion, that is, to a moperad isomorphism $(\tilde{\varphi}^1, \tilde{\varphi}) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaCD}^+$ acting trivially on objects. Accordingly, one obtains the graded cyclotomic Grothendieck–Teichmüller module group

$$\mathsf{GRTM} \cong \mathrm{Aut}_0(\mathsf{PaCD}^+),$$

defined as the group of object-fixing automorphisms of the shifted moperad of parenthesized chord diagrams. This graded group was introduced and studied by Enriquez; see [Enr07, Section 7]. For further discussion of GRTM and related constructions, see [EF12] or [CG20].

3. Genus Zero Kashiwara–Vergne Solutions

This section introduces the necessary background material on genus zero Kashiwara–Vergne (KV) solutions, as developed in the work of Alekseev, Kawazumi, Kuno, and Naef [AKKN18b, AKKN18a]. In particular, we describe tangential and special automorphisms of free Lie algebras, and a moperad of tangential automorphisms, which provides a natural framework in which to study KV solutions. In particular, the moperad of tangential automorphisms mirrors the moperad arising from splittings of the pure braid groups and the braid groups with frozen strands described in the previous section.

Throughout, we write $\mathfrak{ass}_n := \mathbb{K}\langle\langle x_1, \ldots, x_n\rangle\rangle$ for the degree completed *free associative algebra* generated by the symbols $x_1, \ldots, x_n$. The degree completed *free Lie algebra*, $\mathfrak{lie}_n$, is the Lie subalgebra of $\mathfrak{ass}_n$ generated by $x_1, \ldots, x_n$ and with Lie bracket given by the commutator, $[f, g] = fg - gf$. There is a natural identification of the completed universal enveloping algebra of $\mathfrak{lie}_n$ with $\mathfrak{ass}_n$, that is, $\widehat{U}(\mathfrak{lie}_n) = \mathfrak{ass}_n$. We identify the subset of group-like elements in the completed Hopf algebra $\widehat{U}(\mathfrak{lie}_n) = \mathfrak{ass}_n$ with the group $\exp(\mathfrak{lie}_n) \cong \widehat{\mathsf{F}}_n$.

Definition 3.1. A *tangential derivation* of $\mathfrak{lie}_n$ is a derivation u of $\mathfrak{lie}_n$ which acts on the generators x_i by $u(x_i) = [x_i, a_i]$, for some $a_i \in \mathfrak{lie}_n$. A *special derivation* of $\mathfrak{lie}_n$ is a tangential derivation u for which $u(\Sigma_{i=1}^n x_i) = 0$.

The collection of all tangential derivations of $\mathfrak{lie}_n$ forms a Lie algebra which we denote by $\mathfrak{tder}_n$, where the bracket $[u, v]$ is given by $[u, v](x_k) := u(v(x_k)) - v(u(x_k))$ (see [AT12, Proposition 3.4]). There is an isomorphism of vector spaces $\mathfrak{tder}_n \oplus \mathfrak{a}_n \cong \mathfrak{lie}_n^{\oplus n}$ where $\mathfrak{a}_n$ is the n-dimensional abelian Lie algebra generated by $x_1, \ldots, x_n$. As such, we often write tangential derivations as a tuple of Lie words $u := (a_1, \ldots, a_n)$ in $\mathfrak{lie}_n^{\oplus n}$ ([AT12, Definition 3.2]). In this notation, if $u = (a_1, ..., a_n)$ and $v = (b_1, ..., b_n)$, then $([u, v])_k = [a_k, b_k] + u(b_k) - v(a_k)$. The vector space of special derivations $\mathfrak{sder}_n$ is closed under the bracket of $\mathfrak{tder}_n$ and thus forms a Lie subalgebra of $\mathfrak{tder}_n$.

Definition 3.2. The group of *tangential automorphisms* is the prounipotent group $\mathrm{TAut}_n := \exp(\mathfrak{tder}_n)$ associated to the Lie algebra $\mathfrak{tder}_n$. The group of *special automorphisms*, denoted $\mathrm{SAut}_n := \exp(\mathfrak{sder}_n)$ is exponentiated from the Lie algebra $\mathfrak{sder}_n$.

As with tangential derivations, we can identify (as a set) elements of TAut_n with $(\exp(\mathfrak{lie}_n))^{\times n}$, and write $F \in \mathrm{TAut}_n$ as a tuple $F = (f_1, f_2, \ldots, f_n)$, where each $f_i \in \exp(\mathfrak{lie}_n)$. There is an action of TAut_n on $\exp(\mathfrak{lie}_n)$ given by the map $\rho : \mathrm{TAut}_n \to \mathrm{Aut}(\exp(\mathfrak{lie}_n))$ defined on the generators x_i by

$$x_i \mapsto f_i^{-1} x_i f_i.$$

The group law on TAut_n is then given by

$$(F \cdot G)_i := f_i(\rho(F) g_i), \tag{3.1}$$

where $F \cdot G$ stands for the product in TAut_n and $f_i(\rho(F)g_i)$ is calculated in $\exp(\mathfrak{lie}_n)$.

Definition 3.3. The complete, graded vector space of *cyclic words* is the linear quotient

$$\mathrm{cyc}_n := \mathfrak{ass}_n / [\mathfrak{ass}_n, \mathfrak{ass}_n] = \mathfrak{ass}_n / \langle ab - ba \mid \forall\, a, b \in \mathfrak{ass}_n\rangle .$$

We denote the natural projection map by $\mathrm{tr} : \mathfrak{ass}_n \to \mathrm{cyc}_n$.

The adjoint action of $\mathfrak{tder}_n$ on $\mathfrak{lie}_n$ extends to an action of $\mathfrak{tder}_n$ on the vector space $\mathfrak{ass}_n$ by the Leibniz rule. This, in turn, descends to cyc_n. The non-commutative divergence map gives another relationship between $\mathfrak{tder}_n$ and cyc_n. To define it, we note that each element $a \in \mathfrak{ass}_n$ has a unique decomposition

$$a = a_0 + \partial_1(a)x_1 + \ldots + \partial_n(a)x_n = a_0 + \sum_{i=1}^{n} \partial_i(a)x_i$$

for some $a_0 \in \mathbb{K}$ and $\partial_i(a) \in \mathfrak{ass}_n$ for $1 \le i \le n$. In practice, ∂_i picks the words of a sum that end in x_i and deletes their last letter x_i, as well as all other words. This enables the definition of the non-commutative *divergence*, a 1-cocycle of $\mathfrak{tder}_n$ ([AT12, Proposition 3.20]):

Definition 3.4. The non-commutative *divergence* is the linear map $j : \mathfrak{tder}_n \to \mathrm{cyc}_n$ defined on a tangential derivation $u = (a_1, \dots, a_n)$ by

$$j(u) := \mathrm{tr}\left(\sum_{i=1}^{n} \partial_i(a_i)x_i\right).$$

The divergence map is a 1-cocycle of the Lie algebra $\mathfrak{tder}_n$. In particular, for any pair u, v of tangential derivations, we have $j([u,v]) = u \cdot j(v) - v \cdot j(u)$. Here we write $\cdot$ for the natural action of $\mathfrak{tder}_n$ on cyc_n. Integrating the divergence cocycle leads to the non-commutative Jacobian map:

Definition 3.5. The non-commutative *Jacobian* is the map $J : \mathrm{TAut}_n \to \mathrm{cyc}_n$ given by setting

$$J(1) = 0 \quad \text{and} \quad \frac{d}{dt}\Big|_{t=0} J(e^{tu}g) = j(u) + u \cdot J(g)$$

for $g \in \mathrm{TAut}_n$ and $u \in \mathfrak{tder}_n$.

The map J is an additive group 1-cocycle, that is, for any $G, H \in \mathrm{TAut}_n$, we have $J(G \cdot H) = J(G) + G \cdot J(H)$.

Remark 3.6. Our notation (j, J) matches that of [AET10], and corresponds to (div, j) in [AT12].

3.1. Operadic structures on $\mathfrak{lie}$, $\mathfrak{tder}$, $\mathfrak{sder}$, TAut, SAut.

In this section, we define operadic structures on the spaces of tangential and special derivations, and the corresponding groups of tangential and special automorphisms. We summarize the main constructions and results below; detailed proofs and illustrative examples are provided in Appendix A.

We make particular note that while the collection of tangential derivations $\mathfrak{tder} = \{\mathfrak{tder}_n\}$ only assembles into an operad in vector spaces, the collection of special derivations $\mathfrak{sder} = \{\mathfrak{sder}_n\}$ assembles into an operad in Lie algebras. Moreover, in Theorem 3.38, we show that the operadic composition of genus zero KV solutions yields another KV solution, making them a colored operad in sets.

We begin by defining a linear operad structure on the free Lie algebra on n generators:

Definition 3.7. There is a natural right Σ_n action on $\mathfrak{lie}_n$ given by permuting the variables of a Lie word $f = f(x_1, \dots, x_n) \in \mathfrak{lie}_n$. In symbols, for $\sigma \in \Sigma_n$ and $f \in \mathfrak{lie}_n$ we define

$$f^\sigma(x_1, \dots, x_n) := f(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}).$$

Moreover, for $1 \le i \le m$ the partial composition map $\circ_i : \mathfrak{lie}_m \oplus \mathfrak{lie}_n \to \mathfrak{lie}_{m+n-1}$ is defined for $f \in \mathfrak{lie}_m$ and $g \in \mathfrak{lie}_n$ by

$$(f \circ_i g)(x_1, \dots, x_{m+n-1}) := f(x_1, \dots, x_{i-1}, \sum_{j=i}^{i+n-1} x_j, x_{i+n}, \dots, x_{m+n-1}) + g(x_i, \dots, x_{i+n-1}).$$

We denote by $\mathfrak{lie}$ the symmetric sequence of free Lie algebras with these partial compositions, and call this the $\mathbb{K}$-*linear operad of free Lie algebras*, see Proposition 3.8.

Proposition 3.8. *The symmetric sequence of free Lie algebras* $\mathfrak{lie}$ *with the partial compositions above forms a linear operad. The same formulas endow the universal enveloping algebra* $\mathfrak{ass}$ *and the vector space of cyclic words* cyc *with a linear operad structure, such that the quotient map* $\mathrm{tr} : \mathfrak{ass} \to \mathrm{cyc}$ *is a map of linear operads.*

The proof is deferred to Appendix A.

Remark 3.9. Notably $\mathfrak{lie}$ is not an operad in Lie algebras, and as such, the operadic composition is non-canonical: the sum $\sum_{j=i}^{i+n-1} x_j$ could be replaced with any associative Lie expression in the same variables x_j, such as the Baker–Campbell–Hausdorff series. See Remark A.4 for more detail.

Using the linear isomorphism $\mathfrak{tder}_n \oplus \mathfrak{a}_n \cong \mathfrak{lie}_n^{\oplus n}$, the linear operad structure on $\mathfrak{lie} = \{\mathfrak{lie}_n\}_{n\ge 0}$ generalizes to a linear operad of tangential derivations. For $u = (a_1, \dots, a_n) \in \mathfrak{tder}_n$ and $\sigma \in \Sigma_n$ define

$$u^\sigma := (a^\sigma_{\sigma^{-1}(1)}, \dots, a^\sigma_{\sigma^{-1}(n)}),$$

where a_j^σ denotes the Σ_n action on $\mathfrak{lie}$. This is the canonical action induced by the action on $\mathfrak{lie}$ in the sense that it is the unique action which makes the following diagram commute:

$$\begin{tikzcd}
\mathfrak{lie}_n \arrow[r, "(-)^\sigma"] \arrow[d, "u"'] & \mathfrak{lie}_n \arrow[d, "u^\sigma"] \\
\mathfrak{lie}_n \arrow[r, "(-)^\sigma"'] & \mathfrak{lie}_n
\end{tikzcd} \tag{3.2}$$

For $1 \leq i \leq m$, define the composition map

$$\circ_i : \mathfrak{tder}_m \oplus \mathfrak{tder}_n \to \mathfrak{tder}_{m+n-1}$$

to be the $\mathbb{K}$-linear map such that for $u = (a_1, \ldots, a_m)$ and $v = (b_1, \ldots, b_n)$, the composition is

$$u \circ_i v := (a_1 \circ_i 0, \ldots, a_{i-1} \circ_i 0, a_i \circ_i b_1, \ldots, a_i \circ_i b_n, a_{i+1} \circ_i 0, \ldots, a_m \circ_i 0). \tag{3.3}$$

Here, the $\circ_i$ on the right hand side denotes operadic composition in the operad $\mathfrak{lie}$, and 0 stands for $0_n \in \mathfrak{lie}_n$.

The following proposition follows from the arguments presented in [AT12, Section 3.3]. See also [AET10] and [AKKN18b, Section 7].

Proposition 3.10. *The family $\mathfrak{tder} = \{\mathfrak{tder}_n\}_{n\geq 0}$ of tangential derivations endowed with the above $\circ_i$ operations and Σ_n-actions forms a $\mathbb{K}$-linear operad. The non-commutative divergence $j : \mathfrak{tder} \to \mathrm{cyc}$ is then a map of linear operads.*

The collection of special derivations $\mathfrak{sder} = \{\mathfrak{sder}_n\}$ carries a stronger operadic structure than $\mathfrak{tder}$: it forms an operad in Lie algebras. We state the main theorems describing this structure below; full proofs are deferred to Appendix A.

Theorem 3.11. *The family $\mathfrak{sder}$ of special derivations forms a linear suboperad of $\mathfrak{tder}$, and with the operadic structure restricted from $\mathfrak{tder}$ it is an operad in Lie algebras. The family $\mathfrak{t} := \{\mathfrak{t}_n\}_{n\geq 0}$ of infinitesimal braids injects into $\mathfrak{sder}$ as a sub-operad in Lie algebras.*

In analogy with tangential derivations, non-canonical operadic structures can be defined for the groups of tangential automorphisms $\mathrm{TAut} = \{\mathrm{TAut}_n\}_{n\geq 0}$, to produce operads in sets.

Proposition 3.12. *The collection $\mathrm{TAut} = \{\mathrm{TAut}_n\}_{n\geq 0}$ has the structure of an operad in sets. The symmetric group action is given for $F = \exp(u) \in \mathrm{TAut}_n$ and $\sigma \in \Sigma_n$ by $F^\sigma := \exp(u^\sigma)$. The partial composition maps are defined for $F = \exp(u), G = \exp(v)$ with $u \in \mathfrak{tder}_m$ and $v \in \mathfrak{tder}_n$, by the formula*

$$\mathrm{TAut}_m \times \mathrm{TAut}_n \xrightarrow{\circ_i} \mathrm{TAut}_{m+n-1}$$

$$(F, G) \mapsto \exp(u \circ_i 0)\, \exp(0 \circ_i v).$$

In particular, we have $F \circ_i G = (F \circ_i 1)\,(1 \circ_i G)$, which in usual cosimplicial notation reads

$$F^{1,\ldots,i-1,\, i(i+1)\cdots(i+n-1),\, i+n,\ldots,m+n-1} \circ G^{i,i+1,\ldots,i+(n-1)}.$$

As explained in Appendix B, this cosimplicial structure is also present at the level of $\mathfrak{tder}$. In particular, if $F = \exp{(u)} \in \mathrm{TAut}_n, u \in \mathfrak{tder}_n$, then $F^{i,(i+1)\ldots j} = \exp{(u^{i,(i+1)\ldots j})}$, etc. This construction is also natural in the sense that if $a \in \mathfrak{lie}_n$, then we have $(F(a))^{i,(i+1)\ldots j} = F^{i,(i+1)\ldots j}(a^{i,(i+1)\ldots j})$ for instance. Note that even though partial compositions $\circ_i$ are not group homomorphisms in general, composition with the unit $- \circ_i 1$ and $1 \circ_i -$ are, as are the symmetric group actions.

Remark 3.13. The operad structure on TAut defined above agrees with[3] with the operad composition used in [AKKN18b] to construct Kashiwara–Vergne solutions corresponding to genus zero surfaces with more than three boundary components. It is not, however, the operad structure one would obtain by exponentiation of the partial compositions of $\mathfrak{tder}$:

$$\exp(u) \circ_i \exp(v) \neq \exp(u \circ_i v).$$

---

[3] The order of composition in our definition is opposite to that of [AKKN18b], as we follow the definition of the set of KV solutions in [AET10], which is the inverse of that of [AT12] and [AKKN18b].

where $u \in \mathfrak{tder}_n$ and $v \in \mathfrak{tder}_m$, and the composition on the right-hand side is as in Equation (3.3). These notions do, however coincide when restricted to $\mathfrak{sder}$ and SAut, where there are canonical operad structures in the categories of Lie algebras and prounipotent groups, respectively.

Although the operadic structure on TAut is non-canonical, it is natural in the sense of [AT12, Section 3]: for any $F \in \mathrm{TAut}_n$, the identity $J(F^{i,j}) = J(F)^{i,j}$ holds for all pairs i, j. As we compute in Appendix B, this implies that the Jacobian cocycle is compatible with the operadic composition.

Proposition 3.14. *The Jacobian* $J\colon \mathrm{TAut} \to \mathrm{cyc}$ *is a morphism of operads (in sets).*

3.2. **Moperad structures for $\mathfrak{sder}$ and $\mathfrak{tder}$.** Since $\mathfrak{sder}$ is an operad in Lie algebras (Theorem A.3), following the general shift construction presented in Example 2.5, one obtains a shifted $\mathfrak{sder}$-moperad in Lie algebras, denoted $\mathfrak{sder}^+$. The same statement integrates to groups, that is, $\mathrm{SAut} = \{\mathrm{SAut}_n\}_{n\geq 0}$ is an operad in prounipotent groups, and there is a corresponding shifted SAut-moperad, denoted SAut^+. While $\mathfrak{tder}$ is not an operad in Lie algebras, in this section we develop an $\mathfrak{sder}$-moperad in Lie algebras, denoted $\mathfrak{tder}^1$, which injects into the linear moperad $\mathfrak{tder}^+$. The construction is inspired by the $\mathfrak{sder}$-moperad $\mathfrak{sder}^1$, a sub-moperad of $\mathfrak{sder}^+$, below. Both constructions integrate to moperads in prounipotent groups.

Definition 3.15. Let $\mathfrak{sder}^1_n$ denote the semidirect product of Lie algebras $\mathfrak{sder}^1_n := \mathfrak{lie}_n \rtimes \mathfrak{sder}_n$. For $n \geq 1$, write elements of $\mathfrak{sder}^1_n$ as pairs (a, u), where $a = a(x_1, \ldots, x_n)$ is a Lie word in $\mathfrak{lie}_n$ and u is a derivation in $\mathfrak{sder}_n$. Then $[(a, u), (b, v)] = ([a, b] + u(b) - v(a), [u, v])$.

Similarly, let $\mathrm{SAut}^1_n = (\mathsf{F}_n)_{\mathbb{K}} \rtimes \mathrm{SAut}_n$ denote the corresponding semidirect product of groups, consisting of pairs (w, e^u), where $w = w(x_1, ..., x_n)$ is a word in $(\mathsf{F}_n)_{\mathbb{K}}$, and $e^u \in \mathrm{SAut}_n$. If $w, w' \in (\mathsf{F}_n)_{\mathbb{K}}$ and $e^u, e^{u'} \in \mathrm{SAut}_n$ then $(w, e^u)(w', e^{u'}) = \Big(w(e^u \cdot w'), e^u e^{u'}\Big)$.

The symmetric group Σ_n acts on $\mathfrak{sder}^1_n$ by permuting the variables: $\sigma(a, u) := (a^\sigma, u^\sigma)$. This action induces an action of Σ_n on SAut^1_n also by permuting the variables.

We aim to define an $\mathfrak{sder}$-moperad structure on $\mathfrak{sder}^1$. To do so, we define a Lie algebra inclusion

$$\kappa : \mathfrak{lie}_n\langle x_1, \ldots, x_n\rangle \hookrightarrow \mathfrak{sder}_{n+1} = \mathfrak{sder}_{\mathfrak{lie}_{n+1}\langle x_0, x_1, \ldots, x_n\rangle}.$$

We define κ on the generators by setting $\kappa(x_i) = t^{0,i} = (x_i, 0, ..., 0, x_0, 0, ..., 0)$, where x_i is placed in the 0-th component and x_0 in the i-th component. The map κ extends uniquely to a Lie algebra homomorphism on $\mathfrak{lie}_n$. For $a \in \mathfrak{lie}_n$, the 0-th component of $\kappa(a)$ is always a, and a does not involve the variable x_0, hence, the map κ is injective.

Remark 3.16. There is an illuminating visual for understanding the κ-value of a Lie word, shown in Figure 16. Namely, a Lie word in $\mathfrak{lie}_n$ is naturally represented as a rooted binary tree with leaves labeled with the numbers $\{1, 2, ..., n\}$ (each label may be used multiple times), and the root labeled 0. Redefining the root of such a tree to be one of the leaves then gives a new Lie word, this time in the $(n+1)$ variables $\{x_0, x_1, ..., x_n\}$. We obtain an element in $\mathfrak{lie}_{n+1}^{\oplus(n+1)}$ by summing over all ways of rooting the tree, and placing each resulting Lie word in the component numbered by the label of the new root, as shown in Figure 16. To finish, $\mathfrak{lie}_{n+1}^{\oplus(n+1)}$ maps into $\mathfrak{tder}_{n+1}$ in the natural way. This in particular implies that the 0-th component of $\kappa(a)$ is always a. The fact that the image of κ is in $\mathfrak{sder}_{n+1}$ follows from the fact that κ is a Lie algebra map and the images of the generators lie in $\mathfrak{sder}_{n+1}$.

The following proposition follows diagrammatically from [BND17, Theorem 3.28], or algebraically from Drinfeld's Lemma [Dri90, Lemma after Proposition 6.1].

Proposition 3.17. *For* $u = (u_1, ..., u_n) \in \mathfrak{sder}_n$*, denote* $u^+ := (0, u_1, ..., u_n) \in \mathfrak{sder}^+_n \cong \mathfrak{sder}_{n+1}$*. Then, for any* $a \in \mathfrak{lie}_n$*, we have* $\kappa(u(a)) = [u^+, \kappa(a)]$*.*

Example 3.18. We check $\kappa(u(a)) = [u^+, \kappa(a)]$ for $u = (x_2, x_1)$ and $a = x_2$. For this calculation, recall that in $\mathfrak{tder}_n$ if $u = (u_1, ..., u_n)$ and $v = (v_1, ..., v_n)$, then $([u, v])_k = [u_k, v_k] + u(v_k) - v(u_k)$. We compute both

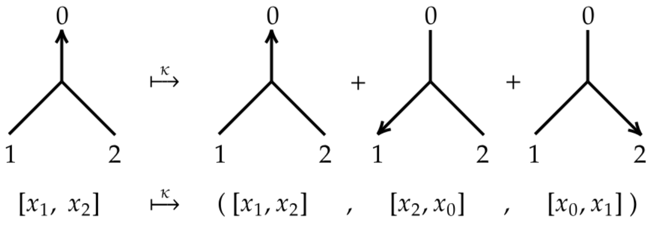


FIGURE 16. The value of $\kappa([x_1, x_2]) \in \mathfrak{sder}_{2+1}$, calculated by re-rooting the tree.

sides:

$$\kappa(u(x_2)) = \kappa([x_2, x_1]) = ([x_2, x_1], [x_0, x_2], -[x_0, x_1]),$$
$$[u^+, \kappa(x_2)] = [(0, x_2, x_1), (x_2, 0, x_0)] = ([x_2, x_1], -[x_2, x_0], [x_1, x_0]),$$

and therefore $\kappa(u(x_2)) = [u^+, \kappa(x_2)]$ as expected.

Lemma 3.19. *There are injective Lie algebra homomorphisms*

$$\iota_n : \mathfrak{sder}_n^1 \hookrightarrow \mathfrak{sder}_n^+ \cong \mathfrak{sder}_{n+1},$$

given on $a \in \mathfrak{lie}_n$ *and* $u = (u_1, \ldots, u_n) \in \mathfrak{sder}_n$ *by*

$$\iota_n(a, u) := \kappa(a) + (0, u_1, \ldots, u_n).$$

Proof. The map ι_n can be written as the sum of $\kappa : \mathfrak{lie}_n \to \mathfrak{sder}_{n+1}$, and the natural inclusion $(\cdot)^+ : \mathfrak{sder}_n \hookrightarrow \mathfrak{sder}_n^+$ given by $(u_1, ..., u_n)^+ := (0, u_1, ...u_n)$. In $\mathfrak{sder}_n^1$, we have $[(a, u), (b, v)] = ([a, b] + u(b) - v(a), [u, v])$. Hence, we have

$$\begin{aligned}\iota_n([(a, u), (b, v)]) &= \kappa([a, b]) + \kappa(u(b)) - \kappa(v(a)) + [u, v]^+ \\ &= [\kappa(a), \kappa(b)] + [u^+, \kappa(b)] + [\kappa(a), v^+] + [u^+, v^+] \\ &= [\kappa(a) + u^+, \kappa(b) + v^+] = [\iota_n(a, u), \iota_n(b, v)].\end{aligned}$$

Here, in the second line we used Proposition 3.17. To show that ι_n is injective, assume that $\iota_n(a, u) = 0$. It follows immediately from the definition and injectivity of κ that $a = 0$, and then we have $\iota_n(0, u) = 0$, which implies $u = 0$ by the injectivity of the shift construction $(\cdot)^+$. □

Theorem 3.20. *The maps* $\{\iota_n\}_{n\geq 0}$ *naturally assemble to a map of symmetric sequences in Lie algebras* $\iota : \mathfrak{sder}^1 \to \mathfrak{sder}^+$*, and the image* $\iota(\mathfrak{sder}^1)$ *of* $\mathfrak{sder}^1$ *under the map* ι *is an* $\mathfrak{sder}$*-submoperad of* $\mathfrak{sder}^+$*.*

Proof. Observe that the map ι_n commutes with the action of Σ_n. We need to show that the image $\iota(\mathfrak{sder}^1)$ is closed under the monoid product and the $\mathfrak{sder}$-action of the moperad $\mathfrak{sder}^+$. Let $v = (v_0, v_1, ..., v_m) = \iota_m((a, u))$ be an arbitrary element in the image of ι. Then, in particular, $v_0 = a \in \mathfrak{lie}\langle x_1, ..., x_m\rangle$, and therefore $v - \kappa(a) \in \mathfrak{sder}_m = \mathfrak{sder}_{\mathfrak{lie}\langle x_1,...,x_m\rangle}$. If $v - \kappa(a) = (0, u_1, ..., u_m)$, then $u = (u_1, ..., u_m)$.

Now for some $w \in \mathfrak{sder}_n$, we compute $v \circ_i w$, where $1 \leq i \leq m$. Recall from Equation (3.3), Definition 3.7, and Section A.2 that

$$v \circ_i w = v \circ_i 0 + 0 \circ_i w = v^{1,\ldots,i-1,i(i+1)\cdots(i+n-1),i+n,\ldots,m+n-1} + w^{i,i+1,\ldots,i+n-1}.$$

Since $v_0 \in \mathfrak{lie}\langle x_1, ..., x_m\rangle$, therefore, $(v \circ_i w)_0 = v_0 \circ_i 0 \in \mathfrak{lie}\langle x_1, ..., x_{m+n-1}\rangle$. Thus, $(v \circ_i w)_0 \in \mathfrak{lie}_{m+n-1}$.

Observe that $\kappa(a \circ_i 0) = \kappa(a) \circ_i 0$. Indeed, since both sides are Lie algebra maps, it is enough to check this for the generators, which in turn is straightforward. Thus, $\kappa(v_0 \circ_i 0) = \kappa(v_0) \circ_i 0$.

Now, we compute

$$v \circ_i w - \kappa(v_0 \circ_i 0) = (v \circ_i 0 - \kappa(v_0) \circ_i 0) + 0 \circ_i w = \iota(0, u) \circ_i 0 + 0 \circ_i w.$$

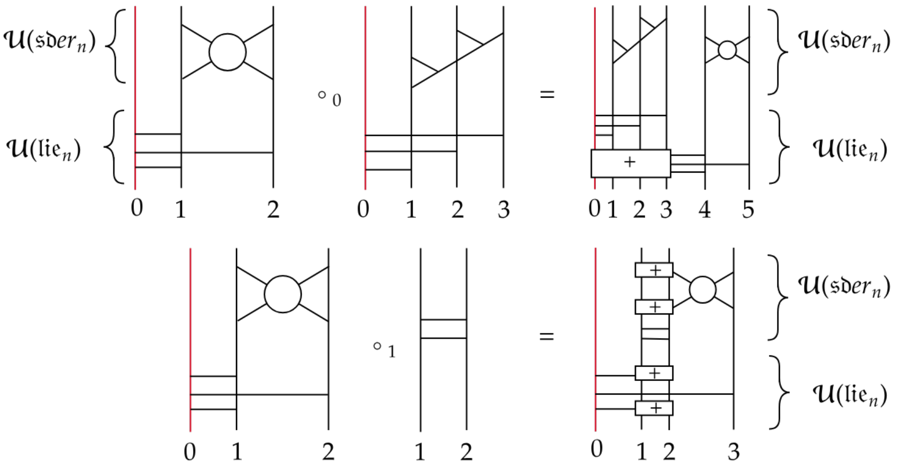


FIGURE 17. Moperad operations for chord diagrams.

Therefore, we have

$$v \circ_i w = \kappa(a \circ_i 0) + \iota(0,u) \circ_i 0 + 0 \circ_i w = \iota(a \circ_i 0, u \circ_i 0 + 0 \circ_i w) = \iota\left(a \circ_i 0, u \circ_i w\right),$$

which is in the image of ι.

Now let $v = (v_0, v_1, ..., v_m) = \iota_m((a,u))$, and $v' = (v'_0, v'_1, ..., v'_n) = \iota_n((a',u'))$. We check that $v \circ_0 v' \in \mathfrak{sder}^+_{m+n}$ is in the image of ι. We have

$$v \circ_0 v' = v \circ_0 0 + (v'_0, v'_1, ..., v'_n, 0, ..., 0).$$

The 0-th component of $v \circ_0 v'$ is $(v_0 \circ_0 0 + v'_0)$. We compute

$$\begin{aligned} v \circ_0 v' - \kappa(v_0 \circ_0 0 + v'_0) &= (v \circ_0 0 - \kappa(v_0 \circ_0 0)) + ((v'_0, ..., v'_n, 0, ..., 0) - \kappa(v'_0)) \\ &= (v - \kappa(v_0)) \circ_0 0 + 0 \circ_0 (v' - \kappa(v'_0)) \\ &= \iota(0,u) \circ_0 0 + 0 \circ_0 \iota(0,u') = \iota(0, u \circ_0 u'). \end{aligned}$$

Therefore, we have $v \circ_0 v' = \iota\left(v_0 \circ_0 0 + v'_0, u \circ_0 u'\right)$, finishing the proof. □

This structure integrates to prounipotent groups, as follows. The operad $\mathfrak{sder} = \{\mathfrak{sder}_n\}_{n\geq 0}$ is an operad in degree completed Lie algebras, meaning that, in each arity, the group-like elements of $\widehat{U}(\mathfrak{sder}_n)$ are identified with $\exp(\mathfrak{sder}_n)$. Let $\mathrm{SAut}^1_n := (\mathsf{F}_n)_{\mathbb{K}} \rtimes \mathrm{SAut}_n$. There is an injective group homomorphism $\exp(\iota) : \mathrm{SAut}^1_n \to \mathrm{SAut}^+_n \cong \mathrm{SAut}_{n+1}$. The symmetric sequence $\{\mathrm{SAut}^+_n\}_{n\geq 0}$ is an SAut-moperad in prounipotent groups, and this restricts to make SAut^1 an SAut-moperad.

Example 3.21. For readers familiar with chord diagrams in the sense of finite type invariants in knot theory, we note that these structures are both part of a moperad in Hopf algebras assembled of chord diagrams on $(n+1)$ strands, as shown in Figure 17. For readers otherwise, this may still be helpful as a visualization.

Expanding on Theorem 3.11, which states that infinitesimal braids form a suboperad of $\mathfrak{sder}$, we have the following.

Lemma 3.22. *The image of* $\mathfrak{t}^+_n \hookrightarrow \mathfrak{sder}^+_n$ *lies in* $\iota(\mathfrak{sder}^1_n)$.

Proof. Since the inclusion $\mathfrak{t}^+_n \hookrightarrow \mathfrak{sder}^+_n$ is a Lie algebra map, it is enough to check this for the generators.

If $0 < i < j$, then $t_{i,j}$ maps to $t^{i,j} = (0, ..., 0, x_j, 0, ..., 0, x_i, 0, ..., 0)$, with x_j in component $i > 0$ and x_i in component $j > 0$. Therefore, $t^{i,j} \in \mathfrak{sder}_n$, so $t^{i,j} = \iota\left(0, t^{i,j}\right)$.

On the other hand, $t_{0,i}$ maps to $t^{0,i} = (x_i, 0, ..., 0, x_0, 0, ..., 0)$, with x_i in the 0-th component and x_0 in the i-th component. Then $t^{0,i} = \kappa(x_i)$, therefore, $t^{0,i} = \iota(x_i, 0)$. □

Finally, we introduce an $\mathfrak{sder}$-moperad in Lie algebras, $\mathfrak{tder}^1$, consisting of tangential derivations. The conceptual difference from the previous construction is that $\mathfrak{tder}$ itself does not form an operad in Lie algebras, and hence there is only a shifted moperad to speak of in a linear sense. We denote this linear shifted moperad by $\mathfrak{tder}^+$. It is the case, however, that tangential derivations admit a right $\mathfrak{sder}$-module structure in Lie algebras.

Proposition 3.23. *The symmetric sequence* $\mathfrak{tder} = \{\mathfrak{tder}_n\}_{n\geq 0}$ *forms a right* $\mathfrak{sder}$*-module in Lie algebras, where the action* $\circ_i : \mathfrak{tder}_m \oplus \mathfrak{sder}_n \to \mathfrak{tder}_{n+m-1}$ *is given by restricting the i-th linear partial compositions of* $\mathfrak{tder}$.

Proof. We have seen in Proposition 3.10 that these partial compositions make $\mathfrak{tder}$ a linear operad. Thus, it is enough to show that the restrictions $\mathfrak{sder}$ in the second component are Lie algebra maps. For $d \in \mathfrak{tder}_m$ and $u \in \mathfrak{sder}_n$, we have

$$d \circ_i u = d \circ_i 0 + 0 \circ_i u = d^{1,\ldots,i-1,i(i+1)\cdots(i+n-1),i+n,\ldots,m+n-1} + u^{i,i+1,\ldots,i+n-1}.$$

Since $(\cdot) \circ_i 0$ and the shift $(\cdot)^{i,i+1,\ldots,i+n-1}$ are both Lie algebra maps, their sum is a Lie algebra map if the images commute in $\mathfrak{tder}_{m+n-1}$. In other words, we need to show $[d \circ_i 0, u^{i,i+1,\ldots,i+n-1}] = 0$ for any $d \in \mathfrak{tder}_m$ and $u \in \mathfrak{sder}_n$. To see this, notice that since $u \in \mathfrak{sder}_n$, the value $u^{i,\ldots,i+n-1}(x_i + \cdots + x_{i+n-1}) = 0$, and therefore $u^{i,\ldots,i+n-1}(d_k \circ_i 0) = 0$ for all $d_k \in \mathfrak{lie}_m$. This implies that $[d \circ_i 0, u^{i,i+1,\ldots,i+n-1}]$ – when written as an $(m + n - 1)$-tuple – is zero in coordinates 1 through $(i - 1)$, and $(i + n)$ through $(m + n - 1)$. For example, if $i > 1$, then the first coordinate of $[d \circ_i 0, u^{i,i+1,\ldots,i+n-1}]$ is computed as follows:

$$[d_1 \circ_i 0, 0] + (d \circ_i 0)(0) - u^{i,\ldots,i+n-1}(d_1 \circ_i 0) = 0 + 0 - 0.$$

Next, if $1 \leq j \leq n$, then the $(i + j - 1)$-st coordinate of $[d \circ_i 0, u^{i,i+1,\ldots,i+n-1}]$ is:

$$[d_i \circ_i 0, u_j^{i,\ldots,i+n-1}] + (d \circ_i 0)(u_j^{i,\ldots,i+n-1}) - 0 = [d_i \circ_i 0, u_j^{i,\ldots,i+n-1}] + [u_j^{i,\ldots,i+n-1}, d_i \circ_i 0] = 0,$$

where the first equality holds because $u_j^{i,\ldots,i+n-1}$ is a Lie word is variables $x_i, .., x_{i+n-1}$, and the tangential derivation $d \circ_i 0$ has identical components $d_i \circ_i 0$ in positions i through $(i + n - 1)$, hence it acts as an inner derivation on $u_j^{i,\ldots,i+n-1}$. This completes the proof. □

Remark 3.24. One can similarly define a right $\mathfrak{sder}$-module structure on the extended symmetric sequence $\mathfrak{tder}^+ = \{\mathfrak{tder}_n^+ = \mathfrak{tder}_{\mathfrak{lie}_{n+1}\langle x_0,\ldots,x_n\rangle}\}$ in which the $\mathfrak{sder}$ action maps are defined by the partial compositions $\circ_i$ for $1 \leq i \leq n$.

Definition 3.25. We define $\mathfrak{tder}_n^1 := \mathfrak{lie}_n \rtimes \mathfrak{tder}_n$. For $n \geq 1$, write elements of $\mathfrak{tder}_n^1$ as pairs (a, u), where $a = a(x_1, \ldots, x_n)$ is a Lie word in $\mathfrak{lie}_n$ and u is a derivation in $\mathfrak{tder}_n$. Then $[(a, u), (b, v)] = ([a, b] + u(b) - v(a), [u, v])$. The corresponding group is $\mathrm{TAut}_n^1 = (\mathsf{F}_n)_{\mathbb{K}} \rtimes \mathrm{SAut}_n$, consisting of pairs (w, e^u), where $w = w(x_1, ..., x_n)$ is a word in $(\mathsf{F}_n)_{\mathbb{K}}$, and $e^u \in \mathrm{TAut}_n$.

The symmetric group Σ_n acts on $\mathfrak{tder}_n^1$ by permuting the variables: $\sigma(a, u) := (a^\sigma, u^\sigma)$. This action integrates to TAut_n^1 as well.

The analogue of κ is a (different) injective Lie algebra map

$$\lambda : \mathfrak{lie}_n\langle x_1, \ldots, x_n\rangle \hookrightarrow \mathfrak{tder}_{n+1} = \mathfrak{tder}_{\mathfrak{lie}\langle x_0,x_1,\ldots,x_n\rangle}.$$

The map λ is simply the inclusion of $\mathfrak{lie}_n$ in the 0-th component, that is, $\lambda(a) = (a, 0, ..., 0, 0, 0, ..., 0)$. The map λ is injective, as a tuple $(a, 0, ..., 0)$ represents the 0 derivation in $\mathfrak{tder}_{n+1}$ if and only if $a = cx_0$ for some scalar c.

Proposition 3.26. *For* $d = (d_1, ..., d_n) \in \mathfrak{tder}_n$ *and any* $a \in \mathfrak{lie}_n$*, we have* $\lambda(d(a)) = [d^+, \lambda(a)]$.

Proof. We compute both sides:

$$\lambda(d(a)) = (d(a), 0, ..., 0),$$
$$[d^+, \lambda(a)] = [(0, d_1, ..., d_n), (a, 0, ...0)] = (d(a), 0, ..., 0).$$

Here in the second line we used the rule for calculating commutators of tangential derivations in the coordinate expression: $([u, v])_k = [u_k, v_k] + u(v_k) - v(u_k)$. □

Lemma 3.27. *For each $n \geq 1$, there is an injective Lie algebra homomorphism*

$$\eta_n : \mathfrak{tder}^1_n \hookrightarrow \mathfrak{tder}^+_n = \mathfrak{tder}_{\mathfrak{lie}_{n+1}\langle x_0, x_1, ..., x_n\rangle} \cong \mathfrak{tder}_{n+1} \tag{3.4}$$

given by

$$\eta_n(a, d) := \lambda(a) + d^+ \quad \textit{where} \quad d^+ = (0, d_1, ..., d_n)$$

for any $a \in \mathfrak{lie}_n$ and $d = (d_1, ..., d_n) \in \mathfrak{tder}_n$.

Proof. The proof of Lemma 3.19 applies verbatim, replacing ι with η. □

The maps $\{\eta_n\}$ assemble to a map of symmetric sequences $\eta : \mathfrak{tder}^1 \to \mathfrak{tder}^+$. As we saw in Proposition 3.23 and Remark 3.24, $\mathfrak{tder}$ and $\mathfrak{tder}^+$ are right $\mathfrak{sder}$-modules in Lie algebras. The next lemma shows that this module structure restricts to $\mathfrak{tder}^1$.

Lemma 3.28. *The image $\eta(\mathfrak{tder}^1)$ in $\mathfrak{tder}^+$ is invariant under the right $\mathfrak{sder}$-action, hence, the action restricts to make $\mathfrak{tder}^1$ a right $\mathfrak{sder}$-module in Lie algebras.*

Proof. For $a \in \mathfrak{lie}_m$, $d \in \mathfrak{tder}_m$, $u \in \mathfrak{sder}_n$, $1 \leq i \leq m$, we have

$$\eta(a, d) \circ_i u = (\lambda(a) + d^+) \circ_i u = \lambda(a) \circ_i 0 + d^+ \circ_i u = \lambda(a \circ_i 0) + (d \circ_i u)^+ = \eta(a \circ_i 0, d \circ_i u).$$

□

Lemma 3.29. *The image $\eta(\mathfrak{tder}^1)$ in $\mathfrak{tder}^+$ is closed under the linear monoid multiplication*

$$\circ_0 : \mathfrak{tder}^+_m \oplus \mathfrak{tder}^+_n \to \mathfrak{tder}^+_{m+n}.$$

Moreover, for each $m, n \geq 1$ restriction of the linear map $\circ_0 : \mathfrak{tder}^+_m \oplus \mathfrak{tder}^+_n \to \mathfrak{tder}^+_{m+n}$ to $\eta(\mathfrak{tder}^1_m) \oplus \eta(\mathfrak{tder}^1_n)$ is a Lie algebra homomorphism.

Proof. For any $a \in \mathfrak{lie}_m$, $a' \in \mathfrak{lie}_n$, $d \in \mathfrak{tder}_m$ and $d' \in \mathfrak{tder}_n$, we compute

$$\begin{aligned}\eta_m(a, d) \circ_0 \eta_n(a', d') &= (a, d_1, ..., d_m) \circ_0 (a', d'_1, \dots, d'_n) \\ &= (a \circ_0 0, \dots, a \circ_0 0, d_1 \circ_0 0, \dots, d_m \circ_0 0) + (a', d'_1, ..., d'_n, 0, ..., 0).\end{aligned}$$

Note that since a and d_j are elements of $\mathfrak{lie}_m\langle x_1, ..., x_m\rangle$, for $1 \leq j \leq m$, it follows that $a \circ_0 0 = a(x_{n+1}, ..., x_{m+n})$, and $d_j \circ_0 0 = d_j(x_{n+1}, ..., x_{m+n})$. Thus, each component of $\eta_m(a, d) \circ_0 \eta_n(a', d')$ is an element of $\mathfrak{lie}_{m+n}$, and therefore, $\eta_m(a, d) \circ_0 \eta_n(a', d') \in \eta_{m+n}(\mathfrak{tder}^1_{m+n})$.

The monoid composition

$$\circ_0 : \eta(\mathfrak{tder}^1_m) \oplus \eta(\mathfrak{tder}^1_n) \to \eta(\mathfrak{tder}^1_{m+n})$$

can be described as the sum of Lie algebra maps $(\cdot) \circ_0 0 = (\cdot)^{01 \cdots n, n+1, \dots, n+m}$ and $(\cdot)^{0, \dots, n}$. It is enough to check that the images of these two maps commute, that is, we need to show that

$$[(a \circ_0 0, \dots, a \circ_0 0, d_1 \circ_0 0, \dots, d_m \circ_0 0), (a', d'_1, ..., d'_n, 0, ..., 0)] = 0 \quad \text{in } \mathfrak{tder}^1_{m+n}$$

for any a, d, a', d'. Since $a \circ_0 0$ and $d_i \circ_0 0$ are Lie words in $\{x_{n+1}, ... x_{m+n}\}$, we compute, for example, the 0-th component of the commutator using $([u, v])_k = [u_k, v_k] + u(v_k) - v(u_k)$. We obtain

$$[a \circ_0 0, a'] + [a', a \circ_0 0] - [a \circ_0 0, 0] = 0,$$

using that the first $n + 1$ components of $(a \circ_0 0, ..., a \circ_0 0, d_1 \circ_0 0, ..., d_n \circ_0 0)$ are identical, and that $a \circ_0 0 = a(x_{n+1}, ..., x_{m+n})$ and all of these variables are killed by $(a', d'_1, ..., d'_n, 0, ..., 0)$. The first to n-th components of the commutator are zero for the same reason, and the last m are zero trivially. This completes the proof. □

In summary, we have proven the following:

Theorem 3.30. *The symmetric sequence* $\mathfrak{tder}^1$ *is an* $\mathfrak{sder}$*-moperad in Lie algebras, with the monoid multiplication and* $\mathfrak{sder}$*-action defined by restriction from the linear moperad* $\mathfrak{tder}^+$*. This structure integrates to groups, making* TAut^1 *an* SAut*-moperad in prounipotent groups.*

Proof. We observe that the map η_n commutes with the action of Σ_n. The result then follows from the combination of Lemma 3.28 and Lemma 3.29. ☐

3.3. **Kashiwara–Vergne (KV) solutions, associators, and symmetry groups.** In this section, we introduce genus zero Kashiwara–Vergne (KV) solutions, a family of higher-arity analogues of the classical KV problem introduced by Alekseev, Kawazumi, Kuno, and Naef [AKKN18b, AKKN20, AKKN18a]. Algebraically, these solutions extend the KV condition to compositions involving multiple inputs, reflecting properties of nested applications of the Baker–Campbell–Hausdorff (BCH) series [AET10, Appendix B.1]. Geometrically, they correspond to homomorphic expansions for the Goldman–Turaev Lie bialgebra associated with free loop spaces on surfaces [AKKN18b, AKKN20], as well as homomorphic expansions for certain four-dimensional tangles [BND17, BND24].

Definition 3.31. A solution to the Kashiwara–Vergne (KV) problem of type $(0, n+1)$ is an element $F \in \mathrm{TAut}_n$ such that

$$F(e^{x_1}\cdots e^{x_n}) = e^{x_1+\cdots+x_n}, \tag{SolKVI}$$

$$\exists h \in z^2\mathbb{K}[[z]] \quad \text{such that} \quad J(F) = \mathrm{tr}\left(h\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} h(x_i)\right). \tag{SolKVII}$$

Remark 3.32. Our definition of KV solutions aligns with the conventions used in [AET10], but corresponds to the inverses of the solutions appearing in [AT12] and [AKKN18b].

The formal power series $h \in z^2\mathbb{K}[[z]]$ appearing in (SolKVII) is called the *Duflo function*. This function is uniquely determined by a given KV solution F [AET10], and is essential data when one aims to glue, or compose, KV solutions. We denote by $\mathsf{SolKV}(n)$ the set of all KV solutions of type $(0, n+1)$, or equivalently, KV solutions of arity n. We refer to elements of $\mathsf{SolKV}(2)$ as “classical KV solutions”, in cases when we need to contrast KV solutions of arity 2 with their higher-arity generalizations.

The existence of classical KV solutions was first established in [AM06]. In [AT12], Alekseev and Torossian gave an alternative proof of the existence of classical KV solutions – that is, of type $(0, 2+1)$ – by exploring the relationship between Drinfeld associators and the associativity of the Baker–Campbell–Hausdorff series:

$$\mathfrak{bch}(x_1, \mathfrak{bch}(x_2, x_3))) = \mathfrak{bch}(\mathfrak{bch}(x_1, x_2), x_3)$$

In particular, if $F \in \mathrm{TAut}_2$ satisfies (SolKVI), we have:

$$F^{12,3}F^{1,2}(\mathfrak{bch}(\mathfrak{bch}(x_1, x_2), x_3) = x_1 + x_2 + x_3 = F^{1,23}F^{2,3}(\mathfrak{bch}(x_1, \mathfrak{bch}(x_2, x_3))).$$

Here we have again used the standard cosimplicial notation for tangential automorphisms, in which the automorphism $F^{1,2}$ denotes the image of $F \in \mathrm{TAut}_2$, viewed as an element of TAut_3 which acts non-trivially on x_1, x_2 and fixes x_3. Similarly, $F^{2,3}$ acts on x_2, x_3, $F^{12,3}$ acts on the pair $(x_1 + x_2, x_3)$, and $F^{1,23}$ acts on $(x_1, x_2 + x_3)$.

The special automorphism

$$G_F := F^{1,23}F^{2,3}(F^{1,2})^{-1}(F^{12,3})^{-1} = (F \circ_2 F)(F \circ_1 F)^{-1} \tag{3.5}$$

satisfies the *pentagon equation* in TAut_4, as we will discuss in Section 5.1:

$$G_F^{1,2,34}G_F^{12,3,4} = G_F^{2,3,4}G_F^{1,23,4}G_F^{1,2,3}. \tag{P}$$

Definition 3.33 ([AT12, Definition 9.1])**.** A special automorphism

$$G_F = F^{1,23}F^{2,3}(F^{1,2})^{-1}(F^{12,3})^{-1} \in \mathrm{SAut}_3$$

as above is a *KV associator* if it satisfies the pentagon (P) equation (P), as well as the inversion and hexagon equations:

$$G^{1,2,3}G^{3,2,1} = 1, \tag{U}$$

$$e^{\frac{\mathfrak{t}^{1,2}+\mathfrak{t}^{1,3}}{2}} = (G^{2,3,1})^{-1}e^{\frac{\mathfrak{t}^{1,3}}{2}}G^{2,1,3}e^{\frac{\mathfrak{t}^{1,2}}{2}}(G^{1,2,3})^{-1}, \tag{H1}$$

$$e^{\frac{\mathfrak{t}^{1,3}+\mathfrak{t}^{2,3}}{2}} = G^{3,1,2}e^{\frac{\mathfrak{t}^{1,3}}{2}}(G^{1,3,2})^{-1}e^{\frac{\mathfrak{t}^{2,3}}{2}}G^{1,2,3}. \tag{H2}$$

There is an injective Lie algebra homomorphism $\iota : \mathfrak{t}_n \to \mathfrak{sder}_n$ given by

$$\iota(t_{ij}) = t^{ij} := (0, \dots, \underbrace{x_j}_{i}, \dots, \underbrace{x_i}_{j}, \dots, 0),$$

see also Example A.5. Exponentiating this map yields an injective group homomorphism $\exp(\mathfrak{t}_n) \to \mathrm{SAut}_n$, and consequently an operad inclusion $\iota : \mathsf{CD} \hookrightarrow \mathrm{SAut}$. It follows that for any Drinfel'd associator $(1, f)$ with $f \in \mathfrak{t}_3$, the image $\iota(f)$ satisfies the defining relations of a KV associator. If the special automorphism G_F is in the image of the group inclusion $\exp(\mathfrak{t}_3) \hookrightarrow \mathrm{SAut}_3$, then we say the KV associator 'is a Drinfeld associator'. In Section 9 of [AT12], the authors use this fact, together with the existence of Drinfeld associators, to prove the Kashiwara–Vergne Theorem:

Theorem 3.34 ([AT12, Theorem 9.6])**.** *The KV problem of type* $(0, 2+1)$ *admits solutions.*

In [AKKN18b, Lemma 7.3], the authors show that solutions of type $(0, n+1)$ can be constructed operadically from those of type $(0, 2+1)$ via a composition procedure. For more on these compositions, see Appendix A.3.

The KV solutions of type $(0, n+1)$ naturally carry the structure of a *bitorsor* under two symmetry groups—the group $\mathsf{KV}(n)$ and its associated graded version $\mathsf{KRV}(n)$. These groups are defined in terms of tangential automorphisms subject to specific exponential and Jacobian constraints, and they act on opposite sides of the solution space.

Definition 3.35. The *graded Kashiwara–Vergne group* of type $(0, n+1)$, denoted $\mathsf{KRV}(n)$, consists of the tangential automorphisms $G \in \mathrm{TAut}_n$ satisfying:

$$G(e^{x_1+\cdots+x_n}) = e^{x_1+\cdots+x_n}, \tag{KRVI}$$

$$\exists\, h(z) \in z^2\mathbb{K}[[z]] \quad \text{such that} \quad J(G) = \mathrm{tr}\left(h\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} h(x_i)\right). \tag{KRVII}$$

Group multiplication is by composition of tangential automorphisms (3.1).

Definition 3.36. The *Kashiwara–Vergne group* of type $(0, n+1)$, denoted $\mathsf{KV}(n)$, consists of the tangential automorphisms $G \in \mathrm{TAut}_n$ satisfying:

$$G(e^{x_1}\cdots e^{x_n}) = e^{x_1}\cdots e^{x_n}, \tag{KVI}$$

$$\exists\, h(z) \in z^2\mathbb{K}[[z]] \quad \text{such that} \quad J(G) = \mathrm{tr}\left(h\left(\mathfrak{bch}(x_1, \dots, x_n)\right) - \sum_{i=1}^{n} h(x_i)\right). \tag{KVII}$$

The group structure is given by composition of tangential automorphisms (3.1).

Theorem 3.37 ([AKKN18a, Theorem 8.4])**.** *The group* $\mathsf{KRV}(n)$ *acts freely and transitively on the right of* $\mathsf{SolKV}(n)$ *by left multiplication by the inverse. The group* $\mathsf{KV}(n)$ *acts freely and transitively on the left of* $\mathsf{SolKV}(n)$ *by right multiplication with the inverse. Moreover, these two actions commute, making the set* $\mathsf{SolKV}(n)$ *into a bitorsor.*

Explicitly, for $F \in \mathsf{SolKV}(n)$ and $G \in \mathsf{KRV}(n)$, the right action is given by

$$F \cdot G := G^{-1}F,$$

while for $G \in \mathsf{KV}(n)$, the left action is

$$G \cdot F := FG^{-1}.$$

Special cases of this bitorsor structure appear in [AKKN18b, Remark 8.9], [AT12, Theorem 5.7], and [AET10, Proposition 8].

Recall that $\mathsf{GT}_1 \subset \mathsf{GT}$ denotes the acyclotomic Grothendieck–Teichmüller group, consisting of elements (λ, f) with $\lambda = 1$, and that GRT_1 is its graded analogue. In [AET10, Theorem 9] and [AT12, Theorem 4.6], the authors construct injective group homomorphisms

$$\mathsf{GT}_1 \hookrightarrow \mathsf{KV}(2) \quad \text{and} \quad \mathsf{GRT}_1 \hookrightarrow \mathsf{KRV}(2).$$

These maps intertwine the pentagon and hexagon identities that define GT_1 and GRT_1 with their analogues arising from nested applications of the Baker–Campbell–Hausdorff formula in the KV framework.

The following theorem establishes that $\{\mathsf{SolKV}(n)\}_{n\geq 2}$ and the corresponding symmetry groups are also equipped with a (non-symmetric) operadic structure. This fact is used in [AKKN18b, Section 7] and [AET10]. We provide further discussion, examples and proofs in Appendix A, see Theorem A.12 and Proposition A.15.

Theorem 3.38.

(i) The family $\mathsf{SolKV} := \{\mathsf{SolKV}(n)\}_{n\geq 2}$ *forms a* $\mathbb{K}[[z]]$*-colored non-symmetric operad. Namely for* $F = (F, h_1) \in \mathsf{SolKV}(m)$ *and* $G = (G, h_2) \in \mathsf{SolKV}(n)$ *two KV solutions with* $h_1 = h_2$*, then* $F \circ_i G$ *is still a KV solution.*

(ii) Similarly, the families $\{\mathsf{KRV}(n)\}_{n\geq 2}$ *and* $\{\mathsf{KV}(n)\}_{n\geq 2}$ *form non-symmetric* $\mathbb{K}[[z]]$*-colored operads in the category of sets.*

(iii) For $F \in \mathsf{SolKV}(2), G \in \mathsf{KRV}(2)$*, given any iterated composition of the form*

$$\tilde{F} = (\cdots((F \circ_{i_1} F) \circ_{i_2} F) \cdots \circ_{i_k} F) \in \mathsf{SolKV}(n), \quad \tilde{G} = (\cdots((G \circ_{i_1} G) \circ_{i_2} G) \cdots \circ_{i_k} G) \in \mathsf{KRV}(n),$$

the action satisfies

$$\tilde{F} \cdot \tilde{G} = (\cdots((F \cdot G \circ_{i_1} F \cdot G) \circ_{i_2} F \cdot G) \cdots \circ_{i_k} F \cdot G) \in \mathsf{SolKV}(n).$$

where $F \cdot G$ *as before denotes the right action of* $\mathsf{KRV}(2)$ *on* $\mathsf{SolKV}(2)$*. An analogous statement holds for* $\mathsf{KV}(n)$ *acting on the left.*

Remark 3.39. Note that by (KRVI), we have that $\mathsf{KRV}(n) \subseteq \mathrm{SAut}_n$, and therefore in that case $\circ_i$ is a group homomorphism, making $\{\mathsf{KRV}(n)\}_{n\geq 2}$ a non-symmetric $\mathbb{K}[[z]]$-colored operad in prounipotent groups. We can achieve the same for $\mathsf{KV}(n)$ by defining $\widetilde{\mathfrak{sDer}} = \{u \in \mathfrak{tDer} \ : \ u(\mathfrak{bch}(x_1, \ldots, x_n)) = 0\}$ with operadic composition defined the same way as in $\mathfrak{sDer}$ but with "$+$" replaced by "$\mathfrak{bch}$", and expontentiating that to get $\widetilde{\mathrm{SAut}}_n$, then $\mathsf{KV}(n) \subseteq \widetilde{\mathrm{SAut}}_n$.

3.3.1. *Symmetric KV solutions.* An *inner derivation* of a Lie algebra $\mathfrak{g}$ is a derivation arising from the adjoint action of $\mathfrak{g}$ on itself. For example, the tangential derivation $t = (x_2, x_1)$ (see also Example A.5) is equivalently described as the inner derivation given by the adjoint action of $(x_1 + x_2) \in \mathfrak{lie}_2$. That is, the inner derivation $\mathbf{t} := \mathrm{ad}_{x_1+x_2}$ acts on a Lie word $a \in \mathfrak{lie}_2$ as

$$\mathbf{t}(a) = [a, x_1 + x_2] = [a, x_1] + [a, x_2].$$

Indeed, $\mathbf{t}(x_1) = [x_1, x_2]$ and $\mathbf{t}(x_2) = [x_2, x_1]$, as in Example A.5.

Denote by b the tangential derivation $b := (0, x_1) \in \mathfrak{tDer}_2$, and set $B := \exp(b)$. The inner automorphism $e^{\mathbf{t}}$, together with the tangential automorphism B, are used to define an involution on the set of KV solutions of type $(0, 2+1)$.

Proposition 3.40 ([AT12, Proposition 8.4])**.** *The automorphism* $\tau : \mathsf{SolKV}(2) \to \mathsf{SolKV}(2)$ *defined by*

$$\tau(F) := e^{-\mathbf{t}/2} F^{2,1} B$$

defines an involution on the set of solutions to the KV problem. That is, if $F \in \mathsf{SolKV}(2)$*, then* $\tau(F) \in \mathsf{SolKV}(2)$ *and* $\tau^2(F) = F$*.*

Remark 3.41. Due to opposite braiding conventions with respect to those in [AT12], the automorphism B here is denoted $R^{2,1}$ in [AT12].

Definition 3.42. A KV solution $F \in \mathsf{SolKV}(2)$ is *symmetric* if $\tau(F) = F$. We write $\mathsf{SolKV}^{\tau}(2) \subseteq \mathsf{SolKV}(2)$ for the set of symmetric KV solutions.

As shown in [AT12, Proposition 8.9], the subgroups

$$\mathsf{KRV}^{\mathrm{sym}}(2) := \{G \in \mathsf{KRV}(2) \mid G^{2,1} = G\} \quad \text{and} \quad \mathsf{KV}^{\mathrm{sym}}(2) := \{G \in \mathsf{KV}(2) \mid G^{2,1} = G\}$$

of $\mathsf{KRV}(2)$ and $\mathsf{KV}(2)$ make $\mathsf{SolKV}^{\tau}(2)$ into a bitorsor. In particular, combining [AT12, Theorem 5.7] and [AT12, Proposition 8.9] we have the following.

Proposition 3.43. *Given two symmetric KV solutions $F_1, F_2 \in \mathsf{SolKV}^{\tau}(2)$, we have*

$$F_1F_2^{-1} \in \mathsf{KRV}^{\mathrm{sym}}(2) \quad \text{and} \quad F_1^{-1}F_2 \in \mathsf{KV}^{\mathrm{sym}}(2).$$

Proof. For convenience, we reprove the $\mathsf{KRV}^{\mathrm{sym}}$ statement directly; the proof for $\mathsf{KV}^{\mathrm{sym}}$ is similar. First, observe that since both F_1 and F_2 satisfy (SolKVI), we have

$$F_1F_2^{-1}(e^{x_1+x_2}) = F_1(e^{x_1}e^{x_2}) = e^{x_1+x_2},$$

which shows that $F_1F_2^{-1}$ satisfies (KRVI). To show that $F_1F_2^{-1}$ satisfies (KRVII), we use the 1-cocycle property of the Jacobian. We have

$$J(F_1F_2^{-1}) = J(F_1) + F_1J(F_2^{-1}) = J(F_1) - F_1F_2^{-1}J(F_2). \tag{3.6}$$

Since F_1 and F_2 are KV solutions, we have $J(F_1) = \mathrm{tr}(h_1(x_1+x_2) - h_1(x_1) - h_1(x_2))$ and $J(F_2) = \mathrm{tr}(h_2(x_1+x_2) - h_2(x_1) - h_2(x_2))$ for some Duflo functions h_1, h_2 in $z^2\mathbb{K}[[z]]$. We claim that the action of $F_1F_2^{-1}$ on $J(F_2)$ is trivial: as $F_1F_2^{-1}$ satisfies (KRVI), it preserves the sum $x_1 + x_2$ and thus the series $h_2(x_1 + x_2)$; moreover, it acts by conjugation on each of $h_2(x_1)$ and $h_2(x_2)$, which cancels under the trace. Therefore, (3.6) becomes

$$J(F_1F_2^{-1}) = J(F_1) - J(F_2) = \mathrm{tr}(h(x_1 + x_2) - h(x_1) - h(x_2)),$$

where $h := h_1 - h_2$ is the difference of the Duflo functions. We conclude that $F_1F_2^{-1}$ satisfies (KRVII).

Thus, we have that $F_1F_2^{-1}$ is in $\mathsf{KRV}(2)$. Using that F_1 and F_2 are symmetric, one can compute $(F_1F_2^{-1})^{2,1}$ directly as follows:

$$(F_1F_2^{-1})^{2,1} = (e^{\mathbf{t}/2}F_1B^{-1})(e^{\mathbf{t}/2}F_2B^{-1})^{-1} = e^{\mathbf{t}/2}F_1F_2^{-1}e^{-\mathbf{t}/2} = F_1F_2^{-1}$$

Thus, $F_1F_2^{-1} \in \mathsf{KRV}^{\mathrm{sym}}(2)$ as required. □

Out of symmetric KV solutions, in [AT12] the authors build special automorphisms which satisfy pentagon and hexagon equations – the defining equations of Drinfeld associators, but in a different space – as follows:

Proposition 3.44 ([AT12, Proposition 7.1])**.** *Let $F \in \mathsf{SolKV}(2)$ be a KV solution. Then, the tangential automorphism*

$$G_F := F^{1,23}F^{2,3}(F^{1,2})^{-1}(F^{12,3})^{-1}$$

is an element of $\mathsf{KRV}(3) \subset \mathrm{SAut}_3$ and satisfies the pentagon equation in TAut_4:

$$G^{1,2,34}G^{12,3,4} = G^{2,3,4}G^{1,23,4}G^{1,2,3}. \tag{P}$$

Proof. We give a short alternative proof. By Theorem 3.38 we know that $F \circ_1 F = F^{12,3}F^{1,2}$ and $F \circ_2 F = F^{1,23}F^{2,3}$ are in $\mathsf{SolKV}(3)$. It follows that $G_F = (F \circ_2 F)(F \circ_1 F)^{-1}$ is an element of $\mathsf{KRV}(3)$. Similarly, we have that both $(F \circ_1 F) \circ_1 F$ and $F \circ_2 (F \circ_2 F)$ are elements of $\mathsf{SolKV}(4)$. Now observe that both sides of (P) are elements of $\mathsf{KRV}(4)$ sending $(F \circ_1 F) \circ_1 F$ to $F \circ_2 (F \circ_2 F)$. Since the $\mathsf{KRV}(4)$ action on $\mathsf{SolKV}(4)$ is free and transitive (Theorem 3.37), the two sides must be equal. □

If F is a symmetric KV solution then more is true: G_F and $e^{\mathbf{t}}$ satisfy all of the remaining defining equations of Drinfeld associators in TAut_3, as follows.

Proposition 3.45 ([AT12, Proposition 8.11])**.** *Let $F \in \mathsf{SolKV}^{\tau}(2)$ be a symmetric KV solution, and let $G := G_F$ be the corresponding solution to the pentagon equation. Then $G_F \in \mathrm{SAut}_3$ is a KV-associator (as in Definition 3.33).*

Proposition 3.46. *Let $F \in \mathsf{SolKV}(2)$ be a KV solution. Then we have,*

$$\tau(F) = F \iff G_F = G_{\tau(F)}.$$

Proof. Suppose that we have $G_F = G_{\tau(F)}$. By [AT12, Proposition 8.5], we have $G_{\tau(F)} = (G_F^{3,2,1})^{-1}$. The fact that F is a symmetric KV solution then follows from [AT12, Proposition 9.3]. ☐

4. CONSTRUCTING KV SOLUTIONS

In [AET10], the authors give an explicit construction of a classical KV solution F_φ associated to a Drinfel'd associator $\varphi : \mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$ with coupling constant $\mu = 1$. In this section, we reinterpret their construction as a procedure that assigns, to each homomorphic expansion of $\mathsf{PaB}^1_{\mathbb{K}}$, or specifically, moperad equivalence $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$, a KV solution of type $(0, 2+1)$. This perspective clarifies how the KV equations emerge naturally from the defining relations of the $\mathsf{PaB}_{\mathbb{K}}$-moperad $\mathsf{PaB}^1_{\mathbb{K}}$.

The construction proceeds as follows. Given a moperad equivalence $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$, we associate to any fully parenthesized permutation w in $\mathrm{Ob}(\mathsf{PaB}^1(n))$ a tangential automorphism $F_w \in \mathrm{TAut}_n$. We will show that the automorphism $F_{0(12)}$ corresponding to the word $w = 0(12)$ is a KV solution. Furthermore, we demonstrate that for a full parenthetization of the identity permutation $12 \cdots n$, $w \in \mathrm{Ob}(\mathsf{PaB}^1(n))$, the tangential automorphism F_{0w} is a KV solution of type $(0, n+1)$. In particular, the collection of all $\{F_{0w}\}$, with w ranging over all full parenthesizations of the identity permutation $12 \cdots n$, defines the suboperad of SolKV (Theorem 3.38) which is freely generated by the classical KV solution $F_{0(12)}$.

Recall from Theorem 2.23, that $\mathsf{PaB}^1_{\mathbb{K}}$ is finitely presented as a PaB-moperad, and hence any equivalence of completed moperads $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ is uniquely determined by the data

$$\varphi(R^{1,2}) = e^{\mu t_{12}/2}, \quad \varphi(\Phi^{1,2,3}) = f(t_{12}, t_{23}), \quad \varphi^1(E^{0,1}) = e^{\mu t_{01}}, \quad \text{and} \quad \varphi^1(\Psi^{0,1,2}) = g(t_{01}, t_{12}),$$

so that the pair $\varphi(R^{1,2})$ and $\varphi(\Phi^{1,2,3})$ determine a Drinfeld associator and $\varphi(R^{1,2})$, $\varphi(\Phi^{1,2,3})$, $\varphi^1(E^{0,1})$ and $\varphi^1(\Psi^{0,1,2})$ satisfy equations (MP) and (O). In fact, for any given φ, there is only a one-parameter family of possible φ^1, which includes the shifted solution given by $g = f$, see Proposition 2.27. The following construction generalizes [AET10, Proposition 28].

Construction 4.1. Fix a moperad equivalence $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ and a parenthesized permutation $w \in \mathrm{Ob}(\mathsf{PaB}^1_{\mathbb{K}}(n))$. We consider the local isomorphism

$$\varphi^1_w : \mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(w) \cong (\mathsf{PB}^1_n)_{\mathbb{K}} \longrightarrow \exp(\mathfrak{t}^1_n) \tag{4.1}$$

of prounipotent groups induced by (φ^1, φ).

(a) Using that $\mathsf{PB}^1_n \cong F_n \rtimes \mathsf{PB}_n$ (Equation (2.10)), we restrict the isomorphism (4.1) to an isomorphism of free groups

$$\bar{\varphi}^1_w : (\mathsf{F}_n)_{\mathbb{K}} \longrightarrow \exp(\mathfrak{lie}_n), \tag{4.2}$$

which is completely determined by the image of the generators $X_1, \ldots, X_n$ of $(\mathsf{F}_n)_{\mathbb{K}}$.

(b) By Lemma 2.21, as a morphism in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(w)$, each generator X_k of $(\mathsf{F}_n)_{\mathbb{K}}$ can be (non-uniquely) represented via operadic, monoidal, and categorical compositions of the generating morphisms of $\mathsf{PaB}^1_{\mathbb{K}}$. In particular, $X_k \in \mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(w)$ is obtained by conjugating $E^{0,k} \in \mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(l^1_n)$ by twists and associativity isomorphisms.

(c) In turn, this determines the values $\bar{\varphi}^1_w$ takes on the generators X_k. Indeed, a straightforward calculation shows that $\bar{\varphi}^1_w(X_k) = e^{-z_k} e^{t_{0k}} e^{z_k}$ for some Lie word $z_k \in \mathfrak{t}^+_n$, as in Section 2.4.

(d) We rewrite these conjugations as $\bar{\varphi}^1_w(X_k) = e^{-z_k} e^{t_{0k}} e^{z_k} = e^{-z'_k} e^{t_{0k}} e^{z'_k}$, where $z'_k \in \mathfrak{lie}_n$. This is done via the homomorphism $\mathfrak{t}^+_n \to \mathfrak{tder}_n$, given by $t_{ij} \mapsto (0, \ldots, x_j, \ldots, x_i, \ldots, 0)$, and $t_{0i} \mapsto (x_i, x_i, \ldots, x_i)$. Here $1 \leq i, j \leq n$, and the values x_j and x_i are placed in position i and j, respectively. Then, the action of $\mathfrak{t}^+_n$ on its subalgebra $\mathfrak{lie}_n$ (generated by $t_{01}, \ldots, t_{0n}$) agrees with the action of the image in $\mathfrak{tder}_n$. Hence, $\exp(t^+_n)$ acts on $(\mathsf{F}_n)_{\mathbb{K}}$ by tangential automorphisms, which gives the second equality above. This is [AET10, Proposition 20].

(e) Completing this construction for each generator $X_1, \dots, X_n$ of $(\mathsf{F}_n)_{\mathbb{K}} \subset \mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(w)$, we find that (φ^1, φ) restricts to a tangential automorphism of $(\mathsf{F}_n)_{\mathbb{K}}$ (using $\exp(\mathfrak{lie}_n) \cong (\mathsf{F}_n)_{\mathbb{K}}, e^{t_{0i}} \mapsto X_i$). Denote this tangential automorphism by

$$F_{w,\varphi^1,\varphi} := (e^{z'_1}, \dots, e^{z'_n}).$$

Notation 4.2. We denote by $F_{w,\varphi^1,\varphi} := (e^{z'_1}, \dots, e^{z'_n})$ the tangential automorphism associated to the moperad equivalence $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ and a choice of parenthesized permutation $w \in \mathrm{Ob}(\mathsf{PaB}^1(n))$. To simplify notation, we will often suppress the dependence on the homomorphic expansion (φ^1, φ) and write F_w in place of $F_{w,\varphi^1,\varphi}$ whenever the context makes the choice of expansion clear.

Example 4.3. Let $l^1_2 = ((01)2)$ denote the leftmost parenthesization of the identity permutation in $\mathrm{Ob}(\mathsf{PaB}^1_{\mathbb{K}}(2))$. The generators of the free group $(\mathsf{F}_2)_{\mathbb{K}}$, viewed as automorphisms in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(2)}(l^1_2)$, can be presented as

$$X_1 = E^{0,1} \quad \text{and} \quad X_2 = (\Psi^{0,1,2})^{-1}(R^{2,1})^{-1}\Psi^{0,2,1}E^{0,2}(\Psi^{0,2,1})^{-1}R^{1,2}(\Psi^{0,1,2}).$$

Applying a moperad equivalence $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$, we compute that

$$\bar{\varphi}^1(X_1) = \varphi(E^{0,1}) = e^{\mu t_{01}} \quad \text{and} \tag{4.3}$$

$$\bar{\varphi}^1(X_2) = \Big(g(t_{02}, t_{12})^{-1} e^{\frac{\mu t_{12}}{2}} g(t_{01}, t_{12})\Big)^{-1} e^{\mu t_{02}} \Big(g(t_{02}, t_{12})^{-1} e^{\frac{\mu t_{12}}{2}} g(t_{01}, t_{12})\Big). \tag{4.4}$$

Where we note that we have written the formula for $\bar{\varphi}^1(X_1)$ in terms of group multiplication.

Since $t_{01} + t_{02} + t_{12}$ is central in $\mathfrak{t}^+_2 \cong \mathfrak{t}_3$, we replace t_{12} by $-t_{01} - t_{02}$ and rewrite (4.4) as

$$\Big(g(t_{02}, -t_{01} - t_{02})^{-1} e^{\frac{\mu(-t_{01}-t_{02})}{2}} g(t_{01}, -t_{01} - t_{02})\Big)^{-1} e^{\mu t_{02}} \Big(g(t_{02}, -t_{01} - t_{02})^{-1} e^{\frac{\mu(-t_{01}-t_{02})}{2}} g(t_{01}, -t_{01} - t_{02})\Big).$$

Thus, we obtain a tangential automorphism $F_{l^1_2,\varphi^1,\varphi} : (\mathsf{F}_2)_{\mathbb{K}} \to \exp(\mathfrak{lie}_2) \cong (\mathsf{F}_2)_{\mathbb{K}}$ defined by restricting (φ, φ^1) to $(\mathsf{F}_2)_{\mathbb{K}}$. As a two-tuple, this is given by

$$F_{l^1_2,\varphi^1,\varphi} = \Big(1, g(t_{02}, -t_{01} - t_{02})^{-1} e^{\frac{-\mu(t_{01}+t_{02})}{2}} g(t_{01}, -t_{01} - t_{02})\Big).$$

□

If we let $r^1_2 = (0(12))$ denote the rightmost parenthesization of the identity permutation in $\mathrm{Ob}(\mathsf{PaB}^1_{\mathbb{K}}(2))$, a similar calculation to that in Example 4.3 gives the tangential automorphism:

$$F_{r^1_2,\varphi^1,\varphi} = \Big(g(t_{01}, -t_{01} - t_{02}), g(t_{02}, -t_{01} - t_{02}) e^{\frac{-\mu(t_{01}+t_{02})}{2}}\Big). \tag{$F_{(0(12)}$}$$

For the remainder of this section, we fix an equivalence of completed moperads $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ given – as usual – by the values

$$\varphi(R^{1,2}) = e^{\frac{t_{12}}{2}}, \quad \varphi(\Phi^{1,2,3}) = f(t_{12}, t_{23}), \quad \varphi^1(E^{0,1}) = e^{t_{01}}, \quad \text{and} \quad \varphi^1(\Psi^{0,1,2}) = g(t_{01}, t_{12}). \tag{4.5}$$

One of the main results of this paper (Theorem 4.11) is that the tangential automorphism $F_{0(12)}$ is a KV solution of type $(0, 2+1)$. This is parallel to [AET10, Theorem 4]. We begin by addressing the first KV equation.

Lemma 4.4. *Let* $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ *be an equivalence of completed moperads as above. Then, the associated tangential automorphism* $F_{0(12)}$ *satisfies the first KV equation from Definition 3.31:*

$$F_{0(12)}(e^{x_1} e^{x_2}) = e^{x_1 + x_2}.$$

Proof. The morphisms

$$X_1 = \Psi^{0,1,2} E^{0,1} (\Psi^{0,1,2})^{-1} \quad \text{and} \quad X_2 = (R^{2,1})^{-1} \Psi^{0,2,1} E^{0,2} (\Psi^{0,2,1})^{-1} R^{1,2}.$$

generate a copy of the free group $(\mathsf{F}_2)_{\mathbb{K}} \subset \mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(2)}(0(12))$. Using first the octagon (O), then the right pentagon (RP) equation we can rewrite the product X_2X_1 as:

$$\begin{aligned} X_2X_1 &= (R^{2,1})^{-1}\Psi^{0,2,1}E^{0,2}(\Psi^{0,2,1})^{-1}R^{1,2}\Psi^{0,1,2}E^{0,1}(\Psi^{0,1,2})^{-1} \\ &\overset{(\mathrm{O})}{=} (R^{2,1})^{-1}\Psi^{0,2,1}E^{0,2}E^{02,1}(\Psi^{0,2,1})^{-1}(R^{1,2})^{-1} \\ &\overset{(\mathrm{RP})}{=} (R^{2,1})^{-1}\Psi^{0,2,1}\left(\Psi^{0,2,1}\right)^{-1}R^{1,2}E^{0,12}R^{2,1}\Psi^{0,2,1}(\Psi^{0,2,1})^{-1}(R^{1,2})^{-1} \\ &= E^{0,12}. \end{aligned}$$

Note that in the application of the RP relation in the second reduction uses that $E^{0,2}$ commutes with $E^{02,1}$: this is an equality in $\mathrm{Aut}_{\mathsf{PaB}^1}((01)2) \cong (\mathsf{PB}^1_2)_{\mathbb{K}} \subseteq (\mathsf{B}^1_2)_{\mathbb{K}}$, given in Remark 2.13, relation (2.6).

Applying Construction 4.16, we obtain $F_{0(12)}(e^{x_1}e^{x_2}) = \bar{\varphi}_{0(12)}(X_2X_1) = \bar{\varphi}_{0(12)}E^{0(12)} = e^{x_1+x_2}$, as claimed. □

Remark 4.5. Since we write function composition left to right, the isomorphism $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(w) \cong (\mathsf{PB}^1_n)_{\mathbb{K}}$ is strictly speaking an anti-isomorphism. This leads to the (somewhat confusing) equality $\bar{\varphi}^1_{0(12)}(X_2X_1) = F_{0(12)}(e^{x_1}e^{x_2})$. On the left-hand side of this equation, X_2X_1 denotes the function composition of morphisms which, under the identification $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(2)}(0(12)) \cong (\mathsf{PB}^1_2)_{\mathbb{K}} \cong (\mathsf{F}_2)_{\mathbb{K}} \rtimes (\mathsf{PB}_2)_{\mathbb{K}}$, is the product of group elements $e^{x_1}e^{x_2}$ in $(\mathsf{F}_2)_{\mathbb{K}}$.

The proof of Lemma 4.4 illustrates how the right pentagon (RP) and octagon (O) relations in the $\mathsf{PaB}_{\mathbb{K}}$-moperad structure on $\mathsf{PaB}^1_{\mathbb{K}}$ correspond to the first KV equation (SolKVI). Our goal is now to prove that $F_{0(12)}$ satisfies the second KV equation (SolKVII). In order to do so, we will use the following generalization of [AET10, Theorem 30]. Recall from Proposition 3.12 the operad structure on TAut, given explicitly for $F \in \mathrm{TAut}_m$ and $G \in \mathrm{TAut}_n$ by the formula

$$F \circ_i G = (F \circ_i 1)(1 \circ_i G) = F^{1,\dots,i-1,i(i+1)\cdots(i+n-1),i+n,\dots,m+n-1} \circ G^{i,i+1,\dots,i+(n-1)} \in \mathrm{TAut}_{m+n-1}\,.$$

Theorem 4.6. *Let $(\varphi^1,\varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ be an equivalence of completed moperads. Given parenthesized words $0(w) \in \mathsf{PaB}^1_{\mathbb{K}}(m)$, $0(w') \in \mathsf{PaB}^1_{\mathbb{K}}(n)$ we have the following equality of the associated tangential automorphisms:*

$$F_{0w} \circ_i F_{0w'} = F_{0(w\circ_i w')},$$

where the composition in the subscript is the partial composition of permutations (1.1).

Proof. Suppose that $w \in \mathsf{PaB}(m)$ and $w' \in \mathsf{PaB}(n)$. We write $F_{0w} = (\tilde{f}_1,\dots,\tilde{f}_m)$ and $F_{0w'} = (\tilde{g}_1,\dots,\tilde{g}_n)$ with $\tilde{f}_j \in \mathfrak{lie}_m$ and $\tilde{g}_k \in \mathfrak{lie}_n$, and denote $f_j := \tilde{f}_j \circ_i 0$ and $g_k := 0 \circ_i \tilde{g}_k$. With this notation, we must prove that

$$F_{0(w\circ_i w')} = (f_1,\dots,f_{i-1},f_i,\dots,f_i,f_{i+1},\dots,f_m)(1,\dots,1,g_1,\dots,g_n,1,\dots,1), \tag{4.6}$$

First observe that since (φ^1,φ) is a morphism of moperads, the following diagram commutes

$$\begin{tikzcd} \mathrm{Aut}_{\mathsf{PaB}^+_{\mathbb{K}}(m)}(0w) \times \mathrm{Aut}_{\mathsf{PaB}_{\mathbb{K}}(n)}(w') \arrow[r, "\varphi^1_{0w}\times\varphi_{w'}"] \arrow[d, "\circ_i"] & \exp(\mathfrak{t}^+_m) \times \exp(\mathfrak{t}_n) \arrow[d, "\circ_i"] \\ \mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(m+n-1)}(0(w\circ_i w')) \arrow[r, "\varphi^1_{0(w\circ_i w')}"] & \exp(\mathfrak{t}^+_{m+n-1}). \end{tikzcd}$$

Restricting to the free group $(\mathsf{F}_m)_{\mathbb{K}} \times \{1\}$ we get the commutative diagram

$$\begin{tikzcd} (\mathsf{F}_m)_{\mathbb{K}} \arrow[r, "\bar{\varphi}^1_{0w}"] \arrow[d] & \exp(\mathfrak{lie}_m) \arrow[d] \\ (\mathsf{F}_{m+n-1})_{\mathbb{K}} \arrow[r, "\bar{\varphi}^1_{0(w\circ_i w')}"] & \exp(\mathfrak{lie}_{m+n-1}). \end{tikzcd}$$

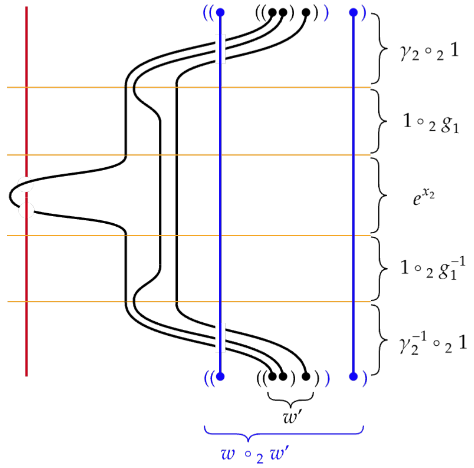


FIGURE 18. An example for showing the formula (4.6) for $j = i$. In this example, $w = ((12)3)$, $w' = ((12)3)$, and $i = 2$.

Here, the vertical arrows are explicitly given by

$$\begin{array}{ccc} (\mathsf{F}_m)_{\mathbb{K}} & \longrightarrow & (\mathsf{F}_{m+n-1})_{\mathbb{K}} \\ X_j & \mapsto & \begin{cases} X_j & \text{if } j < i, \\ X_{i+n-1} \cdots X_i & \text{if } j = i, \\ X_{j+n-1} & \text{if } j > i, \end{cases} \end{array} \quad \text{and} \quad \begin{array}{ccc} \exp(\mathfrak{lie}_m) & \longrightarrow & \exp(\mathfrak{lie}_{m+n-1}) \\ e^{x_j} & \mapsto & \begin{cases} e^{x_j} & \text{if } j < i, \\ e^{\sum_{\ell=i}^{i+n-1} x_\ell} & \text{if } j = i, \\ e^{x_{j+n-1}} & \text{if } j > i. \end{cases} \end{array}$$

Specializing to the generators X_j of $(\mathsf{F}_{m+n-1})_{\mathbb{K}}$ with $j \notin \{i, ..., i+n-1\}$, that is, the images of $X_j \in (\mathsf{F}_m)_{\mathbb{K}}$ for $j \neq i$, we obtain

$$F_{0(w \circ_i w')}(X_j) = \begin{cases} f_j^{-1} e^{x_j} f_j & \text{for } j < i, \\ f_{j-n+1}^{-1} e^{x_j} f_{j-n+1} & \text{for } j > i+n-1. \end{cases}$$

Here the isomorphism $\exp(\mathfrak{lie}_{m+n-1}) \cong (\mathsf{F}_{m+n-1})_{\mathbb{K}}$ identifies e^{x_j} with X_j for $j = 1, \ldots, m+n-1$. Thus, (4.6) holds for all X_j, $j \notin \{i, \ldots, i+n-1\}$.

It remains to show (4.6) for X_j, $j \in \{i, \ldots, i+n-1\}$. For $X_i \in (\mathsf{F}_m)_{\mathbb{K}}$, write

$$F_{0w}(X_i) = \gamma^{-1} e^{x_i} \gamma, \quad \gamma \in \exp(\mathfrak{t}_m^+) \tag{4.7}$$

as given by part (c) of Construction 4.1, and recall that we have denoted by

$$F_{0w}(X_i) = \tilde{f}_i^{-1} e^{x_i} \tilde{f}_i, \quad \tilde{f}_i \in \exp(\mathfrak{lie}_m) \tag{4.8}$$

the action given by part (d) of Construction 4.1.

Observe that $\tilde{f}_i \gamma^{-1}$ commutes with x_i, where the product is taken in $\mathfrak{t}_m^+$ via the inclusion $x_i \mapsto t_{0i}$ of $\mathfrak{lie}_m$ into $\mathfrak{t}_m^+$. The centralizer of t_{0i} in $\mathfrak{t}_m^+$ is the set $\{\lambda t_{0i} + a^{0i,1,\ldots,i-1,i+1,\ldots,m} \mid \lambda \in \mathbb{K} \text{ and } a \in \mathfrak{t}_m\}$. This is a short induction argument using that $\mathfrak{t}_{m+1} \cong \mathfrak{lie}_m \rtimes \mathfrak{t}_m$, as in [AET10, Proposition 51]. Hence, we write

$$\tilde{f}_i \gamma^{-1} = e^{\lambda x_i} \alpha^{0i,1,2,\ldots,i-1,i+1,\ldots,m}, \quad \text{for some } \alpha \in \exp(\mathfrak{t}_m). \tag{4.9}$$

It is possible to choose $\tilde{f}_i$ and γ such that $\log \tilde{f}_i$ and $\log \gamma$ have no linear terms in x_i (as adding a linear term in x_i does not change the action on x_i). Thus, taking log on both sides of Equation 4.9 and comparing linear terms we get $\lambda = 0$.

Further, the morphism X_i in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(m+n)}(0(w \circ_i w'))$, shown in Figure 18 in an example, is obtained by a composition of the action obtained from X_i in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(m)}(0w)$, with $1_{w'}$ inserted into the i-th strand, with the action obtained from X_i in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(0w')$:

$$F_{0(w\circ_i w')}(X_i) = (\gamma\circ_i 1)^{-1} g_1^{-1} e^{x_i} g_1 (\gamma\circ_i 1). \tag{4.10}$$

From (4.9), we have that

$$\gamma\circ_i 1 = (\alpha^{0i(i+1)\dots(i+n-1),1,2,\dots,i-1,i+1,\dots,m})^{-1} f_i.$$

Substituting this into (4.10), and observing that $(\alpha^{0i(i+1)\dots(i+n-1),1,2,\dots,i-1,i+1,\dots,m})$ commutes with each x_j for $i \le j \le i+n-1$, we obtain

$$\begin{aligned} F_{0(w\circ_i w')}(X_i) &= f_i^{-1}\alpha^{0i(i+1)\dots(i+n-1),1,2,\dots,i-1,i+1,\dots,m} g_1^{-1} e^{x_i} g_1 \left(\alpha^{0i(i+1)\dots(i+n-1),1,2,\dots,i-1,i+1,\dots,m}\right)^{-1} f_i \\ &= f_i^{-1} g_1^{-1} e^{x_i} g_1 f_i = (F_{0w}\circ_i F_{0w'})(X_i). \end{aligned}$$

The same argument shows that

$$F_{0(w\circ_i w')}(X_j) = f_i^{-1} g_{j-n+1}^{-1} e^{x_j} g_{j-n+1} f_i = F_{0w}\circ_i F_{0w'}(X_j)$$

for $j = i+1, \dots, i+n-1$. Thus we have $F_{0(w\circ_i w')}(X_j) = (F_{0w}\circ_i F_{0w'})(X_j)$ for all $1 \leqslant j \leqslant m+n-1$, completing the proof. ☐

We illustrate the preceding theorem with an explicit example.

Example 4.7. Let $F = F_{0(12)}$ be the tangential automorphism from Lemma 4.4 and write $r_3^1 = 0(1(23))$ for the rightmost parenthesization of the identity permutation in $\mathsf{PaB}^1_{\mathbb{K}}(3)$. We calculate that

$$F\circ_2 F = F_{0(12)}\circ_2 F_{0(12)} = F_{0(1(23))}.$$

The composition $F\circ_2 F = F^{1,23}\cdot F^{2,3}$ is given by

$$F^{2,3} = \left(1, g(t_{01}, -t_{01}-t_{02}-t_{03}), g(t_{02}, -t_{01}-t_{02}-t_{03})e^{\frac{-\mu(t_{01}+t_{02}+t_{03})}{2}}\right) \quad \text{and} \tag{4.11}$$

$$F^{1,23} = \left(g(t_{01}, -t_{01}-t_{02}-t_{03}), g(t_{02}, -t_{01}-t_{02}-t_{03})e^{\frac{-\mu(t_{01}+t_{02}+t_{03})}{2}}, g(t_{02}, -t_{01}-t_{02}-t_{03})e^{\frac{-\mu(t_{01}+t_{02}+t_{03})}{2}}\right). \tag{4.12}$$

To describe the action of $F\circ_2 F$ on X_1 in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(3)}(r_3^1)$, we first note that $X_1 \in \mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(3)}(r_3^1)$ can be expressed as

$$X_1 = \Psi^{0,1,23} E^{01} (\Psi^{0,1,23})^{-1}$$

Applying the moperad equivalence $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ then gives

$$\begin{aligned} \varphi^1(X_1) &= \varphi^1\left(\Psi^{0,1,23} E^{01} (\Psi^{0,1,23})^{-1}\right) \\ &= g(t_{01}, t_{12}+t_{13})^{-1} e^{t_{01}} g(t_{01}, t_{12}+t_{13}) \\ &= g(t_{01}, -t_{01}-t_{02}-t_{03})^{-1} e^{t_{01}} g(t_{01}, -t_{01}-t_{02}-t_{03}) \end{aligned}$$

Here, the last equality is using that $(t_{12}+t_{13}+t_{23}+t_{01}+t_{02}+t_{03})$ is central in $\mathfrak{t}_3^+$, so $g(t_{01}, t_{12}+t_{13}) = g(t_{01}, -t_{01}-t_{02}-t_{03}-t_{23})$, and t_{23} commutes with t_{01}. Comparing with Formulas (4.11) and (4.12), we see that indeed $\bar{\varphi}^1_{r_3^1}(X_1) = F^{1,23}\cdot F^{2,3}(X_1)$.

Before getting to the proof that $F_{0(12)}$ satisfies (SolKVII), we need one more ingredient.

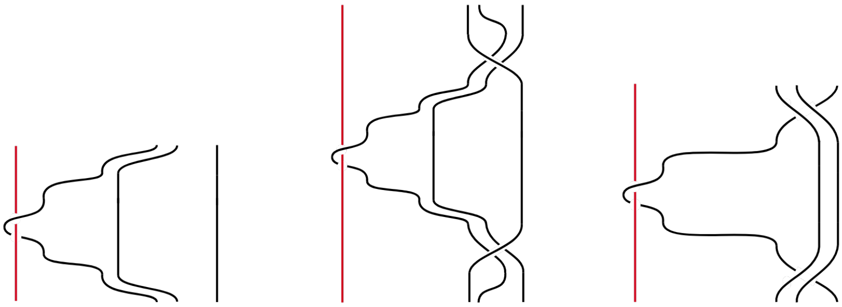
Proposition 4.8. *Let $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ be an equivalence of completed moperads given by the values (4.5). Then, the tangential automorphism $F = F_{0(12)}$ satisfies the identity*

$$f(t_{12}, t_{23})(F\circ_1 F) = (F\circ_2 F).$$

Remark 4.9. The product $f(t_{12}, t_{23})(F\circ_1 F)$ is understood as a composition in TAut_3, where $f(t_{12}, t_{23})$ acts by conjugation on $\exp(\mathfrak{lie}_3) \hookrightarrow \mathsf{CD}^+(3)$, identifying $e^{x_j} = e^{t_{0j}}$. In particular, if $(F\circ_1 F)(e^{t_{0j}}) = z_j^{-1} e^{t_{0j}} z_j$, then

$$\Big(f(t_{12}, t_{23})(F\circ_1 F)\Big)(e^{t_{0j}}) = f(t_{12}, t_{23})^{-1} z_j^{-1} e^{t_{0j}} z_j f(t_{12}, t_{23}).$$

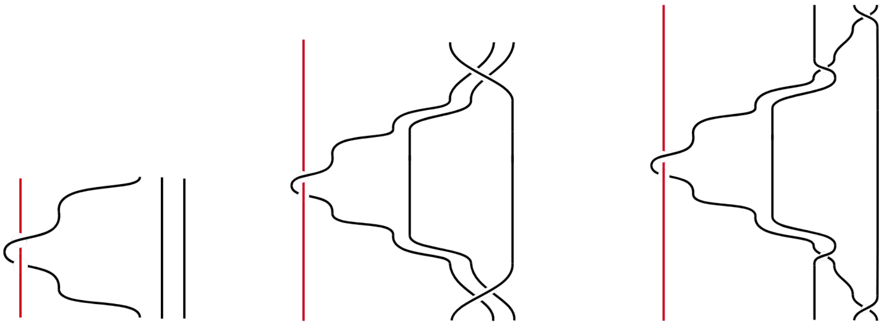
Proof. Theorem 4.6 implies that $F \circ_1 F = F_{0((12)3)}$ and $F \circ_2 F = F_{0(1(23))}$. Our proof strategy is to calculate the images of X_1, X_2 and X_3 under $\bar{\varphi}^1_{0((12)3)}$ and $\bar{\varphi}^1_{0(1(23))}$ and compare the results. To do so, we use specific choices for writing X_1, X_2 and X_3 as products of the generating morphisms $R^{1,2}$, Ψ and Φ. The expressions of the generators are illustrated in Figure 19 for 0((12)3) and in Figure 20 for 0(1(23)), and written out in detail below. We begin with X_1.



X_1 X_2 X_3

FIGURE 19. Expressing X_1, X_2 and X_3 as elements of $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(3)}\Big(0((12)3)\Big)$

.



X_1 X_2 X_3

FIGURE 20. Expressing X_1, X_2 and X_3 as elements of $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(3)}\Big(0(1(23))\Big)$

.

Case of X_1. In $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(3)}\Big(0((12)3)\Big)$, we have the expression

$$X_1 = \Psi^{0,12,3}\Psi^{0,1,2}E^{0,1}\left(\Psi^{0,1,2}\right)^{-1}\left(\Psi^{0,12,3}\right)^{-1} \tag{4.13}$$

From this expression, we follow Construction 4.1 to read off that the action of $(F \circ_1 F)$ on $e^{x_1} = e^{t_{01}}$ is conjugation by

$$z_{1_{(0((12)3))}} = g(t_{01}, t_{12})g(t_{01} + t_{02}, t_{13} + t_{23}).$$

Thus, $f(t_{12}, t_{23})(F \circ_1 F)$ conjugates $e^{x_1} = e^{t_{01}}$ by

$$z_{1_{(0((12)3))}} f(t_{12}, t_{23}) = g(t_{01}, t_{12})g(t_{01} + t_{02}, t_{13} + t_{23})f(t_{12}, t_{23}). \tag{4.14}$$

On the other hand, in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(3)}\Big(0(1(23))\Big)$, we express X_1 as

$$X_1 = \Psi^{0,1,23}E^{01}\left(\Psi^{0,1,23}\right)^{-1}, \tag{4.15}$$

and therefore, $F \circ_2 F$ conjugates e^{x_1} by

$$z_{1_{0(1(23)))}} = g(t_{01}, t_{12} + t_{13}). \tag{4.16}$$

We need to show that the action of $z_{1_{(0((12)3))}} f(t_{12}, t_{23})$ on $e^{t_{01}}$ agrees with the action of $z_{1_{(0(1(23)))}} = g(t_{01}, t_{12} + t_{13})$ on the same. First notice that t_{01} commutes with $(t_{02} + t_{12})$ and with t_{23}, and therefore

$$[g(t_{02} + t_{12}, t_{23}), t_{01}] = 0.$$

Therefore, the action of $z_{1_{(0(1(23)))}} = g(t_{01}, t_{12} + t_{13})$ on t_{01} agrees with the action of $g(t_{02} + t_{12}, t_{23}) g(t_{01}, t_{12} + t_{13})$.

Recall the Mixed Pentagon relation (MP) in $\mathsf{PaB}^1(3)$:

$$\Psi^{0,1,23} \Psi^{01,2,3} = \Phi^{1,2,3} \Psi^{0,12,3} \Psi^{0,1,2}$$

Therefore,

$$g(t_{02} + t_{12}, t_{23}) g(t_{01}, t_{12} + t_{13}) = g(t_{01}, t_{12}) g(t_{01} + t_{02}, t_{13} + t_{23}) f(t_{12}, t_{23}) = z_{1_{(0((12)3))}} f(t_{12}, t_{23}),$$

showing that

$$\Big(f(t_{12}, t_{23})(F \circ_1 F)\Big)(e^{x_1}) = (F \circ_2 F)(e^{x_1}).$$

Case of X_2. Moving on to X_2, in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(3)}\Big(0((12)3)\Big)$, we have the expression

$$X_2 = \left(\Phi^{1,2,3}\right)^{-1} \left(R^{1,23}\right)^{-1} \Psi^{0,23,1} \Psi^{0,2,3} E^{0,2} \left(\Psi^{0,2,3}\right)^{-1} \left(\Psi^{0,23,1}\right)^{-1} R^{1,23} \Phi^{1,2,3},$$

from which we read (noting the cancellation at the end):

$$z_{2_{(0((12)3))}} f(t_{12}, t_{23}) = g(t_{02}, t_{23}) g(t_{02} + t_{03}, t_{12} + t_{23}) e^{-\frac{t_{12} + t_{23}}{2}}$$

On the other hand, in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(3)}\Big(0(1(23))\Big)$, we express X_2 as:

$$X_2 = \left(R^{23,1}\right)^{-1} \Psi^{0,23,1} \Psi^{0,2,3} E^{0,2} \left(\Psi^{0,2,3}\right)^{-1} \left(\Psi^{0,23,1}\right)^{-1} R^{1,23}.$$

Applying the moperad equivalence we get:

$$z_{2_{(0(1(23)))}} = g(t_{02}, t_{23}) g(t_{02} + t_{03}, t_{12} + t_{13}) e^{-\frac{t_{12} + t_{13}}{2}}.$$

Comparing $z_{2_{(0((12)3))}}$ and $z_{2_{(0(1(23)))}}$, we see that $z_{2_{(0((12)3))}} f(t_{12}, t_{23}) = z_{2_{(0(1(23)))}}$, and therefore

$$\Big(f(t_{12}, t_{23})(F \circ_1 F)\Big)(e^{x_2}) = (F \circ_2 F)(e^{x_2}).$$

Case of X_3. Finally, for X_3, we express in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(3)}\Big(0((12)3)\Big)$:

$$X_3 = \left(R^{3,12}\right)^{-1} \Psi^{0,3,12} E^{03} \left(\Psi^{0,3,12}\right)^{-1} R^{12,3},$$

and therefore,

$$z_{3_{(0((12)3))}} f(t_{12}, t_{23}) = g(t_{03}, t_{03} + t_{13}) e^{-\frac{t_{13} + t_{23}}{2}} f(t_{12}, t_{23}).$$

In $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(3)}\Big(0(1(23))\Big)$, we can write

$$X_3 = \left(R^{2,3}\right)^{-1} \Phi^{1,3,2} \left(R^{1,3}\right)^{-1} \Psi^{0,31,2} \Psi^{0,3,1} E^{0,3} \left(\Psi^{0,3,1}\right)^{-1} \left(\Psi^{0,31,2}\right)^{-1} R^{13} \left(\Phi^{1,3,2}\right)^{-1} R^{2,3},$$

and therefore

$$z_{3_{(0(1(23)))}} = g(t_{03}, t_{31}) g(t_{01} + t_{03}, t_{12} + t_{23}) e^{-\frac{t_{13}}{2}} f(t_{13}, t_{23}) e^{-\frac{t_{23}}{2}}$$

To compare $z_{3_{(0((12)3))}} f(t_{12}, t_{23})$ with $z_{3_{(0(1(23)))}}$, recall the Hexagon relation in $\mathsf{PaB}(3)$ (here we permute the indices to match our application):

$$\left(R^{3,12}\right) \Phi^{1,2,3} = \left(\Phi^{3,1,2}\right)^{-1} \left(R^{3,1}\right)^{-1} \Phi^{1,3,2} \left(R^{3,2}\right)^{-1}.$$

Applying this to the last two terms of $z_{3_{(0((12)3))}}f(t_{12},t_{23})$, we obtain:

$$z_{3_{(0((12)3))}}f(t_{12},t_{23}) = g(t_{03},t_{03}+t_{13})f(t_{13},t_{12})^{-1}e^{-\frac{t_{13}}{2}}f(t_{13},t_{23})e^{-\frac{t_{23}}{2}}.$$

Since $(t_{01}+t_{13})$ and t_{12} both commute with t_{03}, it follows that $g(t_{01}+t_{13},t_{12})$ acts trivially on t_{03}, and thus, the action of $z_{3_{(0((12)3))}}f(t_{12},t_{23})$ agrees with the action of

$$g(t_{01}+t_{13},t_{12})g(t_{03},t_{03}+t_{13})f(t_{13},t_{12})^{-1}e^{-\frac{t_{13}}{2}}f(t_{13},t_{23})e^{-\frac{t_{23}}{2}}.$$

In turn, the first three terms of this expression can be reduced via an application of the mixed pentagon relation to arrive at:

$$g(t_{03},t_{13})g(t_{01}+t_{03},t_{12}+t_{23})e^{-\frac{t_{13}}{2}}f(t_{13},t_{23})e^{-\frac{t_{23}}{2}} = z_{3_{(0(1(23)))}}.$$

Thus,

$$\Big(f(t_{12},t_{23})(F\circ_1 F)\Big)(e^{x_3}) = (F\circ_2 F)(e^{x_3}),$$

completing the proof. □

We are now in position to complete the proof that $F_{0(12)}$ is a KV solution of type $(0,2+1)$.

Lemma 4.10. *Let* $(\varphi^1,\varphi):\mathsf{PaB}^1_{\mathbb{K}}\to\mathsf{CD}^+$ *be an equivalence of completed moperads given by the values* (4.5)*. Then, the associated tangential automorphism* $F=F_{0(12)}$ *satisfies the second KV equation:*

$$\exists\, h\in z^2\mathbb{K}[[z]] \quad \textit{such that} \quad J(F)=\operatorname{tr}\left(h\left(x_1+x_2\right)-h(x_1)-h(x_2)\right).$$

Proof. We compute the cosimplicial differential d of $J(F)$ (see Appendix B). We use the following facts:

- the Jacobian is a morphism of operads (Proposition 3.14), and therefore commutes with d;
- $dF=(F\circ_2 F)(F\circ_1 F)^{-1}$, see Example B.2;
- $f(t_{12},t_{23})=(F\circ_2 F)(F\circ_1 F)^{-1}$ from Proposition 4.8;
- $J(f(t_{12},t_{23}))=0$ by [AET10, Proposition 22].

Hence, we obtain

$$d(J(F))=J(dF)=J((F\circ_2 F)(F\circ_1 F)^{-1})=J(f(t_{12},t_{23}))=0.$$

Since the cosimplicial cohomology of cyc vanishes in degree 2 [AT12, Theorem 2.8], there is an $h\in\mathrm{cyc}_1=\mathbb{K}[[z]]$ such that $J(F)=d(h)=\operatorname{tr}\left(h\left(x_1+x_2\right)-h(x_1)-h(x_2)\right)$, as claimed. □

Combining Lemma 4.4 and Lemma 4.10, we obtain the following result:

Theorem 4.11. *Let* $(\varphi^1,\varphi):\mathsf{PaB}^1_{\mathbb{K}}\to\mathsf{CD}^+$ *be an equivalence of completed moperads with coupling constant* $\mu=1$*. Then, the associated tangential automorphism* $F=F_{0(12)}$ *is a symmetric KV solution of type* $(0,2{+}1)$*.*

Proof. The only remaining fact to prove is that F is a symmetric KV solution. Proposition 4.8 shows that its KV associator is equal to $G_F=f(t_{12},t_{23})$. Since f is a Drinfeld associator, it satisfies the equation (I). We deduce that $G_F=(G_F^{3,2,1})^{-1}$. The conclusion then follows from Proposition 3.46. □

In fact, Construction 4.1 defines a *family* of genus zero KV solutions – the suboperad of SolKV generated by the KV solution $F_{0(12)}$. These operadic compositions echo the constructions found in [AT12, Section 3], [AET10, Appendix B1], and especially [AKKN18b, Section 7], where a gluing procedure for genus zero surfaces with boundary is used to produce families of higher-arity KV solutions.

Lemma 4.12. *Let* $(\varphi^1,\varphi):\mathsf{PaB}^1_{\mathbb{K}}\to\mathsf{CD}^+$ *be an equivalence of completed moperads with coupling constant* $\mu=1$*. For any object of the form* $0w=0(w)\in\mathsf{PaB}^1(n)$ *with* w *a parenthesization of the identity permutation* $1\cdots n$*, the associated tangential automorphism* F_{0w} *satisfies the first KV equation*

$$F_{0w}(e^{x_1}\cdots e^{x_n})=e^{x_1+\cdots+x_n}.$$

Proof. First observe that by Theorem 4.6, for any $0w$ as in the statement, the tangential automorphism F_{0w} can be obtained from $F_{0(12)}$ by operadic composition. By Lemma 4.4, $F_{0(12)}$ satisfies (SolKVI), and in Theorem 3.38 we proved that operadic composition preserves the first KV equation. This completes the proof. □

Lemma 4.13. *Let* $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ *be an equivalence of completed moperads with coupling constant* $\mu = 1$*. For any word* $0w \in ob(\mathsf{PaB}^1_{\mathbb{K}}(n))$*, the associated tangential automorphism* F_{0w} *satisfies the second KV equation*

$$\exists\, h \in \mathbb{K}[[z]] \quad \textit{such that} \quad J(F_{0w}) = \operatorname{tr}\left(h\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} h(x_i)\right).$$

Moreover, the Duflo function h is the same for every word.

Proof. By Theorem 4.6, for any $n \geqslant 2$, the tangential automorphism $F_{r^1_n}$ can be obtained from $F_{r^1_2} = F_{0(12)}$ by operadic composition. In Theorem 3.38, we prove that operadic composition preserves the second KV equation as long as the Duflo function $h \in \mathbb{K}[[z]]$ is the same for all tangential automorphisms involved. By Lemma 4.10, $F_{r^1_2}$ satisfies (SolKVII), and therefore so does $F_{r^1_n}$ (see also [AKKN18b, Lemma 7.3]). In particular, any well-defined operadic composition of $F_{r^1_2}$ will have the same Duflo function as $F_{r^1_2}$. Now, any word $0w \in \mathrm{Ob}(\mathsf{PaB}^1_{\mathbb{K}}(n))$ can be obtained from r^1_n by conjugation with the associators Ψ and Φ, and action of the symmetric group Σ^+_n. Since these operations preserve the Jacobian (see Remark A.14), we have $J(F_{0w}) = J(F_{r^1_n})$, and the conclusion follows. □

Combining Lemma 4.12 with Lemma 4.13, we obtain the following theorem, similar in spirit to the arguments in [AET10, Appendix B].

Theorem 4.14. *Let* $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ *be an equivalence of completed moperads with coupling constant* $\mu = 1$*. For any word* $0w \in \mathrm{Ob}(\mathsf{PaB}^1_{\mathbb{K}}(n))$ *with w a parenthesization of the identity permutation* $1 \cdots n$*, the associated tangential automorphism* F_{0w} *is a KV solution of type* $(0, n+1)$*. Moreover, the set of such automorphisms can be identified with the suboperad of* SolKV *generated by the symmetric KV solution* $F_{0(12)}$*.*

Remark 4.15. Tangential automorphisms F_w for other parenthesised words (i.e. not of the form $0w$) are not KV solutions, as they only satisfy (SolKVI) up to conjugation by associators Φ and Ψ. Note for instance that $F_{(01)2}$ does *not* satisfy (SolKVI). In addition, the symmetric group action on KV solutions does not preserve (SolKVI) either, hence SolKV form a non-symmetric operad, see Remark A.14.

4.1. **Actions of GTM on KV solutions.** Recall from Section 2.5 that the the Grothendieck–Teichmüller module groups GTM and GRTM are canonically isomorphic to the groups of object-fixing automorphisms of the moperads $\mathsf{PaB}^1_{\mathbb{K}}$ and PaCD^+, respectively:

$$\mathsf{GTM} \cong \mathrm{Aut}_0(\mathsf{PaB}^1_{\mathbb{K}}), \qquad \mathsf{GRTM} \cong \mathrm{Aut}_0(\mathsf{PaCD}^+).$$

We denote by $\mathsf{GTM}_1 \subset \mathsf{GTM}$ and $\mathsf{GRTM}_1 \subset \mathsf{GRTM}$ the subgroups consisting of elements of the form $(1, f, g)$, that is, those with twisting constant $\lambda = 1$. In this section, we show that the assignment of a KV solution $F_{r^1_2,\varphi^1,\varphi}$ to moperad equivalences $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ intertwines the free and transitive actions of the symmetry groups $\mathsf{GTM}_1, \mathsf{GRTM}_1, \mathsf{KV}(2)$ and $\mathsf{KRV}(2)$.

Applying Theorem 2.33, and setting $\lambda = 1$, we know that an element $(1, f, g) \in \mathsf{GTM}_1$ defines an object-fixing automorphism $(\vartheta^1, \vartheta) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaB}^1_{\mathbb{K}}$ defined on generators by:

(4.17)
$$\vartheta(R^{1,2}) = R^{1,2}, \quad \vartheta(\Phi^{1,2,3}) = \Phi^{1,2,3} \cdot f(x_{12}, x_{23}), \quad \vartheta^1(E^{0,1}) = E^{0,1}, \quad \vartheta^1(\Psi^{0,1,2}) = \Psi^{0,1,2} \cdot g(x_{01}, x_{12}),$$

where x_{ij} denote the pure braid generators as in (2.9).

Given any object-fixing automorphism $(\vartheta^1, \vartheta) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaB}^1_{\mathbb{K}}$ and a parenthesization $0(w)$ of the identity permutation $1, 2, ..., n$, the automorphism (ϑ^1, ϑ) restricts to a local automorphism at the object $0(w) \in \mathrm{Ob}(\mathsf{PaB}^1_{\mathbb{K}}(n))$:

$$\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(0w) \cong (\mathsf{PB}^1_n)_{\mathbb{K}} \xrightarrow{\vartheta^1} (\mathsf{PB}^1_n)_{\mathbb{K}} \cong \mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(0w).$$

Our goal now, given (ϑ^1,ϑ) and a word w, is to construct a corresponding automorphism of the completed free group $(\mathsf{F}_2)_{\mathbb{K}}$, and show that for $w=0(12)$ we obtain elements of $\mathsf{KV}(2)$.

Construction 4.16. Fix a parenthesization of the identity permutation $1,2,...,n$, denoted $0w$, and an object-fixing automorphism of $\mathsf{PaB}^1_{\mathbb{K}}$, denoted $(\vartheta^1,\vartheta):\mathsf{PaB}^1_{\mathbb{K}}\to\mathsf{PaB}^1_{\mathbb{K}}$, given by values as in (4.17).

(a) As in Construction 4.1, we restrict (ϑ^1,ϑ) the local isomorphism

$$\vartheta^1_{0w}:\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(0w)\cong(\mathsf{PB}^1_n)_{\mathbb{K}}\longrightarrow(\mathsf{PB}^1_n)_{\mathbb{K}}\cong\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(0w),$$

to an isomorphism of free groups

$$\bar{\vartheta}^1_{0w}:(\mathsf{F}_n)_{\mathbb{K}}\longrightarrow(\mathsf{F}_n)_{\mathbb{K}},$$

where $(\mathsf{F}_n)_{\mathbb{K}}$ is freely generated by $X_1,X_2,\dots X_n$ in $(\mathsf{PB}^1_n)_{\mathbb{K}}$.

(b) The generators $X_1,X_2,\dots X_n$ of $(\mathsf{F}_n)_{\mathbb{K}}$ are (non-uniquely) represented as products of the generating morphisms $E^{0,i},R^{i,j},\Phi^{i,j,k}$ and $\Psi^{0,i,j}$ in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(0w)$. For example, when $n=2$ and $w=0(12)$, we have

$$X_1=\Psi^{0,1,2}E^{0,1}(\Psi^{0,1,2})^{-1}\quad\text{and}\quad X_2=(R^{2,1})^{-1}\Psi^{0,2,1}E^{0,2}(\Psi^{0,2,1})^{-1}R^{1,2}.$$

This determines the values of $\bar{\vartheta}^1_{0w}(X_i)$ for $i=1,\dots,n$. In particular, the value $\bar{\vartheta}^1_{0w}(X_i)$ is a conjugate of X_i by a braid in $(\mathsf{PB}^1_n)_{\mathbb{K}}$, and therefore also lies in $(\mathsf{F}_n)_{\mathbb{K}}$, since $(\mathsf{F}_n)_{\mathbb{K}}$ is a normal subgroup in $(\mathsf{PB}_n)_{\mathbb{K}}$. This defines an automorphism of the free group

$$G_{0w,\vartheta^1,\vartheta}:(\mathsf{F}_n)_{\mathbb{K}}\to(\mathsf{F}_n)_{\mathbb{K}}, \tag{$G_{0w,\vartheta^1,\vartheta}$}$$

which is given on generators by $X_i\mapsto\bar{\vartheta}^1_{0w}(X_i)$ for $i=1,\dots,n$.

The next lemma will show that when $(\vartheta^1,\vartheta):\mathsf{PaB}^1_{\mathbb{K}}\to\mathsf{PaB}^1_{\mathbb{K}}$ defined on generators via the equations in (4.17) then the induced automorphism of $(\mathsf{F}_2)_{\mathbb{K}}$, $\bar{\vartheta}^1_{0(12)}$, satisfies the first defining equation of the KV symmetry group $\mathsf{KV}(2)$. Before we state and prove the lemma, however, we recall a notational subtlety. The values $\bar{\vartheta}^1_{0w}(X_i)$ are in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}(0w)\cong(\mathsf{PB}^1_n)_{\mathbb{K}}\subset(\mathsf{B}^1_n)_{\mathbb{K}}$. Recall that, in the braid groups, the generators are named according to the positions of strands, rather than their labels. In particular, after pushing forward along the composition $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(2)}(0(12))\cong(\mathsf{PB}^1_2)_{\mathbb{K}}\hookrightarrow(\mathsf{B}^1_2)_{\mathbb{K}}$ we have:

$$\bar{\vartheta}^1_{0(12)}(X_1)=g(x_{01},x_{12})x_{01}g(x_{01},x_{12})^{-1}\quad\text{and}\quad\bar{\vartheta}^1_{0(12)}(X_2)=\beta_1^{-1}g(x_{01},x_{12})x_{01}g(x_{01},x_{12})^{-1}\beta_1.$$

When writing these formulas we have used the fact that the isomorphism $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(n)}((01)2)\cong(\mathsf{PB}^1_2)_{\mathbb{K}}$ identifies $E^{0,1}$ with x_{01} in $(\mathsf{PB}^1_2)_{\mathbb{K}}$. Similarly, the morphism $R^{1,2}$ has been identified with the braid generator β_1 and both β_1 and β_1^{-1} are defined in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(2)}(0(12))\cong(\mathsf{PB}^1_2)_{\mathbb{K}}$ because, while the value $\vartheta^1_{0(12)}(X_2)$ is a pure braid, it is naturally expressed as the multiplication of elements in $(\mathsf{B}^1_2)_{\mathbb{K}}$. Finally, we have omitted the instances of the associativity isomorphism Ψ from the formulas, as these are trivial as elements in $(\mathsf{F}_2)_{\mathbb{K}}\subseteq\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(2)}(0(12))\cong(\mathsf{PB}^1_2)_{\mathbb{K}}$.

Lemma 4.17. *Given an object-fixing automorphism* $(\vartheta^1,\vartheta):\mathsf{PaB}^1_{\mathbb{K}}\to\mathsf{PaB}^1_{\mathbb{K}}$ *defined on generators as in* (4.17)*, and the word* $r^1_2=0(12)$*, then the automorphism*

$$G_{\vartheta^1}=G_{r^1_2,\vartheta^1,\vartheta}:(\mathsf{F}_2)_{\mathbb{K}}\longrightarrow(\mathsf{F}_2)_{\mathbb{K}}$$

is a tangential automorphism. Moreover, G_{ϑ^1} *satisfies equation* (KVI)*,*

$$G_{\vartheta^1}(X_2X_1)=X_2X_1.$$

Proof. The more substantial part of the claim is that G_{ϑ^1} is a tangential automorphism. We computed above that the moperad map defines an action on generators:

$$\bar{\vartheta}^1_{0(12)}(X_1)=g(x_{01},x_{12})x_{01}g(x_{01},x_{12})^{-1}\quad\text{and}\quad\bar{\vartheta}^1_{0(12)}(X_2)=\beta_1^{-1}g(x_{01},x_{12})x_{01}g(x_{01},x_{12})^{-1}\beta_1.$$

First recall that that the square of the full twist is central in a braid group: in this case, the element $z=x_{01}x_{12}x_{02}$ is central in $(\mathsf{B}^1_2)_{\mathbb{K}}$. Therefore, $x_{12}=x_{01}^{-1}zx_{02}^{-1}$. Substituting this into $\vartheta^1_{0(12)}(X_1)$, we obtain

$$\bar{\vartheta}^1_{0(12)}(X_1)=g(x_{01},x_{01}^{-1}zx_{02}^{-1})x_{01}g(x_{01},x_{01}^{-1}zx_{02}^{-1})^{-1}.$$

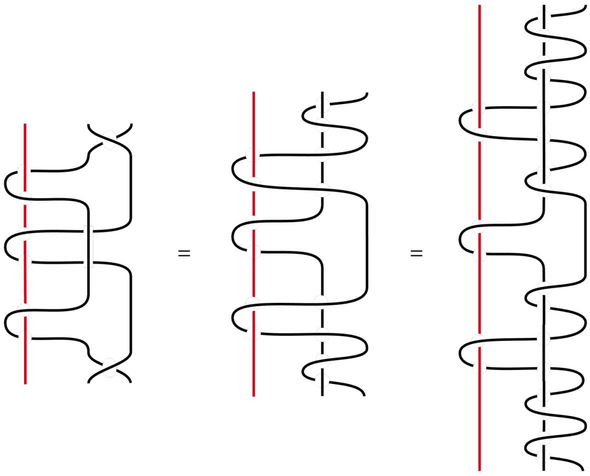


FIGURE 21. Let $\omega(x_{01}, x_{02}) = x_{01}^{-1}x_{02}x_{01}$. We demonstrate the braid equivalence between $\beta_1^{-1}\omega\beta_1$ (displayed on the left), and $x_{12}^{-1}\omega(x_{12}x_{02}x_{12}^{-1}, x_{01})x_{12}$ (displayed on the right).

In the conjugate, powers of z cancel with powers of z^{-1} since z is central, therefore,

$$\bar\vartheta^1_{0(12)}(X_1) = g(x_{01}, x_{01}^{-1}x_{02}^{-1})x_{01}g(x_{01}, x_{01}^{-1}x_{02}^{-1})^{-1}, \tag{4.18}$$

is a conjugate of X_1 by an element of $(\mathsf{F}_2)_{\mathbb{K}}$.

As for $\vartheta^1_{0(12)}(X_2)$, the same trick can be applied to the conjugation by $g(x_{01}, x_{12})$, to obtain

$$\bar\vartheta^1_{0(12)}(X_2) = \beta_1^{-1}g(x_{01}, x_{01}^{-1}x_{02}^{-1})x_{01}g(x_{01}, x_{01}^{-1}x_{02}^{-1})^{-1}\beta_1. \tag{4.19}$$

We observe that given $\omega(x_{01}, x_{02})$ a word in the free group $(\mathsf{F}_2)_{\mathbb{K}} \subseteq (\mathsf{B}_2^1)_{\mathbb{K}}$, we have

$$\beta_1^{-1}\omega(x_{01}, x_{02})\beta_1 = x_{12}^{-1}\omega(x_{12}x_{02}x_{12}^{-1}, x_{01})x_{12}$$

This follows from the braid relations; see Figure 21 for an illustration.

Now using again that $x_{12} = x_{01}^{-1}zx_{02}^{-1}$, we obtain $x_{12}x_{02}x_{12}^{-1} = x_{01}^{-1}x_{02}x_{01}$. Substituting this into the formula (4.19), we get

$$\begin{aligned}\bar\vartheta^1_{0(12)}(X_2) = x_{02}x_{01}g\left(x_{01}^{-1}x_{02}x_{01}, x_{01}^{-1}x_{02}^{-1}x_{01}x_{02}^{-1}\right)x_{01}^{-1} \cdot x_{02} \cdot \\ \cdot x_{01}g\left(x_{01}^{-1}x_{02}x_{01}, x_{01}^{-1}x_{02}^{-1}x_{01}x_{02}^{-1}\right)^{-1}x_{01}^{-1}x_{02}^{-1},\end{aligned} \tag{4.20}$$

completing the proof that G_{ϑ^1} is a tangential automorphism.

To show that $G_{\vartheta^1}(X_2X_1) = X_2X_1$, we recall from Lemma 2.29 that $X_2X_1 = E^{0,12}$, and $\vartheta^1(E^{0,1}) = E^{0,1}$ by assumption. Since $E^{0,12} = E^{0,1} \circ_1 \mathrm{id}_2$ and ϑ^1 is a map of moperads, we have that also $\vartheta^1(E^{0,12}) = E^{0,12}$. The lemma follows. □

Remark 4.18. We note that given an automorphism $(\vartheta^1, \vartheta) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaB}^1_{\mathbb{K}}$ in GTM with twisting constant $\lambda \neq 1$, the associated tangential automorphism G_{ϑ^1} does not satisfy (KVI) because $G_{\vartheta^1}(X_2X_1) = (X_2X_1)^{\lambda}$.

The following theorem shows that the tangential automorphism $G_{\vartheta^1} = G_{r_1^2,\vartheta^1,\vartheta} \in \mathrm{TAut}_2$ from Lemma 4.17 is an element of the KV symmetry group $\mathsf{KV}^{\mathrm{sym}}(2)$. Moreover, the assignment $(\vartheta^1, \vartheta) \mapsto G_{\vartheta^1}$ defines an injective group homomorphism $\mathsf{GTM}_1 \hookrightarrow \mathsf{KV}^{\mathrm{sym}}(2)$.

Theorem 4.19. *Let* $(\vartheta^1, \vartheta) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaB}^1_{\mathbb{K}}$ *be an automorphism in* GTM_1 *and let* $G_{\vartheta^1} = G_{r_1^2,\vartheta^1,\vartheta} \in \mathrm{TAut}_2$ *be the associated tangential automorphism. Let* $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ *be an equivalence of completed moperads with coupling constant* $\mu = 1$ *with associated symmetric KV solution* $F_{\varphi^1} = F_{r_2^1,\varphi^1,\varphi}$. *Then* $F_{\varphi^1} \circ$

G_{ϑ^1} is also a symmetric KV-solution, and $G_{\vartheta^1} \in \mathsf{KV}^{\mathrm{sym}}(2)$. This defines an injective group homomorphism $\mathsf{GTM}_1 \hookrightarrow \mathsf{KV}^{\mathrm{sym}}(2)$.

Proof. Since (ϑ^1, ϑ) is a completed moperad automorphism, the composite $(\varphi^1, \varphi) \circ (\vartheta^1, \vartheta) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ is an equivalence of completed moperads. Furthermore, since (ϑ^1, ϑ) has twisting constant 1, the composite has coupling constant 1.

By construction F_{φ^1} and G_{ϑ^1} are obtained by restricting (φ^1, φ) and (ϑ^1, ϑ), respectively, to $(\mathsf{F}_2)_{\mathbb{K}} \subseteq \mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(2)}(0(12))$. Therefore, $F_{\varphi^1} \circ G_{\vartheta^1} = F_{\varphi^1 \circ \vartheta^1}$. Since $\varphi^1 \circ \vartheta^1$ has coupling constant 1, this is a symmetric KV solution by Theorem 4.11. Moreover, since both F_{φ^1} and $F_{\varphi^1 \circ \vartheta^1}$ are symmetric KV solutions, we have $G_{\vartheta^1} \in \mathsf{KV}^{\mathrm{sym}}(2)$ by Proposition 3.43.

The fact that the resulting map $\mathsf{GTM}_1 \cong \mathrm{Aut}_0(\mathsf{PaB}^1_{\mathbb{K}}) \hookrightarrow \mathsf{KV}(2)$ is a group homomorphism follows from the identity $G_{\vartheta^1} \circ G_{\vartheta'^1} = G_{\vartheta^1 \circ \vartheta'^1}$, which in turn follows from the fact that in Construction 4.16 both of the tangential automorphisms G_{ϑ^1} and $G_{\vartheta'^1}$ are obtained via restriction to $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(2)}(0(12))$.

To show that the map is injective, we assume that $G_{\vartheta^1} = 1$, and aim to show that $(\vartheta^1, \vartheta) = 1$. We have computed in (4.18) that $G_{\vartheta^1}(x_{01}) = g(x_{01}, x_{01}^{-1}x_{02}^{-1})x_{01}g(x_{01}, x_{01}^{-1}x_{02}^{-1})^{-1}$. Since x_{01} and x_{02} are free generators, $g(x_{01}, x_{01}^{-1}x_{02}^{-1})$ commutes with x_{01} only if g does not depend on its second argument, in other words, $g(x, y) = g(x)$. We show that this, however, contradicts the mixed pentagon equation (MP):

$$g(x_{01}, x_{13}x_{12})g(x_{12}x_{02}, x_{23}) = f(x_{12}, x_{23})g(x_{02}x_{01}, x_{23}x_{13})g(x_{01}, x_{12}). \tag{MP}$$

Indeed, if g only depends on its first argument, we obtain

$$g(x_{01})g(x_{12}x_{02})g(x_{01})^{-1}g(x_{02}x_{01})^{-1} = f(x_{12}, x_{23}) \tag{4.21}$$

Here the left hand side is a pure braid on the strands 0, 1 an 2, not involving strand 3. On the right hand side, note that x_{12} and x_{23} generate a copy of $(F_2)_{\mathbb{K}}$ within $(\mathsf{PB}^1_3)_{\mathbb{K}}$. Thus, if $f(x_{12}, x_{23})$ does not involve strand 3, then f also only depends on its first argument: $f(x_{12}, x_{23}) = f(x_{12})$. In that case, the right hand side does not involve strand 0, so the left hand side must remain the same if we delete strand 0. (Deleting strand i is a well-defined group homomorphism of pure braid groups $s_i : \mathsf{PB}_n \to \mathsf{PB}_{n-1}$, which lifts to the completion. In this case $s_0 : \mathsf{PB}^1_3 \to \mathsf{PB}_3$.) By deleting strand 0, we obtain: $g(x_{12}) = f(x_{12})$, so $g = f$. Therefore, we have:

$$g(x_{01})g(x_{12}x_{02})g(x_{01})^{-1}g(x_{02}x_{01})^{-1} = g(x_{12})$$

Similarly, deleting strand 2 on both sides, we get

$$g(x_{01})g(x_{01})^{-1}g(x_{01})^{-1} = 1$$

This implies $g = 1 = f$, so (ϑ^1, ϑ) is the identity, completing the proof. □

Corollary 4.20. *Let $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ be an equivalence of completed moperads with $\mu = 1$ and associated symmetric KV solution $F_{\varphi^1} = F_{r^1_2, \varphi^1, \varphi}$. Let $(\vartheta^1, \vartheta) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaB}^1_{\mathbb{K}}$ be an object-fixing automorphism with $\lambda = 1$ and let $G_{\vartheta^1} = G_{r^1_2, \vartheta^1, \vartheta}$ be the associated element of $\mathsf{KV}^{\mathrm{sym}}(2)$. Then*

$$F_{\varphi^1 \vartheta^1} = G_{\vartheta^1} \cdot F_{\varphi^1},$$

where $\cdot$ denotes the action of $\mathsf{KV}^{\mathrm{sym}}(2)$ on the set of symmetric KV solutions.

Every moperad equivalence $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ extends uniquely to a moperad isomorphism $(\varphi^{1\prime}, \varphi') : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaCD}^+$ which acts as the identity on objects, where PaCD^+ is the shifted moperad associated to the operad PaCD (Definition 1.19). Given a moperad isomorphism $(\varphi^{1\prime}, \varphi') : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaCD}^+$, with $\mu = 1$, and an element of $\mathsf{GTM} \cong \mathrm{Aut}_0(\mathsf{PaB}^1_{\mathbb{K}})$ given by $(\vartheta^1, \vartheta) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaB}^1_{\mathbb{K}}$, with $\lambda = 1$, we construct an element $(\varepsilon^1, \varepsilon) \in \mathrm{Aut}_0(\mathsf{PaCD}^+) \cong \mathsf{GRTM}_1$ via the commutative diagram

$$\begin{tikzcd} \mathsf{PaB}^1_{\mathbb{K}} \arrow[r, "\vartheta^1"] & \mathsf{PaB}^1_{\mathbb{K}} \arrow[d, "\varphi^1"] \\ \mathsf{PaCD}^+ \arrow[u, "(\varphi^1)^{-1}"] \arrow[r, dashed, "\varepsilon^1"] & \mathsf{PaCD}^+. \end{tikzcd} \tag{4.22}$$

Note that since $\mu = 1$ and $\lambda = 1$, we have that $\varepsilon^1(t_{ij}) = t_{ij}$ for all $i, j \geq 0$. This correspondence gives an isomorphism between $\mathrm{Aut}_0(\mathsf{PaB}^1_{\mathbb{K}})$ and the group of completed moperad automorphisms of PaCD^+ which fix the t_{ij}. This group is also known as $\mathrm{Aut}_0(\mathsf{PaCD}^+)$ with $\lambda = 1$, or GRTM_1.

Since all morphisms in the diagram (4.22) fix objects, we can restrict to the local automorphism at $r^1_2 = (0(12))$:

$$\varepsilon^1_{r^1_2} : \mathrm{Aut}_{\mathsf{PaCD}^+(2)}(r^1_2) \cong \exp(\mathfrak{t}^1_2) \to \exp(\mathfrak{t}^1_2) \cong \mathrm{Aut}_{\mathsf{PaCD}^+(2)}(r^1_2).$$

Theorem 4.21. *Let $(\varepsilon^1, \varepsilon) : \mathsf{PaCD}^+ \to \mathsf{PaCD}^+$ be an element of GRTM_1. Then the restriction of $\varepsilon^1_{r^1_2}$ to $\exp(\mathfrak{lie}_2) \subset \exp(\mathfrak{t}^1_2)$ is an element of $\mathsf{KRV}^{\mathrm{sym}}(2)$.*

Proof. The automorphism ε^1 fits into a commutative diagram (4.22). Let $H_{\varepsilon^1} = F_{\varphi^1} G_{\vartheta^1} F^{-1}_{\varphi^1}$ be the element of TAut_2 defined by the restriction of $\varepsilon^1_{r^1_2}$ to $\exp(\mathfrak{lie}_2)$. By Theorem 4.11 and Theorem 4.19 we know that F_{φ^1} and $F_{\varphi^1} G_{\vartheta^1}$ are symmetric KV solutions. Therefore we have $H_{\varepsilon^1} \in \mathsf{KRV}^{\mathrm{sym}}(2)$ by Proposition 3.43. ☐

Corollary 4.22. *Let $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ be an equivalence of completed moperads with $\mu = 1$ and let $F_{\varphi^1} = F_{r^1_2, \varphi^1, \varphi}$ be the associated symmetric KV solution. Let $(\varepsilon^1, \varepsilon) : \mathsf{PaCD}^+ \to \mathsf{PaCD}^+$ be an element of GRTM_1 and let $H_{\varepsilon^1} = H_{r^1_2, \varepsilon^1, \varepsilon}$ be the associated element of $\mathsf{KRV}^{\mathrm{sym}}(2)$. Then, we have*

$$F_{\varepsilon^1 \varphi^1} = F_{\varphi^1} \cdot H_{\varepsilon^1}.$$

Remark 4.23. Just like the family of KV solutions $\mathsf{SolKV} = \{\mathsf{SolKV}(n)\}_{n \geqslant 2}$ forms a colored operad in sets, the families of genus zero symmetry groups $\mathsf{KV} = \{\mathsf{KV}(n)\}_{n \geqslant 2}$ and $\mathsf{KRV} = \{\mathsf{KRV}(n)\}_{n \geqslant 2}$ form colored operads in groups, see Theorem 3.38 and Appendix A.4. There are analogues of Theorem 4.14 for both operads, and Corollary 4.20 and Corollary 4.22 imply that the actions of the suboperads generated by the classical KV and KRV elements associated to the word 0(12) commute with operadic composition of KV solutions.

Remark 4.24. We recover the trees of [AKKN20, Theorem 1.13] by combining operad composition with the diagonal map of GT_1 (resp. GRT):

$$\mathsf{GT}_1 \xrightarrow{\Delta^k} \mathsf{GT}_1 \times \cdots \times \mathsf{GT}_1 \hookrightarrow \mathsf{KV}(2) \times \cdots \times \mathsf{KV}(2) \xrightarrow{(\cdots(\circ_{i_1} \times \circ_{i_2}) \times \cdots \times \circ_{i_{k-1}})} \mathsf{KV}(k+1).$$

Combining Theorem 4.14, Corollary 4.20 and Proposition A.16 with the fact that the action of $\mathsf{KV}(n)$ on $\mathsf{SolKV}(n)$ is free and transitive for each $n \geq 2$, it follows that the assignment

$$(\varphi^1, \varphi) \mapsto \{F_{0w}\}$$

described in Theorem 4.14 defines map of bitorsors which is compatible with operadic composition.

5. Symmetric KV Solutions from Parenthesized Braids

The goal of this section is to explain how symmetric KV solutions give rise to moperad morphisms $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathrm{TAut}^+$, extending the operad map $\varphi : \mathsf{PaB}_{\mathbb{K}} \to \mathrm{SAut}$ defined by the associated KV associator. A key feature of this construction is that each symmetric solution $F \in \mathsf{SolKV}^{\tau}(2)$ determines a KV associator [AT12], which is a special automorphism $G_F \in \mathrm{SAut}_3$ satisfying the pentagon and hexagon equations—properties characteristic of braid associators.

Another essential ingredient in this extension is the behavior of the generator $E^{0,1} \in \mathsf{PaB}^1(2)$, which corresponds to a distinguished tangential automorphism $\Theta \in \mathrm{TAut}^+_2$ defined by conjugation with the Baker–Campbell–Hausdorff series. While Θ does not lie in TAut^1, it arises naturally in the image of $\mathsf{PaB}^1_{\mathbb{K}}$.

Remark 5.1. Much of this section reinterprets, in the context of moperads, the constructions and results of Sections 7 to 9 in [AT12]. However, as mentioned in Remark 3.32, our and definition conventions align with [AET10], in particular what we call a KV solution $F \in \mathsf{SolKV}(2)$ would be the inverse of a KV solution in [AT12], and [AKKN18b].

5.1. **An involution on $\mathsf{SolKV}(2)$.** Recall from Section 3.3.1 that symmetric KV solutions are KV solutions F which are invariant under the involution $\tau : \mathsf{SolKV}(2) \to \mathsf{SolKV}(2)$ defined by $\tau(F) := e^{-\mathbf{t}/2} F^{2,1} B$, where $\mathbf{t} = \mathrm{ad}_{x_1+x_2}$ and $B = \exp(0, x_1)$.

Moreover, their associated associator $G_F = F^{1,23}F^{2,3}(F^{1,2})^{-1}(F^{12,3})^{-1}$ is a *KV associator*, that is satisfies the pentagon, unit and hexagon equations in SAut :

$$G^{1,2,34}G^{12,3,4} = G^{2,3,4}G^{1,23,4}G^{1,2,3}, \tag{P}$$

$$G^{1,2,3}G^{3,2,1} = 1, \tag{U}$$

$$e^{\frac{\mathbf{t}^{1,2}+\mathbf{t}^{1,3}}{2}} = (G^{2,3,1})^{-1} e^{\frac{\mathbf{t}^{1,3}}{2}} G^{2,1,3} e^{\frac{\mathbf{t}^{1,2}}{2}} (G^{1,2,3})^{-1}, \tag{H1}$$

$$e^{\frac{\mathbf{t}^{1,3}+\mathbf{t}^{2,3}}{2}} = G^{3,1,2} e^{\frac{\mathbf{t}^{1,3}}{2}} (G^{1,3,2})^{-1} e^{\frac{\mathbf{t}^{2,3}}{2}} G^{1,2,3}. \tag{H2}$$

Recall also that the image $\iota(f)$ of any Drinfel'd associator $(1, f)$ with $f \in \mathfrak{t}_3$ under the operad inclusion $\iota : \mathsf{CD} \hookrightarrow \mathrm{SAut}$ satisfies the defining relations of a KV associator.

5.2. **Operad and moperad maps from KV solutions.** A map of completed operads $\varphi : \mathsf{PaB}_{\mathbb{K}} \to \mathrm{SAut}$ is uniquely determined by its values on the generators of $\mathsf{PaB}_{\mathbb{K}}$ described in Theorem 1.8. A collection of such values defines an operad map if, and only if, it satisfies the relations (P), (H1), and (H2) in SAut. The following theorem shows that every symmetric KV solution $F \in \mathsf{SolKV}^{\tau}(2)$ gives rise to such a map $\varphi_F : \mathsf{PaB}_{\mathbb{K}} \to \mathrm{SAut}$.

Theorem 5.2. *Let $F \in \mathsf{SolKV}^{\tau}(2)$ be a symmetric KV solution and $G_F := F^{1,23}F^{2,3}(F^{1,2})^{-1}(F^{12,3})^{-1}$ in SAut_3 be the corresponding KV associator. Then the assignment*

$$\varphi_F(\Phi^{1,2,3}) := G_F \quad \textit{and} \quad \varphi_F(R^{1,2}) := e^{\frac{\mathbf{t}}{2}}$$

defines an operad map $\varphi_F : \mathsf{PaB}_{\mathbb{K}} \to \mathrm{SAut}$.

Proof. Since G satisfies the pentagon equation (P) in SAut_4, we have

$$\varphi_F(\Phi^{1,2,34}\Phi^{12,3,4}) = G^{1,2,34}G^{12,3,4} = G^{2,3,4}G^{1,23,4}G^{1,2,3} = \varphi_F(\Phi^{2,3,4}\Phi^{1,23,4}\Phi^{1,2,3})$$

and thus $\varphi_F(\Phi^{1,2,3}) = G$ satisfies (P). Next, using the fact that F is a symmetric KV solution, and thus G satisfies the hexagon equation (H1), we have

$$\begin{aligned} \varphi_F(R^{1,23}) = e^{\frac{\mathbf{t}^{1,2}+\mathbf{t}^{1,3}}{2}} &= (G^{2,3,1})^{-1} e^{\frac{\mathbf{t}^{1,3}}{2}} G^{2,1,3} e^{\frac{\mathbf{t}^{1,2}}{2}} (G^{1,2,3})^{-1} \\ &= \varphi_F\left((\Phi^{2,3,1})^{-1} R^{1,3} (\Phi^{2,1,3}) R^{1,2} (\Phi^{1,2,3})^{-1}\right). \end{aligned}$$

Thus, the morphism φ_F preserves the hexagon equation (H1). The second hexagon equation (H2) follows similarly from (H2). ☐

Corollary 5.3. *Let F be a symmetric KV solution. Then the operad map $\varphi_F : \mathsf{PaB}_{\mathbb{K}} \to \mathrm{SAut}$ factors through CD if and only if the associated KV associator G_F is of the form $\iota(f)$ for some Drinfel'd associator $(1, f) \in \mathbb{K}^{\times} \times \exp(\mathfrak{t}_3)$.*

Proof. Suppose first that φ_F factors through CD. Then $G_F = \iota(f)$ for some $f \in \exp(\mathfrak{t}_3)$, since the image of CD in SAut lies in the image of ι. Because ι is injective and G_F satisfies the defining equations of a KV associator, it follows that f satisfies the pentagon, inversion, and hexagon equations, and hence $(1, f)$ is a Drinfel'd associator.

Conversely, suppose that $G_F = \iota(f)$ for some Drinfel'd associator $(1, f)$. Define a map $\tilde{\varphi}_F : \mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$ by setting $\tilde{\varphi}_F(R^{1,2}) := e^{\frac{t_{12}}{2}}$ and $\tilde{\varphi}_F(\Phi^{1,2,3}) := f$. Then, $\tilde{\varphi}_F$ satisfies the relations of Theorem 1.23, and hence defines an operad morphism $\mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$. Composing with the inclusion $\iota : \mathsf{CD} \to \mathrm{SAut}$ gives $\varphi_F = \iota \circ \tilde{\varphi}_F$, as claimed. ☐

Given $F \in \mathsf{SolKV}^{\tau}(2)$, the operad morphism $\varphi_F : \mathsf{PaB}_{\mathbb{K}} \to \mathrm{SAut}$ of Theorem 5.2 naturally extends to a morphism of moperads $(\varphi_F^1, \varphi_F) : \mathsf{PaB}_{\mathbb{K}}^1 \to \mathrm{TAut}^+$. To describe this extension, we introduce a further inner automorphism.

Definition 5.4. Let $\theta := \mathrm{ad}_{\mathfrak{bch}(x_1,x_2)}$ denote the adjoint action by the element $\mathfrak{bch}(x_1, x_2)$ on $\mathfrak{lie}_2$. That is, θ is the inner derivation of $\mathfrak{lie}_2$ which acts on $a \in \mathfrak{lie}_2$ by

$$\theta(a) = [a, \mathfrak{bch}(x_1, x_2)].$$

We denote by $\Theta := \exp(\theta)$ its associated tangential automorphism.

The inner automorphism Θ is closely related to the tangential automorphism $B = \exp((0, x_1)) \in \mathrm{TAut}_2$. The following proposition summarizes Propositions 8.1–8.3 of [AT12].

Proposition 5.5. *The following equalities hold for the automorphisms* Θ *and* B*:*

$$\Theta = B^{2,1}B^{1,2}, \tag{5.1}$$

$$B^{1,2}B^{1,3}B^{2,3} = B^{2,3}B^{1,3}B^{1,2}, \tag{5.2}$$

$$B^{1,23} = B^{1,2}B^{1,3}. \tag{5.3}$$

Furthermore, given $F \in \mathrm{TAut}_2$ *satisfying the first KV equation* (SolKVI)*, we have*

$$e^{\mathbf{t}} = F\Theta F^{-1}, \tag{5.4}$$

$$(F^{1,2})^{-1}B^{12,3}F^{1,2} = B^{1,3}B^{2,3}. \tag{5.5}$$

Theorem 5.6. *Let* $F \in \mathsf{SolKV}^{\tau}(2)$ *be a symmetric KV solution. Then the assignment*

$$\varphi^1_F(E^{0,1}) := \Theta^{0,1} \quad \textit{and} \quad \varphi^1_F(\Psi^{0,1,2}) := F^{1,2}.$$

defines a map of PaB*-modules in pro-unipotent groups and* PaB*-moperads in sets:*

$$(\varphi^1_F, \varphi_F) : \mathsf{PaB}^1_{\mathbb{K}} \longrightarrow \mathrm{TAut}^+ .$$

Proof. The presentation of PaB^1 as a PaB-moperad in Theorem 2.17 says that, in order to extend the operad map $\varphi_F : \mathsf{PaB}_{\mathbb{K}} \to \mathrm{SAut}$ defined in Theorem 5.2 to a map $(\varphi^1_F, \varphi_F) : \mathsf{PaB}^1_{\mathbb{K}} \longrightarrow \mathrm{TAut}^+$, it is sufficient to define values $\varphi^1_F(E^{0,1}) := \Theta$ and $\varphi^1_F(\Psi^{0,1,2}) := F$ and verify that these values satisfy the defining equations (MP), (O) and (RP) of PaB^1. The mixed pentagon equation (MP) gives

$$\varphi^1_F\left(\Psi^{0,1,23}\Psi^{01,2,3}\right) = F^{1,23}F^{2,3} = G^{1,2,3}_F F^{12,3}F^{1,2} = \varphi^1_F\left(\Phi^{1,2,3}\Psi^{0,12,3}\Psi^{0,1,2}\right).$$

Thus φ^1_F preserves the mixed pentagon equation (MP).

To verify that the values $\varphi^1_F(E^{0,1})$ and $\varphi^1_F(\Psi^{0,1,2})$ preserve the octagon equation (O), we will use the hexagon identities from Proposition 3.45. In particular, if we write out a cyclic permutation of (H1),

$$e^{\frac{\mathbf{t}^{3,12}}{2}} = (G^{1,2,3})^{-1}\, e^{\frac{\mathbf{t}^{3,2}}{2}}\, G^{1,3,2}\, e^{\frac{\mathbf{t}^{3,1}}{2}}\, (G^{3,1,2})^{-1} \tag{5.6}$$

and multiplying $e^{\frac{\mathbf{t}^{3,12}}{2}} e^{\frac{\mathbf{t}^{12,3}}{2}} = e^{\frac{\mathbf{t}^{3,1}+\mathbf{t}^{3,2}}{2}} e^{\frac{\mathbf{t}^{1,3}+\mathbf{t}^{2,3}}{2}} = e^{\mathbf{t}^{1,3}+\mathbf{t}^{2,3}}$, we get:

$$\begin{aligned} e^{\mathbf{t}^{1,3}+\mathbf{t}^{2,3}} &= \left((G^{1,2,3})^{-1}e^{\frac{\mathbf{t}^{3,2}}{2}}G^{1,3,2}e^{\frac{\mathbf{t}^{3,1}}{2}}(G^{3,1,2})^{-1}\right)\left(G^{3,1,2}e^{\frac{\mathbf{t}^{1,3}}{2}}(G^{1,3,2})^{-1}e^{\frac{\mathbf{t}^{2,3}}{2}}G^{1,2,3}\right) \\ &= (G^{1,2,3})^{-1}e^{\frac{\mathbf{t}^{3,2}}{2}}G^{1,3,2}e^{\frac{\mathbf{t}^{3,1}}{2}}e^{\frac{\mathbf{t}^{1,3}}{2}}(G^{1,3,2})^{-1}e^{\frac{\mathbf{t}^{2,3}}{2}}G^{1,2,3}. \end{aligned}$$

If we expand out each of the instances of G, we arrive at

$$\begin{aligned} e^{\mathbf{t}^{1,3}+\mathbf{t}^{2,3}} = {} & \left(F^{12,3}\,F^{1,2}\,(F^{2,3})^{-1}\,(F^{1,23})^{-1}\right)\ e^{\frac{\mathbf{t}^{3,2}}{2}}\ \left(F^{1,32}\,F^{3,2}\,(F^{1,3})^{-1}\,(F^{13,2})^{-1}\right) \\ & e^{\frac{\mathbf{t}^{3,1}}{2}}\ e^{\frac{\mathbf{t}^{1,3}}{2}}\ \left(F^{13,2}\,F^{1,3}\,(F^{3,2})^{-1}\,(F^{1,32})^{-1}\right)\ e^{\frac{\mathbf{t}^{2,3}}{2}} \\ & \left(F^{1,23}\,F^{2,3}\,(F^{1,2})^{-1}\,(F^{12,3})^{-1}\right). \end{aligned} \tag{5.7}$$

After conjugating by $F^{12,3}$ this becomes:

$$\begin{aligned}(F^{12,3})^{-1}e^{\mathbf{t}^{1,3}+\mathbf{t}^{2,3}}F^{12,3} = {} & F^{1,2}(F^{2,3})^{-1}(F^{1,23})^{-1}e^{\frac{\mathbf{t}^{3,2}}{2}} \\ & F^{1,32}F^{3,2}(F^{1,3})^{-1}(F^{13,2})^{-1}e^{\frac{\mathbf{t}^{3,1}}{2}}e^{\frac{\mathbf{t}^{1,3}}{2}} \\ & F^{13,2}F^{1,3}(F^{3,2})^{-1}(F^{1,32})^{-1}e^{\frac{\mathbf{t}^{2,3}}{2}} \\ & F^{1,23}\,F^{2,3}\,(F^{1,2})^{-1}.\end{aligned} \tag{5.8}$$

Since $F^{1,2}$ commutes with $(F^{12,3})^{-1}e^{\mathbf{t}^{1,3}+\mathbf{t}^{2,3}}F^{12,3}$, we can conjugate (5.8) by $F^{1,2}$ to get

$$\begin{aligned}(F^{12,3})^{-1}e^{\mathbf{t}^{1,3}+\mathbf{t}^{2,3}}F^{12,3} = {} & (F^{2,3})^{-1}(F^{1,23})^{-1}e^{\frac{\mathbf{t}^{3,2}}{2}} \\ & F^{1,32}F^{3,2}(F^{1,3})^{-1}(F^{13,2})^{-1}e^{\frac{\mathbf{t}^{3,1}}{2}}e^{\frac{\mathbf{t}^{1,3}}{2}} \\ & F^{13,2}F^{1,3}(F^{3,2})^{-1}(F^{1,32})^{-1}e^{\frac{\mathbf{t}^{2,3}}{2}} \\ & F^{1,23}\,F^{2,3}.\end{aligned} \tag{5.9}$$

Further, since $e^{\frac{\mathbf{t}^{i,j}}{2}}$ commutes with the $F^{k,ij}$ and $F^{ij,k}$, this can be simplified to

$$(F^{12,3})^{-1}e^{\mathbf{t}^{1,3}+\mathbf{t}^{2,3}}F^{12,3} = (F^{2,3})^{-1}e^{\frac{\mathbf{t}^{3,2}}{2}}F^{3,2}(F^{1,3})^{-1}e^{\frac{\mathbf{t}^{3,1}}{2}}e^{\frac{\mathbf{t}^{1,3}}{2}}F^{1,3}(F^{3,2})^{-1}e^{\frac{\mathbf{t}^{2,3}}{2}}F^{2,3} \tag{5.10}$$

Using that $e^{\frac{\mathbf{t}^{3,1}}{2}}e^{\frac{\mathbf{t}^{1,3}}{2}} = e^{\mathbf{t}^{1,3}}$, we can now apply (5.4) on both sides to arrive at

$$\Theta^{12,3} = \big((F^{2,3})^{-1}e^{\frac{\mathbf{t}^{2,3}}{2}}F^{3,2}\big)\Theta^{1,3}\big((F^{3,2})^{-1}e^{\frac{\mathbf{t}^{2,3}}{2}}F^{2,3}\big). \tag{5.11}$$

Passing through the isomorphism $\mathrm{TAut}_3 \cong \mathrm{TAut}_2^+$, we can now check:

$$\begin{aligned}\varphi_F^1(E^{01,2}) = \Theta^{01,2} &= B^{2,1}\Theta^{0,2}B^{1,2} \\ &= (F^{1,2})^{-1}e^{\mathbf{t}^{2,1}/2}F^{2,1}\Theta^{0,2}(F^{2,1})^{-1}e^{\mathbf{t}^{1,2}/2}F^{1,2} \\ &= \varphi_F^1\left((\Psi^{0,1,2})^{-1}R^{2,1}\Psi^{0,2,1}E^{0,2}(\Psi^{0,2,1})^{-1}R^{1,2}\Psi^{0,1,2}\right).\end{aligned}$$

Therefore, φ_F^1 preserves the octagon equation (O).

It remains to see that it preserves the right pentagon (RP). We first rewrite the expression $\Theta^{12,3}\Theta^{1,2}$ as follows

$$\begin{aligned}\Theta^{12,3}\Theta^{1,2} &= B^{3,2}B^{3,1}\left(B^{1,3}B^{2,3}B^{2,1}\right)B^{1,2} \\ &\overset{(1)}{=} B^{3,2}B^{3,1}B^{2,1}\left(B^{2,3}B^{1,3}B^{1,2}\right) \\ &\overset{(2)}{=} B^{3,2}B^{3,1}B^{2,1}\left(B^{1,3}B^{1,2}\right)B^{2,3} \\ &\overset{(3)}{=} B^{3,2}B^{3,1}B^{2,1}B^{1,2}B^{1,3}B^{2,3} \\ &\overset{(4)}{=} B^{3,2}\left((F^{3,2})^{-1}B^{32,1}F^{3,2}\right)\left((F^{3,2})^{-1}B^{1,32}F^{3,2}\right)B^{2,3} \\ &= B^{3,2}(F^{3,2})^{-1}\Theta^{1,32}F^{3,2}B^{2,3}\end{aligned}$$

The equality (1) and (2) are repeated use of the Yang–Baxter equation (5.2), (3) uses the commutativity $B^{1,3}B^{1,2} = B^{1,2}B^{1,3}$. Finally the equality (4) uses (5.5) and the relation $F^{2,3}B^{1,2}B^{1,3}(F^{2,3})^{-1} = B^{1,23}$ permuted by (2 3).

Because F is symmetric, we have $B^{3,2} = (F^{2,3})^{-1}e^{\mathbf{t}^{3,2}/2}F^{3,2}$ and $B^{1,2} = (F^{2,1})^{-1}e^{\mathbf{t}^{1,2}/2}F^{1,2}$. Putting this together, and passing through the isomorphism $\mathrm{TAut}_3 \cong \mathrm{TAut}_2^+$ to relabel the variables, we have:

$$\begin{aligned}\varphi_F^1(E^{01,2}E^{0,1}) = \Theta^{01,2}\Theta^{0,1} &= B^{2,1}(F^{2,1})^{-1}\Theta^{0,21}F^{2,1}B^{1,2} \\ &= (F^{1,2})^{-1}e^{\mathbf{t}^{2,1}/2}F^{2,1}(F^{2,1})^{-1}\Theta^{0,21}F^{2,1}(F^{2,1})^{-1}e^{\mathbf{t}^{1,2}/2}F^{1,2} \\ &= (F^{1,2})^{-1}e^{\mathbf{t}^{2,1}/2}\Theta^{0,21}e^{\mathbf{t}^{1,2}/2}F^{1,2} \\ &= \varphi_F^1((\Psi^{0,1,2})^{-1}R^{2,1}E^{0,21}R^{1,2}\Psi^{0,1,2}).\end{aligned}$$

It follows that φ^1_F preserves the right pentagon equation (RP).

In summary $(\varphi^1_F, \varphi_F) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathrm{TAut}^+$ is a map of moperads (in sets) and PaB-modules in pro-unipotent groups. ☐

In the case where the KV associator $G_F := F^{1,23}F^{2,3}(F^{1,2})^{-1}(F^{12,3})^{-1}$ is a Drinfel'd associator, the corresponding moperad morphism $(\varphi^1_F, \varphi_F) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathrm{TAut}^+$ factors through an isomorphism of moperads $\mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$. The converse also holds, as the following corollary shows.

Corollary 5.7. *Let F be a symmetric KV solution. Then the moperad map $(\varphi^1_F, \varphi_F) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathrm{TAut}^+$ factors through CD^+ if, and only if, the associated KV associator G_F is in the ι-image of a Drinfel'd associator.*

Proof. Assume that (φ^1_F, φ_F) is obtained as a composition with ι of a moperad equivalence $(\varphi'^1_F, \varphi'_F) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$:

$$\begin{tikzcd} \mathsf{PaB}^1_{\mathbb{K}} \arrow[r, "\varphi'^1_F"] \arrow[rr, dashed, bend right, "\varphi^1_F"'] & \mathsf{CD}^+ \arrow[r, hook, "\iota"] & \mathrm{TAut}^+ . \end{tikzcd}$$

Since $\varphi'_F : \mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$ is an equivalence, $\iota^{-1}(G_F) = \varphi'_F(\Phi^{1,2,3})$ satisfies the defining identities of a Drinfel'd associator by Theorem 1.23.

Conversely, suppose that $G_F = \iota(f)$, where $(1, f)$ is a Drinfel'd associator. By Corollary 5.3, the operad map $\varphi_F : \mathsf{PaB}_{\mathbb{K}} \to \mathrm{SAut}$ factors through CD. Applying the shifting construction to $\varphi_F : \mathsf{PaB}_{\mathbb{K}} \to \mathsf{CD}$ then yields a moperad isomorphism $(\varphi^+_F, \varphi_F) : \mathsf{PaB}^+_{\mathbb{K}} \xrightarrow{\cong} \mathsf{CD}^+$. Precomposing with the inclusion $(\rho^1, \mathrm{id}) : \mathsf{PaB}^1_{\mathbb{K}} \hookrightarrow \mathsf{PaB}^+_{\mathbb{K}}$ defined in Lemma 2.24, we obtain an isomorphism

$$(\varphi'^1_F, \varphi_F) : \mathsf{PaB}^1_{\mathbb{K}} \xrightarrow{\rho^1} \mathsf{PaB}^+_{\mathbb{K}} \xrightarrow{\varphi^+_F} \mathsf{CD}^+.$$

Lemma 3.22 shows that, under the inclusion $\mathfrak{t}^+_n \hookrightarrow \mathfrak{tder}^+_n$, we have

$$\varphi'^1_F(E^{0,1}) = \varphi'^1_F(R^{1,0}R^{0,1}) = B^{1,0}B^{0,1} = \varphi^1_F(E^{0,1}), \quad \text{and} \quad \varphi'^1_F(\Psi^{0,1,2}) = \varphi^1_F(\Psi^{0,1,2}) = F,$$

which coincide with the values assigned by φ^1_F in Theorem 5.6. Since $\varphi'^1_F = \varphi^1_F$, it follows that the original map (φ^1_F, φ_F) indeed factors through CD^+. ☐

Example 5.8. In [ŠW11b, Thm. 1 (3)], it is shown that the Alekseev–Torossian associator is in fact a Drinfeld associator; i.e. that the corresponding moperad map $\mathsf{PaB}^1_{\mathbb{K}} \to \mathrm{TAut}^+$ factors through CD^+.

5.3. Actions of GTM on KV associators.

As we have seen in Theorem 4.19, the group $\mathsf{GTM} \cong \mathrm{Aut}_0(\mathsf{PaB}^1_{\mathbb{K}})$ acts on symmetric KV solutions which come from moperad isomorphisms $\mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ by precomposition. This action coincides with the usual $\mathsf{KV}(2)$ action on $\mathsf{SolKV}(2)$, via the inclusion $\mathsf{GTM} \hookrightarrow \mathsf{KV}^{\mathrm{sym}}(2)$.

The group GTM acts similarly on the set of module maps $(\varphi^1, \varphi) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathrm{TAut}^+$, as we shall see now. Let F be a KV solution, and let $G \in \mathsf{KV}(2)$. Then, the KV associator of $G \cdot F$ reads

$$\begin{aligned} \Phi_{G\cdot F} &= (FG^{-1})^{1,23}(FG^{-1})^{2,3}((FG^{-1})^{1,2})^{-1}((FG^{-1})^{12,3})^{-1} \\ &= F_1^{1,23}(G^{-1})^{1,23}F_1^{2,3}(G^{-1})^{2,3}G^{1,2}(F_2^{1,2})^{-1}G^{12,3}(F_2^{12,3})^{-1} \\ &= F_1^{1,23}F_1^{2,3}\Phi_{G^{-1}}(F_2^{1,2})^{-1}(F_2^{12,3})^{-1} \end{aligned} \tag{5.12}$$

This defines an action of $\mathsf{KV}(2)$ on the set of KV associators. A similar definition gives a $\mathsf{KRV}(2)$ action as well [AT12, Equation (27)], called *Drinfeld twist.*

Proposition 5.9. *Let F be a symmetric KV solution and $(\varphi^1_F, \varphi_F) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathrm{TAut}^+$ be the associated module map. Let G be an element of GTM_1 and let $(\vartheta^1_G, \vartheta_G) : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaB}^1_{\mathbb{K}}$ be the associated automorphism with $\lambda = 1$. Then, the composite*

$$(\varphi^1_F \circ \vartheta^1_G, \varphi_F \circ \vartheta_G) : \mathsf{PaB}^1_{\mathbb{K}} \xrightarrow{\vartheta^1} \mathsf{PaB}^1_{\mathbb{K}} \xrightarrow{\varphi^1} \mathrm{TAut}^+$$

is such that $\varphi^1_F \circ \vartheta^1_G(\Psi^{0,1,2}) = G \cdot F$ *and* $\varphi_F \circ \vartheta_G(\Phi^{1,2,3}) = \Phi_{G\cdot F}$*. In other words, the induced action of* GTM *on the symmetric KV solution* F *and its KV associator* Φ_F *coincides with the natural* $\mathsf{KV}^{\mathrm{sym}}(2)$ *action.*

Proof. As we have seen in (4.17), given $(1, f, g) \in \mathsf{GTM}$, the action of the automorphism ϑ^1_G is given by

$$\vartheta^1_G(\Psi^{0,1,2}) = \Psi^{0,1,2} \cdot g(x_{01}, x_{12}) \quad \text{and} \quad \vartheta^1_G(\Phi^{1,2,3}) = \Phi^{1,2,3} \cdot f(x_{12}, x_{23}).$$

Therefore, the further composition with φ^1_F is given by

$$\varphi^1_F \circ \vartheta^1_G(\Psi^{0,1,2}) = F^{1,2} \cdot \varphi^1_F(g(x_{01}, x_{12})) = FG^{-1} = G \cdot F$$

and

$$\varphi^1_F \circ \vartheta^1_G(\Phi^{1,2,3}) = \Phi_F \cdot \varphi^1_F(f(x_{12}, x_{23})) = \cdots = \Phi_{G\cdot F}.$$

This finishes the proof. □

When restricting to moperad maps $\mathsf{PaB}^1_{\mathbb{K}} \to \mathrm{TAut}^+$ that arise from Drinfeld associators, i.e., those that factor through an isomorphism $\mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$ (Corollary 5.7), the GTM-action coincides with the classical action of $\mathsf{GT}_1 \cong \mathrm{Aut}_0(\mathsf{PaB}_{\mathbb{K}})$ on associators (cf. Equation (1.5).

The reader may want to compare this with [AT12, Proposition 9.13].

Proposition 5.10. *Let* $\varphi^1 : \mathsf{PaB}^1_{\mathbb{K}} \to \mathrm{TAut}^+$ *and* $\vartheta^1 : \mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{PaB}^1_{\mathbb{K}}$ *be as in the previous Proposition. Suppose moreover that* φ^1 *factors through* CD^+*. Then, the composite*

$$(\varphi^1 \circ \vartheta^1, \varphi \circ \vartheta) : \mathsf{PaB}^1_{\mathbb{K}} \xrightarrow{\vartheta^1} \mathsf{PaB}^1_{\mathbb{K}} \xrightarrow{\varphi^1} \mathsf{CD}^+ \hookrightarrow \mathrm{TAut}^+$$

is such that $\varphi^1 \circ \vartheta^1(\Psi^{0,1,2})$ *and* $\varphi \circ \vartheta(\Phi^{1,2,3})$ *are both given by Equation* (1.5)*. In other words, the induced action of* GTM *on Drinfeld associators coincides with the classical* GT_1 *action.*

Proof. By Proposition 2.26, we know that (φ^1, φ) factors through a composition

$$\mathsf{PaB}^1_{\mathbb{K}} \xrightarrow{\rho^1} \mathsf{PaB}^+_{\mathbb{K}} \xrightarrow{\varphi^+} \mathsf{CD}^+,$$

where ρ^1 sends the associativity isomorphism $\Psi^{0,1,2}$ to the shifted associativity isomorphism $\Phi^{1,2,3}$, and the generators of the $\mathsf{PaB}_{\mathbb{K}}$-moperad $\mathsf{PaB}^1_{\mathbb{K}}$ are defined to be

$$\varphi^1(\Psi^{0,1,2}) = f_\varphi(t_{01}, t_{12}), \quad \varphi^1(E^{0,1}) = e^{t_{01}}.$$

Precomposing with (ϑ^1, ϑ) gives a new $\mathsf{PaB}_{\mathbb{K}}$-moperad map $(\varphi^1 \circ \vartheta^1, \varphi \circ \vartheta)$, whose underlying operad map is:

$$(\varphi \circ \vartheta)(R^{1,2}) = e^{\frac{t_{12}}{2}} \quad \text{and} \quad (\varphi \circ \vartheta)(\Phi^{1,2,3}) = f_\varphi(t_{12}, t_{23}) \cdot f_\vartheta(e^{t_{12}}, f_\varphi(t_{12}, t_{23})^{-1} e^{t_{23}} f_\varphi(t_{12}, t_{23})).$$

This corresponds to the action of an element $(1, f_\vartheta) \in \mathsf{GT}_1$ and a Drinfeld associator $(1, f_\varphi)$ as in (1.5).

At the moperad level, we also have:

$$(\varphi^1 \circ \vartheta^1)(E^{0,1}) = e^{t_{01}} \quad \text{and} \quad (\varphi^1 \circ \vartheta^1)(\Psi^{0,1,2}) = f_\varphi(t_{01}, t_{12}) \cdot f_\vartheta(\varphi^1(x_{01}), \varphi^1(x_{12})),$$

where the pure braids $x_{01}, x_{12} \in \mathsf{PB}^1_2$ are represented as morphisms in $\mathrm{Aut}_{\mathsf{PaB}^1_{\mathbb{K}}(2)}((01)2)$ as:

$$x_{01} = \mathrm{id}_{01} \circ_1 E^{0,1} \quad \text{and} \quad x_{12} = \Psi^{-1}_{0,1,2} \cdot (\mathrm{id}_{01} \circ_1 (R^{2,1} R^{1,2})) \cdot \Psi_{0,1,2}.$$

The images under φ^1 are:

$$\varphi^1(x_{01}) = e^{t_{01}} \quad \text{and} \quad \varphi^1(x_{12}) = f_\varphi(t_{01}, t_{12})^{-1} e^{t_{12}} f_\varphi(t_{01}, t_{12}).$$

Hence,

$$(\varphi^1 \circ \vartheta^1)(\Psi^{0,1,2}) = f_\varphi(t_{01}, t_{12}) \cdot f_\vartheta(e^{t_{01}}, f_\varphi(t_{01}, t_{12})^{-1} e^{t_{12}} f_\varphi(t_{01}, t_{12})).$$

This shows that precomposition with (ϑ^1, ϑ) gives the expected GT_1-action on the associator f_φ, when viewed as a moperad map $\mathsf{PaB}^1_{\mathbb{K}} \to \mathsf{CD}^+$. □

Appendix A. Operadic Structures and Kashiwara–Vergne Solutions

A.1. **Operadic structure on Lie algebras and $\mathfrak{tder}$.** In this section, we provide proofs and examples of some of the statements discussed in Section 3. Recall Definition 3.7, which gives a right Σ_n action on $\mathfrak{lie}_n$ as well as a partial composition map, which are given by, for any $f \in \mathfrak{lie}_m, g \in \mathfrak{lie}_n, \sigma \in \Sigma_m, 1 \leq i \leq m$,

$$f^\sigma(x_1, \dots, x_n) := f(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)}),$$

$$(f \circ_i g)(x_1, \dots, x_{m+n-1}) := f(x_1, \dots, x_{i-1}, \sum_{j=i}^{i+n-1} x_j, x_{i+n}, \dots, x_{m+n-1}) + g(x_i, \dots, x_{i+n-1}).$$

As we show below in Proposition 3.8, the symmetric sequence of free Lie algebras with these partial compositions is a linear operad denoted $\mathfrak{lie}$.

Example A.1. Consider the Lie monomials $f(x_1, x_2) = [x_1, x_2] \in \mathfrak{lie}_2$ and $g(x_1, x_2) = [x_1, [x_1, x_2]] \in \mathfrak{lie}_2$. Then $f \circ_1 g \in \mathfrak{lie}_3$ is computed by substituting $x_1 \mapsto x_1 + x_2$ in f and adding $g(x_1, x_2)$ to the result:

$$(f \circ_1 g)(x_1, x_2, x_3) = f(x_1 + x_2, x_3) + g(x_1, x_2) = [x_1 + x_2, x_3] + [x_1, [x_1, x_2]].$$

Similarly, composing at the second input gives:

$$(f \circ_2 g)(x_1, x_2, x_3) = f(x_1, x_2 + x_3) + g(x_2, x_3) = [x_1, x_2 + x_3] + [x_2, [x_2, x_3]].$$

Proposition A.2. *The symmetric sequence of free Lie algebras* $\mathfrak{lie}$ *with the partial compositions above forms a linear operad.*

Proof. Observe first that both the Σ_n action and the $\circ_i$ operations are linear maps. We proceed to verify that they satisfy axioms (i)–(iii) from Definition 1.1. For (i), we observe that the element $0 \in \mathfrak{lie}_1$ is such that $f \circ_i 0 = 0 \circ_1 f = f$ for any $f \in \mathfrak{lie}$. To check (ii), associativity, let $f \in \mathfrak{lie}_m$, $g \in \mathfrak{lie}_n$, and $h \in \mathfrak{lie}_k$ be three Lie words, and let $1 \leq j \leq m$. For $1 \leq i \leq j-1$, one finds that $(f \circ_j g) \circ_i h$ and $(f \circ_i h) \circ_{j+k-1} g$ are both equal to

$$f(x_1, \dots, x_{i-1}, \sum_{\ell=i}^{i+k-1} x_\ell, x_{i+k}, \dots, \sum_{\ell=j+k-1}^{j+k+n-2} x_\ell, \dots, x_{m+n+k-2}) + g(x_{j+k-1}, \dots, x_{j+k+n-2}) + h(x_i, \dots, x_{i+k-1}).$$

For $j \leq i \leq j+n-1$, one finds that $(f \circ_j g) \circ_i h)$ and $f \circ_j (g \circ_{i-j+1} h)$ are both equal to

$$f(x_1, \dots, x_{j-1}, \sum_{\ell=j}^{j+n+k-2} x_\ell, \dots, x_{m+n+k-2}) + g(x_j, \dots, \sum_{\ell=i}^{i+k-1} x_\ell, \dots, x_{j+n+k-2}) + h(x_i, \dots, x_{i+k-1}).$$

Finally, if $j+k \leq i \leq m+n-1$, then $(f \circ_j g) \circ_i h$ and $(f \circ_{i-n+1} h) \circ_j g$ are both equal to

$$f(x_1, \dots, x_{j-1}, \sum_{\ell=j}^{j+n-1} x_\ell, x_{j+n}, \dots, x_{i-1}, \sum_{\ell=i}^{i+k-1} x_\ell, x_{i+k}, \dots, x_{m+n+k-2}) + g(x_j, \dots, x_{j+n-1}) + h(x_i, \dots, x_{i+k-1}).$$

It remains to check (iii), equivariance. Let f, g be two Lie words as before, let $\sigma \in \Sigma_m$ and $\tau \in \Sigma_n$ be two permutations, and let $1 \leq i \leq m$. Then, one computes that

$$\begin{aligned} (f^\sigma \circ_{\sigma^{-1}(i)} g^\tau)(x_1, \dots, x_{m+n-1}) &= f(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(m)}) \circ_{\sigma^{-1}(i)} g(x_{\tau^{-1}(1)}, \dots, x_{\tau^{-1}(n)}) \\ &= (f \circ_i g)(x_{(\sigma \circ_i \tau)^{-1}(1)}, \dots, x_{(\sigma \circ_i \tau)^{-1}(m+n-1)}) \\ &= (f \circ_i g)^{\sigma \circ_i \tau}(x_1, \dots, x_{m+n-1}). \end{aligned}$$

We conclude that $\mathfrak{lie}$ is a linear operad. □

Proof of Proposition 3.8. The same formulas and the same proof work *mutatis mutandis* for the universal enveloping algebra $\mathfrak{ass}$ and the vector space of cyclic words cyc. Checking that the quotient map $\mathrm{tr} : \mathfrak{ass} \to \mathrm{cyc}$ is a morphism of linear operads can be verified directly: for $f \in \mathfrak{ass}_m$, $g \in \mathfrak{ass}_n$, $1 \leqslant i \leqslant m$ and $\sigma \in \Sigma_m$, we clearly have $\mathrm{tr}(f^\sigma) = \mathrm{tr}(f)^\sigma$ and $\mathrm{tr}(f \circ_i g) = \mathrm{tr}(f) \circ_i \mathrm{tr}(g)$. □

As discussed in Section 3.1, this in turn gives an operadic structure on $\mathfrak{tder}$. Namely, for any $u = (a_1, \dots, a_m) \in \mathfrak{tder}_m, v = (b_1, \dots, b_n) \in \mathfrak{tder}_n, \sigma \in \Sigma_m, 1 \leq i \leq m$, we have

$$u^\sigma := (a^\sigma_{\sigma^{-1}(1)}, \dots, a^\sigma_{\sigma^{-1}(n)}),$$
$$u \circ_i v := (a_1 \circ_i 0, \dots, a_{i-1} \circ_i 0, a_i \circ_i b_1, \dots, a_i \circ_i b_n, a_{i+1} \circ_i 0, \dots, a_m \circ_i 0).$$

Here a^σ_j denotes the Σ_m action on $\mathfrak{lie}$, $\circ_i$ on the right hand side above denotes operadic composition in the operad $\mathfrak{lie}$, and 0 stands for $0_n \in \mathfrak{lie}_n$. We next focus on the operadic structure of $\mathfrak{sder}_n \subseteq \mathfrak{tder}_n$.

A.2. **The operad of special derivations $\mathfrak{sder}$.** While tangential derivations naturally form an operad in vector spaces, they do not preserve a Lie algebra structure under operadic composition. In contrast, the subspace of special derivations forms an operad in Lie algebras, as the direct sum of special derivations is closed under the Lie bracket.

Theorem A.3. *The family $\mathfrak{sder}$ of special derivations forms a linear suboperad of $\mathfrak{tder}$, and $\mathfrak{sder}$ is, moreover, an operad in Lie algebras.*

Proof. We need to show that $\mathfrak{sder}$ is stable under symmetric group actions, partial compositions, and that it contains the unit of $\mathfrak{tder}$. The fact that $0 \in \mathfrak{tder}_1$ is in $\mathfrak{sder}_1$ is immediate. To check the stability under the Σ_n action, let $u \in \mathfrak{sder}_n$ and observe that since the sum $\sum_{i=1}^n x_i \in \mathfrak{lie}_n$ is invariant under the action of Σ_n, diagram (3.2) gives the equality for any $\sigma \in \Sigma_n$,

$$u\left(\sum_{i=1}^n x_i\right)^\sigma = u^\sigma\left(\sum_{i=1}^n x_i\right).$$

Since the left hand side is zero by hypothesis, it is also the case of the right hand side, and we have $u^\sigma \in \mathfrak{sder}_n$. It remains to show stability under partial composition. Let $u \in \mathfrak{sder}_m$ and $v \in \mathfrak{sder}_n$ be two special derivations. Making the change of variables $y_1 := x_1, \dots, y_i := \sum_{\ell=i}^{i+n-1} x_\ell, \dots, y_m := x_{m+n-1}$ and $z_1 := x_i, \dots, z_n := x_{i+n-1}$, we have

$$u \circ_i v\left(\sum_{j=1}^{m+n-1} x_j\right) = u \circ_i 0\left(\sum_{j=1}^{m+n-1} x_j\right) + 0 \circ_i v\left(\sum_{j=1}^{m+n-1} x_j\right) = u\left(\sum_{k=1}^m y_k\right) + v\left(\sum_{\ell=1}^n z_\ell\right) = 0.$$

Therefore, $u \circ_i v \in \mathfrak{sder}_{m+n-1}$ and $\mathfrak{sder}$ forms a linear suboperad of $\mathfrak{tder}$.

To show that $\mathfrak{sder}$ is an operad in the category of (degree complete) Lie algebras, we need to show that the $\circ_i$ composition maps and the Σ_n actions are morphisms of Lie algebras. The fact that the Σ_n actions are morphisms of Lie algebras is straightforward to check. For the partial compositions, given $u, u' \in \mathfrak{tder}_m$ and $v, v' \in \mathfrak{tder}_n$, we want to show that

$$[u \circ_i v, u' \circ_i v'] = [u, u'] \circ_i [v, v']. \tag{A.1}$$

Writing $u \circ_i v = u \circ_i 0 + 0 \circ_i v$, the left hand side of (A.1) gives four terms, while the right hand side gives two. We first show that $[u \circ_i 0, u' \circ_i 0] = [u, u'] \circ_i 0$; a similar argument shows that $[0 \circ_i v, 0 \circ_i v'] = 0 \circ_i [v, v']$. Writing $u = (a_1, \dots, a_m)$ and $u' = (a'_1, \dots, a'_m)$, we have

$$[u, u'] = uu' - u'u = (c_1, \dots, c_m) \quad \text{where} \quad c_j = u(a'_j) - u'(a_j) + [a_j, a'_j].$$

Then, we have

$$[u, u'] \circ_i 0 = (\tilde{c}_1, \dots, \tilde{c}_{i-1}, \tilde{c}_i, \dots, \tilde{c}_i, \tilde{c}_{i+1}, \dots, \tilde{c}_m)$$

where $\tilde{c}_j = c_j(x_1, \dots, \sum_{j=i}^{i+n-1} x_j, x_{i+n}, \dots, x_{m+n-1})$. Similarly we have

$$u \circ_i 0 = (\tilde{a}_1, \dots, \tilde{a}_{i-1}, \tilde{a}_i, \dots, \tilde{a}_i, \tilde{a}_{i+1}, \dots, \tilde{a}_m), \quad u' \circ_i 0 = (\tilde{a}'_1, \dots, \tilde{a}'_{i-1}, \tilde{a}'_i, \dots, \tilde{a}'_i, \tilde{a}'_{i+1}, \dots, \tilde{a}'_m)$$

and

$$[u \circ_i 0, u' \circ_i 0] = (\tilde{d}_1, \dots, \tilde{d}_{i-1}, \tilde{d}_i, \dots, \tilde{d}_i, \tilde{d}_{i+1}, \dots, \tilde{d}_m) \quad \text{where} \quad \tilde{d}_j = (u \circ_i 0)(\tilde{a}'_j) - (u' \circ_i 0)(\tilde{a}_j) + [\tilde{a}_j, \tilde{a}'_j].$$

Thus our assertion is equivalent to the equalities $\tilde{c}_j = \tilde{d}_j$ for all j. We readily have that

$$[a_j, a'_j](x_1, \dots, \sum_{j=i}^{i+n-1} x_j, x_{i+n}, \dots, x_{m+n-1}) = [\tilde{a}_j, \tilde{a}'_j].$$

It remains to show that

$$u(a'_j)(x_1,\dots,\sum_{j=i}^{i+n-1}x_j,x_{i+n},\dots,x_{m+n-1})=(u\circ_i 0)(\tilde{a}'_j),$$

a similar statement holds for $u'(a_j)$. Indeed, $u(a'_j(x_1,\dots,x_m))$ is the sum over all occurrences of generators x_k in the word a'_j, of the word a'_j with this occurrence replaced by the commutator $[x_k,a_k]$. Performing the substitution of variables

$$(x_1,\dots,x_m)\mapsto(x_1,\dots,\sum_{j=i}^{i+n-1}x_j,x_{i+n},\dots,x_{m+n-1})$$

in this Lie word gives the new Lie word $(u\circ_i 0)(\tilde{a}'_j)$.

It remains to show that $[u\circ_i 0, 0\circ_i v']=0$; the proof that $[0\circ_i v, u'\circ_i 0]=0$ is similar. Let us write $u\circ_i 0=(\tilde{a}_1,\dots,\tilde{a}_{i-1},\tilde{a}_i,\dots,\tilde{a}_i,\tilde{a}_{i+1},\dots,\tilde{a}_m)$ where $\tilde{a}_j=a_j(x_1,\dots,\sum_{j=i}^{i+n-1}x_j,x_{i+n},\dots,x_{m+n-1})$ and $0\circ_i v'=(0,\dots,0,\tilde{b}'_1,\dots,\tilde{b}'_n,0,\dots,0)$ where $\tilde{b}'_j=b'_j(x_i,\dots,x_{i+n-1})$. We have

$$\begin{aligned}[u\circ_i 0,0\circ_i v'] \quad = \quad & ((u\circ_i 0)(0)-(0\circ_i v')(\tilde{a}_1)+[\tilde{a}_1,0]),\\ & \dots,\\ & ((u\circ_i 0)(\tilde{b}'_1)-(0\circ_i v')(\tilde{a}_i)+[\tilde{a}_i,\tilde{b}'_1])\\ & \dots,\\ & ((u\circ_i 0)(\tilde{b}'_n)-(0\circ_i v')(\tilde{a}_i)+[\tilde{a}_i,\tilde{b}'_n])\\ & \dots,\\ & ((u\circ_i 0)(0)-(0\circ_i v')(\tilde{a}_m)+[\tilde{a}_m,0])\end{aligned}$$

The terms of the form $((u\circ_i 0)(0)-(0\circ_i v')(\tilde{a}_j)+[\tilde{a}_j,0])$ are equal to $-(0\circ_i v')(\tilde{a}_j)$. This is in turn the negative sum over all occurrences of generators in $\tilde{a}_j$ of the bracket with the corresponding entry of $0\circ_i v'$. These terms are zero, including the occurrences of the sum $x_i+\dots+x_{i+n-1}$ since v' is a special derivation (i.e. we have $(0\circ_i v')(\sum_{j=i}^{i+n-1}x_j)=0$). Considering the terms of the form $((u\circ_i 0)(\tilde{b}'_j)-(0\circ_i v')(\tilde{a}_i)+[\tilde{a}_i,\tilde{b}'_j])$, a similar phenomenon occurs for $(0\circ_i v')(\tilde{a}_i)$ which is equal to 0. Then, the remaining terms $((u\circ_i 0)(\tilde{b}'_j)+[\tilde{a}_i,\tilde{b}'_j])$ cancel thanks to the Jacobi identity. Therefore, we have $[u\circ_i 0,0\circ_i v']=0$ and the proof is complete. ☐

Remark A.4. The following observation was communicated to us by Pavol Ševera. Let us choose a family $l=\{l_n(x_1,\dots,x_n)\}_{n\geqslant 2}$ of Lie words $l_n\in\mathfrak{lie}_n$ such that

$$l_m(x_1,\dots,x_{i-1},l_n(x_i,\dots,x_{i+n-1}),x_{i+n},\dots,x_{m+n-1})=l_{m+n-1}(x_1,\dots,x_{m+n-1}).$$

Then, one can define an alternative linear operad structure on $\mathfrak{lie}$ by the formula

$$(f\circ_i g)(x_1,\dots,x_{m+n-1}):=f(x_1,\dots,x_{i-1},l_n(x_i,\dots,x_{i+n-1}),x_{i+n},\dots,x_{m+n-1})+g(x_i,\dots,x_{i+n-1}).$$

We write $\mathfrak{lie}^l$ for this operad. It induces an operad structure $\mathfrak{tder}^l$ on $\mathfrak{tder}$ by the same formulas as above. Moreover, one can define the Lie algebra $\mathfrak{sder}^l$ of *special derivations* with respect to l, which are tangential derivations $u\in\mathfrak{tder}_n$ such that

$$u(l_n(x_1,\dots,x_n))=0.$$

The proof of Theorem A.3 carries *mutatis mutandis* and shows that $\mathfrak{sder}^l$ is a linear operad in Lie algebras. Taking $l_n=x_1+\dots+x_n$, one recovers the above operad structures, while taking $l_n=\mathfrak{bch}(x_1,\dots,x_n)$, one gets other operad structures that we denote $\mathfrak{lie}^{\mathfrak{bch}}$, $\mathfrak{tder}^{\mathfrak{bch}}$ and $\mathfrak{sder}^{\mathfrak{bch}}$.

We next describe how some special derivations are obtained from braids. The natural isomorphism $\mathsf{PB}_{n+1}\cong\mathsf{F}_n\rtimes\mathsf{PB}_n$ is preserved under prounipotent completion (cf. [Fre17, Proposition 8.5.3]) and thus induces an isomorphism of Lie algebras

$$\mathfrak{t}_{n+1}\cong\mathfrak{lie}_n\rtimes\mathfrak{t}_n. \tag{A.2}$$

The action of $\mathfrak{t}_n$ on the free Lie algebra $\mathfrak{lie}_n$ is a source of special derivations of the free Lie algebra. To make this precise consider the following example.

Example A.5. The element $t = (x_2, x_1) \in \mathfrak{tder}_2$ is the tangential derivation that acts on the generators of $\mathfrak{lie}_2$ via:

$$t(x_1) = [x_1, x_2] \quad \text{and} \quad t(x_2) = [x_2, x_1].$$

The element t is a special derivation since $t(x_1 + x_2) = [x_1, x_2] + [x_2, x_1] = 0$.

Generalizing Example A.5, we let t^{ij} denote the special derivation

$$t^{ij} = (0, \dots, \underbrace{x_j}_{i}, \dots, \underbrace{x_i}_{j}, \dots, 0) \in \mathfrak{tder}_n,$$

$1 \leq i < j \leq n$ that acts as $t^{ij}(x_i) = [x_i, x_j]$ and $t^{ij}(x_j) = [x_j, x_i]$. The derivations $t^{ij} \in \mathfrak{sder}_n$ span a Lie subalgebra isomorphic to the Drinfeld–Kohno Lie algebra $\mathfrak{t}_n$ described in Section 1.4 ([AT12, Proposition 3.11]). The following proposition shows that the inclusion of the Lie subalgebra $\mathfrak{t}_n \hookrightarrow \mathfrak{sder}_n$ extends to an inclusion of the operad $\mathfrak{t}$ as a sub-operad of $\mathfrak{sder}$.

Proposition A.6. *The family* $\mathfrak{t} := \{\mathfrak{t}_n\}_{n \geq 0}$ *of infinitesimal braids is a suboperad of* $\mathfrak{sder}$.

Proof. We need to check that each $\mathfrak{t}_n$ is stable under the action of Σ_n and that $\mathfrak{t}$ is stable under operadic composition. It suffices to check these properties on generators. Let t_{ij} be a generator of $\mathfrak{t}_n$, and let σ be a permutation in Σ_n. A direct computation shows that $(t^{ij})^\sigma = t^{\sigma^{-1}(i)\sigma^{-1}(j)}$ and thus $\mathfrak{t}$ is stable under the symmetric group actions.

Given $t^{ij} \in \mathfrak{sder}_m$, $t^{kl} \in \mathfrak{sder}_n$ and $\alpha \in \{1, \dots, m\}$, the composition rule in the operad $\mathfrak{sder}$ says that $t^{ij} \circ_\alpha t^{kl} = t^{ij} \circ_\alpha 0 + 0 \circ_\alpha t^{kl}$ with

$$t^{ij} \circ_\alpha 0 = \begin{cases} t^{(i+n-1)(j+n-1)} & \text{if } \alpha < i, \\ \sum_{\beta=1}^n t_{(i+\beta-1)(j+n-1)} & \text{if } \alpha = i, \\ t^{i(j+n-1)} & \text{if } i < \alpha < j, \\ \sum_{\beta=1}^n t^{i(j+\beta-1)} & \text{if } \alpha = j, \\ t^{ij} & \text{if } \alpha > j, \end{cases}$$

and $0 \circ_\alpha t^{kl} = t^{(k+\alpha-1)(l+\alpha-1)}$ for all α. But this is precisely the definition of $t_{ij} \circ_\alpha t_{kl}$ in $\mathfrak{t}$. It follows that the inclusion $\mathfrak{t} \hookrightarrow \mathfrak{sder}$ is stable under operadic composition and completes the proof. ☐

Remark A.7. Related results appear in [Wil15] and [ŠW11b], where analogous structures are studied in the context of graph complexes.

Consider the linearization of $\mathsf{KRV}(n)$, the Lie subalgebra $\mathfrak{krv}_n$ of $\mathfrak{tder}_n$, called the *graded Kashiwara-Vergne Lie algebra*, which consists of pairs $(u, r) \in \mathfrak{tder}_n \times \mathbb{K}[[z]]$ satisfying the equations

$$\text{(krv1)} \qquad u\left(\sum_{i=1}^n x_i\right) = 0\ ,$$

$$\text{(krv2)} \qquad j(u) = \operatorname{tr}\left(r\left(\sum_{i=1}^n x_i\right) - \sum_{i=1}^n r(x_i)\right)\ .$$

Proposition A.8. *The family* $\mathfrak{krv} := \{\mathfrak{krv}_n\}_{n \geq 0}$ *of graded Kashiwara–Vergne Lie algebras is a colored suboperad of* $\mathfrak{sder}$.

Proof. We need only check that the symmetric group action and operadic composition preserve the second equation (krv2). Let $u \in \mathfrak{krv}_n$ and $\sigma \in \Sigma_n$. Since the Jacobian j is a morphism of operads (Proposition 3.14), we have $j(u^\sigma) = j(u)^\sigma = j(u)$ since the RHS of (krv2) is invariant under the action of Σ_n.

Let $u \in \mathfrak{krv}_m$ and $v \in \mathfrak{krv}_n$ and $1 \leqslant i \leqslant m$. Suppose moreover that both u and v share the same Duflo function r. Then, writing $\omega := x_i + \cdots x_{i+n-1}$ and using again the fact that j is a morphism of operads, we have

$$\begin{aligned} j(u \circ_i v) &= j(u \circ_i 0 + 0 \circ_i v) \\ &= j(u) \circ_i 0 + 0 \circ_i j(v) \\ &= \operatorname{tr}\left(r\left(\sum_{j=1}^{m+n-1} x_j\right) - \sum_{j=1}^{i-1} r(x_j) - r(\omega) - \sum_{j=i+n}^{m+n-1} r(x_j)\right) + \operatorname{tr}\left(r(\omega) - \sum_{j=i}^{i+n-1} r(x_j)\right) \\ &= \operatorname{tr}\left(r\left(\sum_{j=i}^{m+n-1} x_j\right) - \sum_{j=i}^{m+n-1} r(x_j)\right), \end{aligned}$$

which finishes the proof. □

This operad integrates to a group, giving the operad in groups KRV described in Appendix A.4.

Remark A.9. The same works for the linearization $\mathfrak{kv}$ of the family of groups KV; they form a suboperad in Lie algebras of $\mathfrak{sder}^{\mathfrak{bch}}$. However it does not form a suboperad of $\mathfrak{sder}$. This explains the apparent dissymmetry between the operads KRV and KV presented below in Appendix A.4.

The isomorphism $\mathsf{PB}_n^1 \cong \mathsf{PB}_{n+1} \cong \mathsf{F}_n \rtimes \mathsf{PB}_n$ described in Lemma 2.14 induces, after prounipotent completion, an isomorphism of Lie algebras

$$\mathfrak{t}_n^1 \cong \mathfrak{t}_{n+1} \cong \mathfrak{lie}_n\langle t_{01}, \ldots, t_{0n}\rangle \rtimes \mathfrak{t}_n. \tag{A.3}$$

Notation A.10. There is an isomorphism of Lie algebras $\mathfrak{t}_n^1 \cong \mathfrak{t}_{n+1}$ obtained by reindexing the generators $t_{0i} \mapsto t_{1i}$ and $t_{ij} \mapsto t_{i+1,j+1}$ for $1 \leq i < j \leq n$. To align with our convention of denoting moperads obtained via a shifting construction using a superscript $(-)^+$, we will write $\mathfrak{t}_n^+$ in place of $\mathfrak{t}_n^1$ throughout this paper.

A.3. **Operadic composition of KV solutions.** We first consider the lift to TAut of the operad structure on $\mathfrak{tder}$, and show that it is indeed an operad.

Proof of Proposition 3.12. We proceed to verify that the Σ_n action and the $\circ_i$ operations satisfy axioms (i)–(iii) from Definition 1.1. For (i), we observe that the element $1 = \exp(0) \in \mathrm{TAut}_1$ is such that $F \circ_i 1 = 1 \circ_1 F = F$ for any $F \in \mathrm{TAut}$. To check (ii), associativity, let $F = \exp(u) \in \mathrm{TAut}_m$, $G = \exp(v) \in \mathrm{TAut}_n$, and $H = \exp(w) \in \mathrm{TAut}_k$ be three tangential automorphisms, and let $1 \leq j \leq m$. For $1 \leq i \leq j-1$, we compute

$$\begin{aligned} \mathfrak{bch}(\mathfrak{bch}(u \circ_j 0, 0 \circ_j v) \circ_i 0, 0 \circ_i w) &= \mathfrak{bch}(\mathfrak{bch}((u \circ_j 0) \circ_i 0, (0 \circ_j v) \circ_i 0), 0 \circ_i w) \\ &= \mathfrak{bch}(\mathfrak{bch}((u \circ_i 0) \circ_{j+k-1} 0, (0 \circ_i 0) \circ_{j+k-1} v), 0 \circ_i w) \\ &= \mathfrak{bch}((u \circ_i 0) \circ_{j+k-1} 0, \mathfrak{bch}(0 \circ_{j+k-1} v, 0 \circ_i w)) \\ &= \mathfrak{bch}((u \circ_i 0) \circ_{j+k-1} 0, \mathfrak{bch}(0 \circ_i w, 0 \circ_{j+k-1} v)) \\ &= \mathfrak{bch}(\mathfrak{bch}((u \circ_i 0) \circ_{j+k-1} 0, (0 \circ_i w) \circ_{j+k-1} 0), 0 \circ_{j+k-1} v) \\ &= \mathfrak{bch}(\mathfrak{bch}(u \circ_i 0, 0 \circ_i w) \circ_{j+k-1} 0, 0 \circ_{j+k-1} v), \end{aligned}$$

which shows that $(F \circ_j G) \circ_i H$ and $(F \circ_i H) \circ_{j+k-1} G$ are equal. Here, we made use of the following properties: compositions with 1 are group homomorphisms in TAut, operadic composition in $\mathfrak{tder}$ is associative, the $\mathfrak{bch}$ product is associative (twice), and the fact that we have $[0 \circ_{j+k-1} v, 0 \circ_i w] = 0$ in $\mathfrak{tder}_{m+n+k-2}$.

For $j \leq i \leq j+n-1$, a similar, slightly simpler computation gives

$$\begin{aligned} \mathfrak{bch}(\mathfrak{bch}(u \circ_j 0, 0 \circ_j v) \circ_i 0, 0 \circ_i w) &= \mathfrak{bch}(\mathfrak{bch}((u \circ_j 0) \circ_i 0, (0 \circ_j v) \circ_i 0), 0 \circ_i w) \\ &= \mathfrak{bch}(\mathfrak{bch}(u \circ_j (0 \circ_{i-j+1} 0), 0 \circ_j (v \circ_{i-j+1} 0)), 0 \circ_i w) \\ &= \mathfrak{bch}(u \circ_j 0, \mathfrak{bch}(0 \circ_j (v \circ_{i-j+1} 0), 0 \circ_j (0 \circ_{i-j+1} w))) \\ &= \mathfrak{bch}(u \circ_j 0, 0 \circ_j \mathfrak{bch}(v \circ_{i-j+1} 0, 0 \circ_{i-j+1} w), \end{aligned}$$

which shows that $(F \circ_j G) \circ_i H = F \circ_j (G \circ_{i-j+1} H)$. Finally, if $j + k \leq i \leq m + n - 1$, a computation similar to the first one above shows that $(F \circ_j G) \circ_i H = (F \circ_{i-n+1} H) \circ_j G$.

It remains to check (iii), equivariance. Let f, g be two Lie words as before, let $\sigma \in \Sigma_m$ and $\tau \in \Sigma_n$ be two permutations, and let $1 \leq i \leq m$. Using the fact that the Σ_n-actions on TAut are group homomorphisms, and that operadic composition in $\mathfrak{tDer}$ is Σ_n-equivariant, we have

$$\mathfrak{bch}(u \circ_i 0, 0 \circ_i v)^{\sigma \circ_i \tau} = \mathfrak{bch}((u \circ_i 0)^{\sigma \circ_i \tau}, (0 \circ_i v)^{\sigma \circ_i \tau}) = \mathfrak{bch}(u^{\sigma} \circ_{\sigma^{-1}(i)} 0, 0 \circ_{\sigma^{-1}(i)} v^{\tau}),$$

which shows that $(F \circ_i G)^{\sigma \circ_i \tau} = (F^{\sigma} \circ_{\sigma^{-1}(i)} G^{\tau})$, as desired. □

In [AKKN18b, Section 7] and [AET10] the authors use the operadic structure above on tangential automorphisms to show that by operadically composing a KV solution of type $(0, 2+1)$ with itself, one obtains KV solutions of type $(0, n+1)$. More generally, we can form a group-colored, non-symmetric operad of KV solutions.

Recall from Definition 3.31 that each KV solution F of type $(0, n+1)$ determines a Duflo function $h \in \mathbb{K}[[z]]$ which arises in the second KV equation (SolKVII). Let

$$\mathsf{SolKV}(n) := \{(F, h) \mid F \in \mathrm{TAut}_n \text{ satisfies (SolKVI) and (SolKVII) for } h \in \mathbb{K}[[z]]\}\,.$$

The sequence $\mathsf{SolKV} = \{\mathsf{SolKV}(n)\}_{n \geq 2}$, forms a $\mathbb{K}[[z]]$-colored sequence in $\mathbb{K}$-vector spaces. Given a pair $F = (F, h_1) \in \mathsf{SolKV}(m)$ and $G = (G, h_2) \in \mathsf{SolKV}(n)$ one can define the operadic composition $F \circ_i G \in \mathsf{SolKV}(m+n-1)$ if $h_1 = h_2$. Before we state this as a theorem, let us illustrate with an example.

Example A.11. Let $F \in \mathrm{TAut}_2$ be a KV solution of type $(0, 2+1)$. Then the operadic composition inherited from TAut defines

$$F \circ_2 F = (F \circ_2 1)(1 \circ_2 F) = F^{1,23} F^{2,3}.$$

Since F satisfies the first KV equation (SolKVI), i.e. $F(e^{x_1} e^{x_2}) = e^{x_1 + x_2}$, the composition $F \circ_2 F$ also satisfies it. Explicitly, we compute:

$$(F \circ_2 F)(e^{x_1} e^{x_2} e^{x_3}) = F^{1,23} F^{2,3}(e^{x_1} e^{x_2} e^{x_3}) = F^{1,23}\left(e^{x_1} e^{x_2 + x_3}\right) = e^{x_1 + x_2 + x_3}.$$

To show that $F \circ_2 F$ satisfies (SolKVII), we recall that since J is a 1-cocycle on TAut, we have:

$$J(F^{1,23} F^{2,3}) = J(F^{1,23}) + F^{1,23} \cdot J(F^{2,3}).$$

Then, since

$$\begin{aligned} J(F^{2,3}) &= \operatorname{tr}\left(h(x_2 + x_3) - h(x_2) - h(x_3)\right) \quad \text{and} \\ J(F^{1,23}) &= \operatorname{tr}\left(h(x_1 + (x_2 + x_3)) - h(x_1) - h(x_2 + x_3)\right), \end{aligned}$$

the sum becomes:

$$J(F^{1,23}) + F^{1,23} \cdot J(F^{2,3}) = \operatorname{tr}\left(h(x_1 + x_2 + x_3) - h(x_1) - h(x_2) - h(x_3)\right).$$

Here, we used the fact that $h \in z^2 \mathbb{K}[[z]]$ is conjugation-equivariant, that $F^{1,23}$ conjugates x_2 and x_3 by the same element, and the fact that the trace is additive. More generally, given a KV solution F of type $(0, 2+1)$, one can then consider the non-symmetric operad generated by this solution (See [AKKN18b, Lemma 7.3].).

Theorem A.12. *The family* $\mathsf{SolKV} := \{\mathsf{SolKV}(n)\}_{n \geq 2}$ *forms a* $\mathbb{K}[[z]]$*-colored non-symmetric operad.*

Proof. Let $F = (F, h_1) \in \mathsf{SolKV}(m)$ and $G = (G, h_2) \in \mathsf{SolKV}(n)$ be two KV solutions with $h_1 = h_2$. It suffices to show that the operadic composition $F \circ_i G$ is still a KV solution. We need to check that the composite $F \circ_i G$ satisfies the two equations in Definition 3.31. To simplify notation, let us write $\omega := x_i + \ldots + x_{i+n-1}$. Since both F and G satisfy (SolKVI), we have

$$\begin{aligned} F \circ_i G(e^{x_1} \cdots e^{x_{m+n-1}}) &= (F \circ_i 1)(1 \circ_i G)(e^{x_1} \cdots e^{x_{m+n-1}}) \\ &= (F \circ_i 1)(e^{x_1} \cdots e^{x_{i-1}} e^{\omega} e^{x_{i+n}} \cdots e^{x_{m+n-1}}) \\ &= e^{x_1 + \cdots + x_{m+n-1}}. \end{aligned}$$

Here, we are using the fact that $(1 \circ_i G)$ acts trivially on the basis elements x_j, for $j \notin \{i, i+1, \ldots, i+n-1\}$. It follows that $F \circ_i G$ satisfies (SolKVI).

Part II

Grothendieck–Teichmüller Symmetries of Cyclic Operads

GROTHENDIECK-TEICHMÜLLER GROUP THROUGH CYCLIC RIBBON OPERAD AND ITS ACTION ON TANGLES

MARCY ROBERTSON AND CHANDAN SINGH

Abstract. We characterize the prounipotent Grothendieck–Teichmüller group $\widehat{\mathsf{GT}}_{\mathbb{K}}$ as the group of object-preserving automorphisms of the prounipotent completion of the cyclic operad of parenthesized ribbon braids. This extends the classical description of $\widehat{\mathsf{GT}}_{\mathbb{K}}$ to the cyclic setting, and provides a conceptual operadic framework for understanding arithmetic symmetries in low-dimensional topology. As an application, we use the $\widehat{\mathsf{GT}}_{\mathbb{K}}$-action to give a streamlined proof of the formality of the cyclic framed little disks operad. We further construct a lift of the $\widehat{\mathsf{GT}}_{\mathbb{K}}$-action from parenthesized braids to parenthesized framed tangles, offering a new operadic interpretation of a conjectural Galois action outlined by Kassel–Turaev, and connecting it to the broader theory of Grothendieck–Teichmüller symmetries in tensor categories.

Introduction

In his *Esquisse d'un Programme* Grothendieck suggested probing the structure of the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ through its actions on the fundamental groupoids of moduli spaces of curves [Gro97]. The paradigm case is Belyi's faithful action of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\pi_1(\mathcal{M}_{0,4}) \cong \pi_1(\mathcal{P}^1 \setminus \{0,1,\infty\})$ (cf. [Bel79, Iha94]). Drinfeld later isolated a family of *Grothendieck–Teichmüller groups* that governs these "Galois-type" actions [Dri90], and Ihara showed that there is an injection $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{\mathsf{GT}}$ into the profinite version [Iha94]. Over a characteristic–zero field $\mathbb{K}$ one obtains the prounipotent version of the Grothendieck–Teichmüller group, denoted $\widehat{\mathsf{GT}}_{\mathbb{K}}$, which acts faithfully on prounipotent braid groups; studying these $\widehat{\mathsf{GT}}_{\mathbb{K}}$-actions therefore retains information about the original Galois actions (see also [Bro12]). The present paper aims to develops an operadic mechanism that transports the $\widehat{\mathsf{GT}}_{\mathbb{K}}$-action from braids to the richer setting of framed tangles as a means for studying Galois actions on tangles.

Bar-Natan recast Drinfeld's theory in categorical terms, highlighting the universal action of $\widehat{\mathsf{GT}}_{\mathbb{K}}$ on braided monoidal categories and exhibiting its link with universal finite-type knot invariants in [BN98]. Subsequent work of Fresse [Fre17], Willwacher [Wil15] and others refined this viewpoint operadically: the prounipotent Grothendieck–Teichmüller group appears as the object-fixing automorphisms of the completed operad of parenthesised braids,

$$\widehat{\mathsf{GT}}_{\mathbb{K}} \cong \mathrm{Aut}^{+}(\widehat{\mathsf{PaB}}_{\mathbb{K}}).$$

Here, $\mathsf{PaB} = \{\mathsf{PaB}(n)\}_{n\geq 0}$, is the operad of parenthesized braids. Each $\mathsf{PaB}(n)$ is a groupoid whose objects are fully parenthesized words in the symbols $\{1,\dots,n\}$ and whose morphisms are given by elements of the Artin braid group B_n. Topologically, PaB serves as a combinatorial model for the little discs operad E_2, encoding the structure of braided, but not necessarily symmetric, operations (see, for example, [Fre17]). Categorically, algebras over PaB are braided monoidal categories (cf. [Wah01] or [Yau19]). The action of $\widehat{\mathsf{GT}}_{\mathbb{K}}$ on the prounipotent completion of $\widehat{\mathsf{PaB}}_{\mathbb{K}}$ induces a coherent action on the braiding and associativity contraints of prounipotent $\mathbb{K}$-linear braided monoidal categories.

In this paper we study a hierarchy of operadic structures refining the operad of parenthesized braids, each layer capturing progressively finer topological and categorical data. The first such refinement is the operad of parenthesized ribbon braids, PaRB, whose n-ary operations are isotopy classes of braids on n labelled strands, each equipped with a framing—that is, a chosen longitudinal twist. Topologically, PaRB is weakly equivalent to the operad of framed little discs FD_2 [SW03], and therefore provides a combinatorial model for framed E_2-structures. Categorically, algebras over PaRB are balanced braided monoidal categories [Wah01]: braided monoidal categories equipped with a natural self-twist $\theta_X \colon X \to X$ satisfying appropriate compatibility conditions (see Section 1).

The prounipotent Grothendieck-Teichmüller group $\widehat{\mathsf{GT}}_{\mathbb{K}}$ also acts on the operad PaRB. Indeed, it was show in by Boavida, Horel, and the second author [BdBHR19] that the homotopy automorphisms of the profinite operad recover the profinite Grothendieck–Teichmüller group:

$$\widehat{\mathsf{GT}} \cong \mathrm{Aut}^{+}\big(\widehat{\mathsf{PaRB}}\big).$$

These techniques similarly apply in the prounipotent setting, yielding the identification

$$\widehat{\mathsf{GT}}_{\mathbb{K}} \cong \mathrm{Aut}^{+}\big(\widehat{\mathsf{PaRB}}_{\mathbb{K}}\big),$$

where $\widehat{\mathsf{PaRB}}_{\mathbb{K}}$ denotes the prounipotent completion of the parenthesized ribbon braid operad. In both cases, the Grothendieck–Teichmüller group appears as the group of object-preserving automorphisms, and thereby governs the universal symmetries of balanced braided monoidal categories.

Cyclic operads, introduced by Getzler and Kapranov [GK95], generalize ordinary operads by allowing the inputs and output of an operation to be permuted cyclically. More precisely, a cyclic operad is an operad $\mathcal{O}$ in which each space $\mathcal{O}(n)$ carries a right action of the symmetric group Σ_{n+1} (see Definition 2.6), extending the standard Σ_n-action on the inputs.

The framed little discs operad FD_2 admits such a cyclic structure, due to Budney [Bud08], and this structure descends to groupoid models of FD_2. In [CIW19] and [MW22] the authors transfer this cyclic structure to PaRB, and we give an explicit account in Lemma 3.7, verifying that PaRB is indeed a cyclic operad. One of the main results of this paper is that $\widehat{\mathsf{GT}}_{\mathbb{K}}$ acts on the automorphisms of PaRB^{cyc} and, indeed, we can identify $\widehat{\mathsf{GT}}_{\mathbb{K}}$ with the group of object-fixing automorphisms of the cyclic operad of parenthesized ribbon braids.

Theorem A (Theorem 7.4)**.** *There exists an isomorphism of prounipotent groups*

$$\widehat{\mathsf{GT}}_{\mathbb{K}} \cong \mathrm{Aut}^{+}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc}).$$

This realization confirms a suggestion made by Kontsevich in [Kon17, Section 2.4]. As an application, we show the cyclic operad PaRB^{cyc} is rationally formal in Section 7.2. This is an alternative proof of the formality of cyclic PaRB from [CIW19, Thoerem 3.1] that completely relies on $\widehat{\mathsf{GT}}_{\mathbb{K}}$-actions.

The completed parenthesized ribbon braid operad $\widehat{\mathsf{PaRB}}_{\mathbb{K}}$ admits a natural associated graded operad: the operad of parenthesized ribbon chord diagrams, PaRCD. Algebras over PaRCD are symmetric monoidal categories equipped with an infinitesimal braiding, reflecting the linearized structure of braids. This operad inherits a canonical cyclic structure [CIW19, Wil24], making it a natural recipient for graded versions of $\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc}$.

For each Drinfeld associator, one can construct an isomorphism of cyclic operads from $\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc}$ to PaRCD, yielding a framed, cyclic refinement of the identification of associators with isomorphisms between the operad of parenthesized braids and the operad of parenthesized chord diagrams $\widehat{\mathsf{PaB}}_{\mathbb{K}} \to \mathsf{PaCD}$ ([BN98], [Fre17], [CiL24]). The prounipotent Grothendieck–Teichmüller group $\widehat{\mathsf{GT}}_{\mathbb{K}}$ acts freely and transitively on the set of Drinfeld associators from the left and the associated graded group, $\widehat{\mathsf{GRT}}_{\mathbb{K}}$ acts freely and transitively from the right. We show that this bitorsor structure lifts to the setting of cyclic operads (Proposition 8.6).

Theorem B.

- *The set of Drinfeld associators is in bijection with the set of object-fixing isomorphisms of cyclic operads:*

$$\mathrm{Iso}^{+}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}} \to \mathsf{PaRCD}^{\mathrm{cyc}}).$$

- *There is an isomorphism of prounipotent groups:*

$$\widehat{\mathsf{GRT}}_{\mathbb{K}} \cong \mathrm{Aut}^{+}(\mathsf{PaRCD}^{\mathrm{cyc}}).$$

As a second application of Theorem 7.4, we study how the action of $\widehat{\mathsf{GT}}_{\mathbb{K}}$ extends to categories of tangles. The framed tangle category $\mathbf{T}$ is a $\mathbb{K}$-linear symmetric monoidal category whose objects are finite words in the symbols $\{+,-\}$, representing oriented endpoints, and whose morphisms are isotopy classes of framed tangles between such words.

The category $\mathbf{T}$ carries the structure of a ribbon category, which is a balanced braided monoidal category equipped with duals (Definition 1.3). A key result of Shum [Shu94] and Reshetikhin–Turaev [RT90] states that $\mathbf{T}$ is the free ribbon category on one object. That is, for any ribbon category $\mathbf{R}$ and any object $X \in \mathbf{R}$, there exists a unique strict ribbon functor

$$F\colon \mathbf{T} \longrightarrow \mathbf{R}$$

such that $F(+) = X$.

The tangle category $\mathbf{T}$ and its weak, or non-associative, version, denoted $q\mathbf{T}$ play a central role in the theory of finite-type invariants. In particular, Kassel and Turaev [KT$^+$95] constructed a completed version of the tangle category, $\widehat{\mathbf{T}}_{\mathbb{K}}$, and showed that it is isomorphic to the category $\widehat{\mathbf{A}}(\mathbb{K})$ of chord diagrams. This isomorphism realizes the Kontsevich integral as a universal invariant for framed tangles, meaning that all finite-type (Vassiliev) invariants factor uniquely through this functor.

To every operad, one can associate a strict symmetric monoidal category—called a *prop*. Similarly, to every cyclic operad, Hinich and Vaintrob [HV02] associate a strict symmetric monoidal category with self-duality, known as a metric prop (Section 2.3). Using this framework, we relate the cyclic operads $\mathsf{CoRB}^{\mathrm{cyc}}$ and $\mathsf{PaRB}^{\mathrm{cyc}}$ to the categories of framed tangles. We show that the metric prop associated to $\mathsf{CoRB}^{\mathrm{cyc}}$ is isomorphic to the full subcategory $\mathbf{T}' \subset \mathbf{T}$ consisting of self-dual objects, and similarly that

$$\Pi(\mathsf{CoRB}^{\mathrm{cyc}}) \cong \mathbf{T}' \quad \text{and} \quad \Pi(\mathsf{PaRB}^{\mathrm{cyc}}) \cong q\mathbf{T}',$$

where $q\mathbf{T}'$ denotes the corresponding subcategory of the non-strict tangle category $q\mathbf{T}$ (Proposition 5.7, Corollary 5.8).

Combining these ideas, we show that the $\widehat{\mathsf{GT}}_{\mathbb{K}}$-action on the cyclic operad $\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc}$ extends naturally to its associated metric prop $\Pi(\mathsf{PaRB}^{\mathrm{cyc}})$, and hence to the category $q\mathbf{T}'$ of framed tangles with self-dual, parenthesized boundaries. This provides a conceptual and operadic framework for understanding the arithmetic symmetries of tangles. In particular, our construction recovers the $\widehat{\mathsf{GT}}_{\mathbb{K}}$-action on the completed category of framed tangles $\widehat{\mathbf{T}}_{\mathbb{K}}$ conjectured in Appendix D of [KT$^+$95], where Kassel and Turaev propose the existence of a Galois action on tangles compatible with the Kontsevich integral. Our result gives a precise operadic model for this action and confirms that it arises naturally from the $\widehat{\mathsf{GT}}_{\mathbb{K}}$-symmetry of the cyclic operad PaRB.

This perspective also connects with the work of Furusho [Fur17, Fur20], who studied the Galois symmetries of braids and tangles via the so-called ABC construction. Our approach provides an alternative derivation of these symmetries, grounded in the formalism of cyclic operads and metric props, and embeds them into a broader framework that unifies topological, algebraic, and arithmetic structures.

Organisation of the paper. Section 1 and Section 2 provide background on monoidal categories, operads, cyclic operads, envelopes, and (metric) props.

In Section 3, we show that the operad of parenthesized ribbon braids admits a cyclic structure (Lemma 3.7). A parallel discussion of the cyclic structure on the operad of ribbon chord diagrams appears in Section 4.

Section 5 establishes the connection between the categories of tangles and chord diagrams and the metric props associated to the cyclic operads PaRB and PaRCD (Proposition 5.7, Corollary 5.8).

In Section 7, we characterize the Grothendieck–Teichmüller group $\widehat{\mathsf{GT}}_{\mathbb{K}}$ as the automorphism group of the completed cyclic operad $\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}$ (Proposition 7.4), prove the formality of $\mathsf{FD}_2^{\mathrm{cyc}}$ (Lemma 7.6), and describe the induced $\widehat{\mathsf{GT}}_{\mathbb{K}}$-action on the tangle category $\mathbf{T}$ (Proposition 7.10).

Section 8 turns to the graded setting: we identify the group $\widehat{\mathsf{GRT}}_{\mathbb{K}}$ as the automorphism group of the completed cyclic operad of parenthesized ribbon chord diagrams (Theorem 8.2) and describe the corresponding action on the category $\mathbf{A}(\mathbb{K})$ (Proposition 8.9).

Finally, Appendix A reviews the necessary adjunctions between operads and cyclic operads.

Contents

Introduction 80
  Organisation of the paper 82
1. Categorical Background 84
  1.1. Monoidal Categories 84
  1.2. Infinitesimal Symmetric Category 86
  1.3. Completion of categories 87
2. Operads, Cyclic operads, Props 87
  2.1. Operads 87
  2.2. Cyclic operads 89
  2.3. Props and the envelope of an operad 90
  2.4. Metric props and the envelope of a cyclic operad 91
  2.5. Completion of Operads and Props 92
3. Cyclic operad of ribbon braids 93
  3.1. Braids and ribbon braids 93
  3.2. The operad of parenthesized ribbon braids 93
  3.3. The cyclic operad of ribbon braids 95
4. Cyclic operad of chord diagrams 98
  4.1. Drinfeld-Kohno Lie algebras 99
5. Tangles and Chord diagrams 101
  5.1. Tangles 101
  5.2. Chord diagrams 105
6. Drinfeld's Associators 107
7. Grothendieck-Teichmüller group $\widehat{\mathsf{GT}}_{\mathbb{K}}$ 110
  7.1. Maps out of the cyclic PaRB 110
  7.2. Rational formality of the cyclic framed little discs operad 113
  7.3. Homotopy Automorphisms of PaRB^{cyc} 114
  7.4. GT action on Tangles 115
8. Graded Grothendieck-Teichmüller group $\widehat{\mathsf{GRT}}_{\mathbb{K}}$ 117
  8.1. Graded Grothendieck-Teichmüller group - GRT 117
  8.2. GRT action on Chord diagrams 120
Appendix A. Adjunctions between operads and cyclic operads 121
  A.1. Operad for cyclic operads 122
  A.2. Operads and cyclic operads 123
References 123

1. Categorical Background

Throughout, we work over a fixed field $\mathbb{K}$ of characteristic zero, and all categories are assumed to be $\mathbb{K}$-linear. A monoidal category $(\mathbf{C}, \otimes, \mathbb{1})$ is said to be $\mathbb{K}$-linear if each hom-set $\mathrm{Hom}_{\mathbf{C}}(X, Y)$ is a $\mathbb{K}$-module and both the composition of morphisms and the monoidal product

$$(f, g) \mapsto g \circ f \quad \text{and} \quad (f, g) \mapsto f \otimes g$$

are bilinear over $\mathbb{K}$. A functor between $\mathbb{K}$-linear categories is said to be $\mathbb{K}$-linear if it induces $\mathbb{K}$-linear maps on hom-sets.

1.1. **Monoidal Categories.**

We work various types of monoidal categories throughout this paper and take [Mac71], [Kas12] and [EGNO15] as references. We briefly recall that a *monoidal category* $(\mathbf{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$ consists of a category $\mathbf{C}$ equipped with a tensor product $\otimes$, a unit object $\mathbb{1}$, and natural isomorphisms: an associator $\alpha_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ and left and right unitors $\lambda_V : \mathbb{1} \otimes V \to V$ and $\rho_V : V \otimes \mathbb{1} \to V$, subject to coherence conditions expressed via commutative diagrams (the pentagon and triangle identities). A monoidal category is called *strict* if the associativity and unit isomorphisms are identities, that is, $(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z)$ and $\mathbb{1} \otimes X = X = X \otimes \mathbb{1}$ hold strictly for all objects X, Y, Z.

A *braiding* on a strict monoidal category $(\mathbf{C}, \otimes, \mathbb{1})$ is a family of natural isomorphisms

$$c_{V,W} : V \otimes W \to W \otimes V$$

satisfying the following conditions for all objects $U, V, W, V', W' \in \mathbf{C}$ and morphisms $f : V \to V'$, $g : W \to W'$:

$$c_{V',W'}(f \otimes g) = (g \otimes f)\, c_{V,W}, \tag{1.1}$$

$$c_{U,V\otimes W} = (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W), \tag{1.2}$$

$$c_{U\otimes V,W} = (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}). \tag{1.3}$$

A *strict braided monoidal category* is a strict monoidal category equipped with a braiding. A *strict symmetric monoidal category* is a braided monoidal category with a symmetric braiding, meaning

$$c_{V,W} \circ c_{W,V} = \mathrm{id}_{V\otimes W}$$

for all objects V, W.

A *balancing* on a strict braided monoidal category $(\mathbf{C}, \otimes, \mathbb{1}, c)$ is a natural family of automorphisms $\Theta_V : V \to V$ satisfying

$$\Theta_{V\otimes W} = (\Theta_V \otimes \Theta_W) \circ c_{W,V} \circ c_{V,W}. \tag{1.4}$$

A *strict balanced monoidal category* is a braided monoidal category equipped with a balancing. A *strong monoidal functor* between monoidal categories $(\mathbf{C}, \otimes, \mathbb{1}_{\mathbf{C}})$ and $(\mathbf{D}, \otimes, \mathbb{1}_{\mathbf{D}})$ is a functor $F : \mathbf{C} \to \mathbf{D}$ equipped with natural isomorphisms

$$F(X) \otimes F(Y) \xrightarrow{\cong} F(X \otimes Y) \quad \text{and} \quad \mathbb{1}_{\mathbf{D}} \xrightarrow{\cong} F(\mathbb{1}_{\mathbf{C}})$$

satisfying the standard coherence conditions for associativity and unit compatibility. A functor between strict braided, symmetric, or balanced monoidal categories is a strong monoidal functor that also preserves the additional structure: it intertwines the braidings (respectively, symmetries) and, in the balanced case, satisfies $F(\Theta_V) = \Theta_{F(V)}$ for all objects V.

Example 1.1. Let $\mathbf{C} = \mathrm{Rep}(U_q(\mathfrak{sl}_2))$ be the category of finite-dimensional representations of the Drinfeld–Jimbo quantum group $U_q(\mathfrak{sl}_2)$ over $\mathbb{K} = \mathbb{C}(q)$. Then $\mathbf{C}$ is a braided monoidal category, where the braiding $c_{V,W}$ is defined via the universal R-matrix. This braiding is not symmetric when q is not a root of unity. The category becomes balanced by defining the twist Θ_V using the ribbon element, which acts as a scalar on weight spaces and satisfies

$$\Theta_{V\otimes W} = (\Theta_V \otimes \Theta_W) \circ c_{W,V} \circ c_{V,W}.$$

Definition 1.2. Let $\mathbf{C} = (\mathbf{C}, \otimes, \mathbb{1})$ be a strict monoidal category. We say that $\mathbf{C}$ has (left) *duality*if, for every object $V \in \mathbf{C}$, there exists an object V^* together with morphisms

$$b_V : \mathbb{1} \to V \otimes V^*, \qquad d_V : V^* \otimes V \to \mathbb{1}$$

such that

$$(\mathrm{id}_V \otimes d_V) \circ (b_V \otimes \mathrm{id}_V) = \mathrm{id}_V, \qquad (d_V \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{V^*} \otimes b_V) = \mathrm{id}_{V^*}.$$

The pair (d_V, b_V) implies that the dual V^* is unique up to a unique isomorphism, if it exists. In a strict monoidal category with left duals, the *transpose* of a morphism $f : V \to W$ is defined as

$$f^* := (d_W \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{W^*} \otimes f \otimes \mathrm{id}_{V^*}) \circ (\mathrm{id}_{W^*} \otimes b_V).$$

Definition 1.3. A *strict ribbon category* is a strict balanced monoidal category $(\mathbf{C}, \otimes, \mathbb{1}, c, \Theta)$ equipped with duals, such that the twist satisfies

$$\Theta_{V^*} = (\Theta_V)^*$$

for all objects V.

A *ribbon monoidal functor*, $F : \mathbf{C} \to \mathbf{D}$ is a strong monoidal functor equipped with a monoidal natural isomorphism

$$F(X) \otimes F(Y) \xrightarrow{\cong} F(X \otimes Y), \quad \mathbb{1}_{\mathbf{D}} \xrightarrow{\cong} F(\mathbb{1}_{\mathbf{C}}),$$

that is compatible with the braiding, twist, and duality structures of $\mathbf{C}$ and $\mathbf{D}$.

Let $\mathbf{E} = (\mathbf{E}, \otimes, \mathbb{1})$ be a strict symmetric monoidal category. An object $A \in \mathbf{E}$ admits a *non-degenerate, symmetric form* if there exist morphisms

$$d : A \otimes A \to \mathbb{1}, \qquad b : \mathbb{1} \to A \otimes A$$

such that the composites

$$\mathbb{1} \otimes A \xrightarrow{b \otimes \mathrm{id}_A} A \otimes A \otimes A \xrightarrow{\mathrm{id}_A \otimes d} A \otimes \mathbb{1} \tag{1.5}$$

and

$$A \otimes \mathbb{1} \xrightarrow{\mathrm{id}_A \otimes b} A \otimes A \otimes A \xrightarrow{d \otimes \mathrm{id}_A} \mathbb{1} \otimes A \tag{1.6}$$

are both equal to the identity morphism on A. The condition is equivalent to saying that A is self-dual. See [HV02, 2.1.1] or [Luk10] for further discussion.

All definitions and constructions above extend naturally to this setting by inserting the appropriate associators and unitors into composition formulas. A non-degenerate, symmetric form with weak associativity, say $\alpha : (A \otimes A) \otimes A \longrightarrow A \otimes (A \otimes A)$, gives rise to isomorphism of the identities (1.5) and (1.6) to id_A. That is

$$(b \otimes \mathrm{id}_A)\alpha(\mathrm{id}_A \otimes d) \cong \mathrm{id}_A \cong (\mathrm{id}_A \otimes b)\alpha^{-1}(d \otimes \mathrm{id}_A).$$

We note that, by Mac Lane's Coherence Theorem [ML63], every monoidal category is monoidally equivalent to a strict one. Moreover, the strictification process extends to braided, balanced, and ribbon monoidal categories; see [JS93, Shu94].

Example 1.4. The category $\mathrm{Vect}_{\mathbb{K}}$ of finite-dimensional $\mathbb{K}$-vector spaces is a symmetric monoidal category with the usual tensor product and symmetry

$$c_{V,W}(v \otimes w) = w \otimes v.$$

The symmetry satisfies $c_{W,V} \circ c_{V,W} = \mathrm{id}_{V \otimes W}$, and the balancing $\Theta_V = \mathrm{id}_V$ is trivial.

Example 1.5. The category $\mathrm{Rep}(U_q(\mathfrak{sl}_2))$ of finite-dimensional representations of the quantum group $U_q(\mathfrak{sl}_2)$ over $\mathbb{K} = \mathbb{C}(q)$, for generic q, is a ribbon category. The braiding is induced by the universal R-matrix, and the twist (balancing) is given by the ribbon element. Each object has a dual given by the contragredient representation, and the twist satisfies $\Theta_{V^*} = (\Theta_V)^*$.

Remark 1.6. If $\mathbf{E} = (\mathsf{Cat}, \times, *)$ is the category of small categories, whose monoidal structure is given by the cartesian product of categories, then the only category $\mathbf{C} \in \mathsf{Cat}$ which admits a nondegenerate, symmetric form is the trivial category, i.e. $\mathbf{C} \simeq *$, see Remark 2.15 of [Luk10].

1.2. **Infinitesimal Symmetric Category.** Infinitesimal symmetric categories abstract the notion of a "first-order" or "linearized" deformation of braiding in a symmetric monoidal category. More precisely, if one wishes to deform a symmetric monoidal category into a braided monoidal category, the infinitesimal braiding provides a tangent vector or direction for such a deformation: it captures how the symmetric structure begins to deviate from commutativity at first order.

Let $(\mathbf{S}, \otimes, \mathbb{1})$ be a strict symmetric $\mathbb{K}$-linear category with symmetric braiding

$$\{c_{X,Y} : X \otimes Y \to Y \otimes X\}_{X,Y \in \mathbf{S}}.$$

The category $\mathbf{S}[[\hbar]]$ which has the same objects as those of $\mathbf{S}$ and whose morphism sets are defined by

$$\mathrm{Hom}_{\mathbf{S}[[\hbar]]}(X, Y) := \mathrm{Hom}_{\mathbf{S}}(X, Y) \otimes_{\mathbb{K}} \mathbb{K}[[\hbar]],$$

for all objects $X, Y \in \mathbf{S}$. That is, $\mathbf{S}[[\hbar]]$ is the category obtained by extending the morphisms of $\mathbf{S}$ to formal power series in a deformation parameter $\hbar$. The monoidal structure on $\mathbf{S}$ naturally extends to $\mathbf{S}[[\hbar]]$ by $\mathbb{K}[[\hbar]]$-linearity.

Definition 1.7. An *infinitesimal braiding* on $\mathbf{S}$ is a family of endomorphisms

$$\{t_{X,Y} : X \otimes Y \longrightarrow X \otimes Y\}_{X,Y \in \mathbf{S}}$$

of $\mathbf{S}$ which satisfy the following conditions:

$$(f \otimes g) \circ t_{X,Y} = t_{X',Y'} \circ (f \otimes g), \quad \text{for all } f : X \to X',\ g : Y \to Y'. \tag{1.7}$$

$$t_{X,Y} = c_{X,Y} \circ t_{Y,X} \circ c^{-1}_{X,Y}, \quad \text{for all } X, Y \in \mathbf{S}. \tag{1.8}$$

$$t_{X \otimes Y,Z} = (\mathrm{id}_X \otimes t_{Y,Z}) + (\mathrm{id}_X \otimes c_{Y,Z}) \circ (t_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,Z})^{-1}. \tag{1.9}$$

In particular, the $t_{X,Y}$ define a braiding

$$\{c_{X,Y}(\mathrm{id}_{X \otimes Y} + \frac{\hbar}{2} t_{X,Y})\}_{X,Y \in \mathbf{S}}$$

on $\mathbf{S}[[\hbar]]$ which satisfies (1.8) (see [Car93]).

We note that equation (1.9) admits several equivalent reformulations via graphical calculus (cf. [KT+95, Prop. B.1]):

$$t_{X \otimes Y,Z} = (c_{X,Y} \otimes \mathrm{id}_Z) \circ (\mathrm{id}_Y \otimes t_{X,Z}) \circ (c_{X,Y} \otimes \mathrm{id}_Z)^{-1} + (\mathrm{id}_X \otimes t_{Y,Z}), \tag{1.10}$$

$$t_{X,Y \otimes Z} = (t_{X,Y} \otimes \mathrm{id}_Z) + (\mathrm{id}_X \otimes c_{Y,Z}) \circ (t_{X,Z} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,Z})^{-1}. \tag{1.11}$$

Moreover, the infinitesimal braiding satisfies a compatibility condition resembling a flatness equation: the following commutator vanishes,

$$[t_{X,Y} \otimes \mathrm{id}_Z, c \circ (t_{X,Y} \otimes \mathrm{id}_Y) \circ c^{-1} + \mathrm{id}_X \otimes t_{Y,Z}] = 0,$$

where $c = \mathrm{id}_X \otimes c_{Y,Z}$, and $[f, g] = fg - gf$.

Definition 1.8. A *strict infinitesimal symmetric category* is a strict symmetric $\mathbb{K}$-linear category equipped with an infinitesimal braiding.

If $\mathbf{S}$ is not strict, the compatibility of the infinitesimal braiding with associativity is slightly more complicated.

Proposition 1.9. *Let* $\mathbf{S}$ *be a symmetric monoidal* $\mathbb{K}$*-linear category. An* infinitesimal braiding *on* $\mathbf{S}$ *is a family of endomorphisms*

$$\{t_{X,Y} : X \otimes Y \to X \otimes Y\}_{X,Y \in \mathbf{S}}$$

satisfying the symmetry condition (1.8) *and the semi-classical pentagon:*

$$t_{X \otimes Y,Z} = \alpha_{X,Y,Z} \circ (\mathrm{id}_X \otimes t_{Y,Z}) \circ \alpha^{-1}_{X,Y,Z} + \beta_{X,Y} \circ (t_{X,Z} \otimes \mathrm{id}_Y) \circ \beta^{-1}_{X,Y}, \tag{1.12}$$

where $\beta_{X,Y} = \alpha^{-1}_{X,Y,Z} \circ (\mathrm{id}_X \otimes c_{Y,Z}) \circ \alpha_{X,Y,Z}$ *and* $\alpha_{X,Y,Z}$ *is the associator isomorphism.*

Remark 1.10. Equation (1.12) is equivalent to the semi-classical pentagon relation described in [Fre17, Section 10.3.3].

Example 1.11. Let $\mathfrak{g}$ be a quadratic Lie algebra over $\mathbb{K}$. The classic example of a symmetric monoidal category with infinitesimal braiding is the category of (complex) representations of $\mathfrak{g}$ has an infinitesimal braiding $t_{x,x'}$ given by the action of $t \in \mathrm{Sym}^2\mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$ on $x \otimes x'$.

1.3. **Completion of categories.** We briefly recall the completion of $\mathbb{K}$-linear category following [KT+95]. Let $(\mathbf{M}, \circ, \otimes)$ be a $\mathbb{K}$-linear monoidal category.

Definition 1.12. An *ideal* $\mathbf{I}$ in $\mathbf{M}$ is a $\mathbb{K}$-linear subcategory of $\mathbf{M}$ consisting of morphisms such that for any $f, g \in \mathrm{Hom}_{\mathbf{M}}(-,-)$, the composite $f \circ g$ and the tensor product $f \otimes g$ belong to $\mathbf{I}$ whenever $f \in \mathbf{I}$ or $g \in \mathbf{I}$.

An ideal $\mathbf{I}^{n+1}$ is generated by the composition of at least $n+1$ morphisms of $\mathbf{I}$. Now, the prounipotent completion of the $\mathbb{K}$-linear monoidal category $\mathbf{M}$ is defined by the inverse limit of the projective system of categories $\{\mathbf{M}/\mathbf{I}^{n+1}\}_{n \geq 0}$,

$$\widehat{\mathbf{M}}_{\mathbb{K}} := \lim_{n} \mathbf{M}/\mathbf{I}^{n+1},$$

where $\mathbf{M}/\mathbf{I}^{n+1}$ is a $\mathbb{K}$-linear category whose objects $\mathrm{ob}(\mathbf{M}/\mathbf{I}^{n+1}) = \mathrm{ob}(\mathbf{I})$ and

$$\mathrm{Hom}_{\mathbf{M}/\mathbf{I}^{n+1}}(X, Y) = \mathrm{Hom}_{\mathbf{M}}(X, Y)/\mathbf{I}^{n+1}(X, Y),$$

for any $X, Y \in \mathrm{ob}(\mathbf{M})$.

Example 1.13. For a $\mathbb{K}$-linear braided monoidal category $\mathbf{M}$, its prounipotent completion $\widehat{\mathbf{M}}_{\mathbb{K}}$ is the inverse limit of the filtration $\{\mathbf{M}/\mathbf{I}^{n+1}\}_{n \geq 0}$, of the $\mathbb{K}$-linear quotient categories $\mathbf{M}/\mathbf{I}^{n+1}$ with respect to the ideal $\mathbf{I}$ generated by $c_{X,Y} \circ c_{Y,X} - \mathrm{id}_{X \otimes Y}$ with $c_{X,Y} : X \otimes Y \longrightarrow Y \otimes X$ the braiding constraints of $\mathbf{M}$.

Remark 1.14. The prounipotent completion of a $\mathbb{K}$-linear monoidal category and the formal power series completion $\mathbf{M}[[\hbar]]$ are closely related constructions, both designed to capture infinitesimal or formal deformations of categorical structures. When the ideal $\mathbf{I}$ is generated by a first-order infinitesimal–such as a braiding defect of the form $c_{X,Y}c_{Y,X} - \mathrm{id}$–the prounipotent completion $\widehat{\mathbf{M}}_{\mathbb{K}}$ can be viewed as the $\hbar$-adic completion of $\mathbf{M}$, and thus equivalent to $\mathbf{M}[[\hbar]]$ under suitable identifications. For example, the prounipotent completion of $\mathbb{Z}$ over $\mathbb{Q}$ is $\mathbb{Q}$. Completing the group algebra $\mathbb{Q}[[\mathbb{Z}]]$, which is isomorphic to the ring $\mathbb{Q}[[s, s^{-1}]]$, is the formal power series $\mathbb{Q}[[S]]$ with the augmentation ideal generated by $S - 1$. The set of grouplike elements of $\mathbb{Q}[[\mathbb{Z}]]$ is $\mathbb{Q}$.

Example 1.15. The prounipotent completion $\widehat{\mathbf{R}}_{\mathbb{K}}$ of a $\mathbb{K}$-linear ribbon monoidal category $\mathbf{R}$ is the prounipotent completion of its underlying braided monoidal category, as in Example 1.13, together with additional compatibility with twists $\Theta : X \to X$ and duality pairing (d, b). We note that $\Theta^2 - \mathrm{id}_X$ belongs to the augmentation ideal of $\mathbf{R}$ due to Lemma 3.4 of [KT+95].

2. Operads, Cyclic operads, Props

This section recalls the algebraic frameworks of operads, cyclic operads, and props used to model structured families of operations.

2.1. **Operads.** Throughout, we will let $\mathbf{E} := (\mathbf{E}, \otimes, \mathbb{1})$ denote a closed, symmetric monoidal category where the tensor product $\otimes$ commutes with all colimits. We will write $\mathrm{Hom}_{\mathbf{E}}(X, Y)$ for both the set of maps $X \to Y$ in $\mathbf{E}$ and for the internal hom of $\mathbf{E}$, that is the object in $\mathbf{E}$ fits into the tensor-hom adjunction

$$\mathrm{Hom}_{\mathbf{E}}(X \otimes Y, Z) \cong \mathrm{Hom}_{\mathbf{E}}(X, \mathrm{Hom}_{\mathbf{E}}(Y, Z)).$$

Definition 2.1. A *symmetric sequence* in $\mathbf{E}$ consists of an $\mathbb{N}$-graded sequence $\mathcal{O} = (\mathcal{O}(0), \mathcal{O}(1), \ldots, \mathcal{O}(n), \ldots)$ in which each $\mathcal{O}(n) \in \mathbf{E}$ is equipped with a right action

$$\mathcal{O}(n) \times \Sigma_n \xrightarrow{\sigma^*} \mathcal{O}(n).$$

An *operad* in $\mathbf{E}$ consists of a symmetric sequence $\mathcal{O} = \{\mathcal{O}(n)\}_{n \geq 0}$ in $\mathbf{E}$ together with:

(1) a distinguished operation $1 \in \mathcal{O}(1)$ called the *unit*;

(2) a family of associative, equivariant, and unital partial compositions

$$\mathcal{O}(n) \times \mathcal{O}(m) \xrightarrow{\circ_i} \mathcal{O}(n+m-1),$$

where $1 \leq i \leq n$.

The composition operation $\circ_i$ represents the substitution of an m-ary operation into the i^{th} input of an n-ary operation. This composition is often depicted by grafting rooted trees as in Figure 1. Associativity and equivariance ensure the coherence of multiple such substitutions, while the unit provides an identity for composition. A morphism $f : \mathcal{O} \to \mathcal{P}$ of operads is a map of the underlying symmetric sequences that commutes with additional operad structure. The category of operads in $\mathbf{E}$ is denoted by $\mathbf{Op}(\mathbf{E})$. For further details see [Fre17, Chapter 1] or [MSS02].

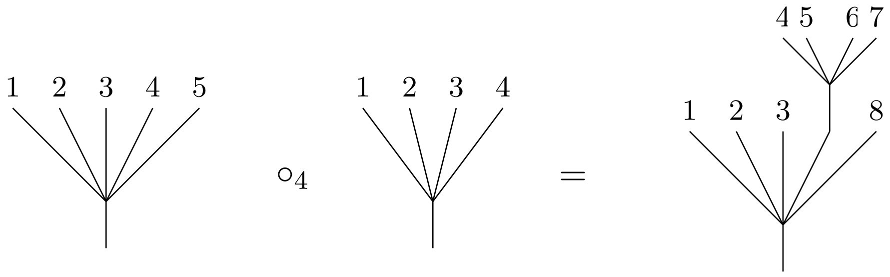


FIGURE 1. The partial composition $\circ_4 : \mathcal{O}(5) \times \mathcal{O}(4) \longrightarrow \mathcal{O}(8)$

Example 2.2. Fix an object $X \in \mathbf{E}$ and define $\mathrm{End}(X)(n) := \mathrm{Hom}_{\mathbf{E}}(X^{\otimes n}, X)$. This gives a symmetric sequence $\mathrm{End}(X) = \{\mathrm{End}(X)(n)\}_{n \geq 1}$, with a natural Σ_n-action given by

$$\mathrm{End}(X)(n) \times \Sigma_n \xrightarrow{\sigma^*} \mathrm{End}(X)(n)$$

where $\sigma^*(f) := f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. The composition maps are defined by substitution:

$$(f(x_1, \ldots, x_n); g(y_1, \ldots, y_m)) \mapsto f(x_1, \ldots, x_{i-1}, g(y_1, \ldots, y_m), x_{i+1}, \ldots, x_n).$$

The identity morphism id_X serves as the operadic unit.

An *algebra* over an operad $\mathcal{O}$ is an operad morphism $\rho : \mathcal{O} \to \mathrm{End}(A)$. Note that this is equivalent to giving an action

$$\mathcal{O}(n) \otimes A^{\otimes n} \xrightarrow{\rho^*} A,$$

for each $n \geq 1$. A morphism of $\mathcal{O}$-algebras is a map $f : A \to B$ in $\mathbf{E}$ which is compatible with the $\mathcal{O}$-action. We write $\mathrm{Alg}_{\mathcal{O}}(\mathbf{E})$ for the category of $\mathcal{O}$-algebras in $\mathbf{E}$.

Example 2.3. The framed little 2-disks operad FD_2 is an operad given in positive arity n by the space of all smooth, orientation-preserving embeddings $\coprod_{k=1}^{n} D_k^2 \longrightarrow D^2$ of the disjoint union of n disks into the unit disk such that the embeddings are compositions of translation, dilation, and rotation. We define $\mathsf{FD}(0)$ as the empty space. The operadic composition is given by the composition of embeddings (see, for example, [Get94, pg. 20]). FD_2 has same homotopy type as the configuration space of the framed points in $\mathbb{R}^2$. By dropping the rotation or framing data, one obtains the operad of little 2-disks D_2, in particular, these two operads are closely related by the semidirect product $\mathsf{FD}_2 \simeq \mathsf{D}_2 \rtimes SO(2)$, see [Wah01], [SW03].

Example 2.4. The operad $\mathcal{M}$ is a collection $\{\mathcal{M}_{0,l+1}\}_{l \geq 1}$ of moduli spaces of framed curves of genus 0 with tangent rays at $l+1$ ordered marked points and compactify to adhere to the reference, upto biholomorphic automorphism. The moduli space $\mathcal{M}_{0,l+1}$ parameterizes the configuration of distinct $l+1$ marked points on the sphere. The operadic composition $\circ_i : \mathcal{M}_{0,l+1} \times \mathcal{M}_{0,k+1} \longrightarrow \mathcal{M}_{0,l+k+1}$ is given by gluing two such configurations and forgetting the glued points to obtain a new configuration; see for example [Get94] for more details. $\mathcal{M}$ is homotopy equivalent to the operad FD_2 [GS12, Proposition 2.1].

Example 2.5. The Batalin–Vilkovisky operad BV is a differential graded operad whose algebras are precisely Batalin–Vilkovisky algebras, in particular, a quadruple $(A, \wedge, [-,-], \Delta,)$ in which A is a graded vector space, $\wedge$ is a graded commutative product in degree 0, $[-,-]$ is a lie bracket on A of degree -1, and $\Delta : A \to A$ is a BV operator of degree -1 such that $\wedge$ makes A into a graded commutative algebra together with the following relations:

$$\Delta^2 = 0 \quad \text{and} \quad [x, y \wedge z] = [x, y] \wedge z + (-1)^{(|x|+1)|y|} y \wedge [x, z], \tag{2.1}$$

$$[x, y] = \Delta(x \wedge y) - (\Delta x) \wedge y - (-1)^{|x|} x \wedge (\Delta y) \tag{2.2}$$

The operad BV generated by the graded commutative algebra product $\wedge$ in arity 2, and the differential operator Δ in arity 1. Note that the bracket $[-,-]$ is implicitly given by the second relation (2.2), which reduces the number of generators to two. The operad BV can be equivalently defined as the homology operad of the framed little 2-disk operad. See, for example, [Get94] for more details.

2.2. **Cyclic operads.** Let $\Sigma_n := \mathrm{Aut}(\{1, \dots, n\})$ denote the symmetric group on n-letters and $\Sigma_n^+ = \mathrm{Aut}(\{0, 1, \dots, n\})$ for the *extended symmetric group*. We note that Σ_n^+ contains Σ_n as the subgroup of automorphisms that fix zero. Let z_{n+1} denote the cyclic permutation $z_{n+1}(i) = i + 1 \pmod{n+1}$ that generates a copy of the cyclic group $C_{n+1} \subset \Sigma_n^+$.

A cyclic structure on an operad is an additional structure which encodes the data needed to speak about invariant bilinear forms on the algebras over that operad. More precisely, a *cyclic structure* on an operad $\mathcal{O}$ consists of action maps

$$\mathcal{O}(n) \times \Sigma_n^+ \xrightarrow{\sigma^*} \mathcal{O}(n),$$

$n \geq 1$, which satisfy the following properties.

(1) If $\sigma \in \Sigma_n \subset \Sigma_n^+$, then the action $\mathcal{O}(n) \times \Sigma_n^+ \xrightarrow{\sigma^*} \mathcal{O}(n)$, is the action coming from the operad structure on $\mathcal{O}$.

(2) For every $x \in \mathcal{O}(n)$ and $y \in \mathcal{O}(m)$, we have the following:

$$z_{n+m}^*(x \circ_i y) = \begin{cases} z_{n+1}^*(x) \ \circ_{i-1} \ y & \text{if } 2 \leq i \leq n \\ z_{m+1}^*(y) \ \circ_m \ z_{n+1}^*(x) & \text{if } i = 1 \text{ and } n \neq 0. \end{cases} \tag{2.3}$$

The identity (2.3) ensures that rotating an operation and then composing is the same as composing and then rotating, guaranteeing compatibility between cyclic symmetry and operadic substitution.

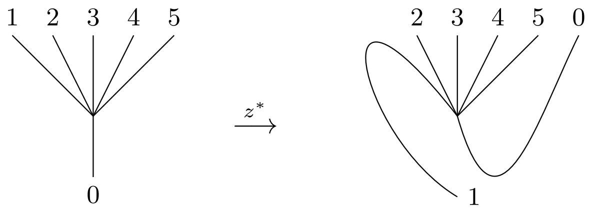


FIGURE 2. A cyclic action z^* that permutes the first input into the output and the output with the last input.

Definition 2.6. A *cyclic operad* is an operad $\mathcal{O} = \{\mathcal{O}(n)\}_{n \geq 1}$ equipped with a cyclic structure.

A map $f : \mathcal{O} \to \mathcal{P}$ of cyclic operads is a morphism of the underlying operads that commutes with the additional cyclic structure. We denote the category of cyclic operads in $\mathbf{E}$ by $\mathbf{Cyc}(\mathbf{E})$.

Example 2.7 (Cyclic endomorphism operad)**.** Let $A \in \mathbf{E}$ be an object equipped with a non-degenerate symmetric bilinear form $d : A \otimes A \to \mathbb{1}$. The usual endomorphism operad $\mathrm{End}(A)$ has

$$\mathrm{End}(A)(n) := \mathrm{Hom}_{\mathbf{E}}(A^{\otimes n}, A),$$

with partial compositions given by substitution. The bilinear form d allows us to identify $\mathrm{Hom}_{\mathbf{E}}(A^{\otimes n}, A)$ with $\mathrm{Hom}_{\mathbf{E}}(A^{\otimes n+1}, \mathbb{1})$, by setting

$$f \mapsto d \circ (f \otimes \mathrm{id}) : A^{\otimes n} \otimes A \to \mathbb{1}.$$

This identification equips $\mathrm{End}(A)$ with a Σ_n^+-action, where the generator z_{n+1} of the cyclic group C_{n+1} acts by cyclically permuting the inputs and the "output" position. The resulting operad is called the *cyclic endomorphism operad* $\mathrm{End}^{\mathrm{cyc}}(A)$.

An *algebra* over a cyclic operad $\mathcal{O}^{\mathrm{cyc}}$ is morphism of cyclic operads $\mathcal{O}^{\mathrm{cyc}} \to \mathrm{End}^{\mathrm{cyc}}(A)$. Equivalently, an algebra over $\mathcal{O}^{cyc}$ in $\mathbf{E} := (\mathbf{E}, \otimes, \mathbb{1})$ consists of $\mathcal{O}$-algebra A together with a non-degenerate symmetric form $d : A \otimes A \to \mathbb{1}$ which is *$\mathcal{O}$-invariant* in the sense that the composites

$$\mathcal{O}(n) \otimes A^{\otimes n+1} \xrightarrow{\rho_n \otimes \mathrm{id}_A} A \otimes A \xrightarrow{d} \mathbb{1} \tag{2.4}$$

are Σ_n^+-equivariant for all $n \geq 1$.

Example 2.8. The operads FD_2, $\mathcal{M}$, and BV, as in Examples 2.3, 2.4, and 2.5, are all examples of cyclic operads. Their explicit cyclic structures are discussed in [Bud08], [GK95], and [Get94].

2.3. **Props and the envelope of an operad.** Every operad $\mathcal{O}$ admits a canonical extension to a strict symmetric monoidal category called its *envelope*, denoted $\mathrm{Env}(\mathcal{O})$. This envelope is a prop: its objects are finite sequences (or words), and morphisms describe coherent compositions of operations from $\mathcal{O}$ along both input and output directions. Props thus provide a natural categorical framework for comparing operads with $\mathbb{K}$-linear tensor categories.

Definition 2.9. A *prop* $\mathcal{P}$ in a symmetric monoidal category $\mathbf{E}$ is a strict symmetric monoidal category whose objects are the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$, with monoidal product given by addition: $m \boxtimes n := m + n$. The morphism set $\mathcal{P}(m, n)$ is interpreted as the space of operations with m inputs and n outputs. A *morphism of props* is a strict symmetric monoidal functor between such categories.

Every operad can be embedded into a prop via a universal construction called the *envelope.*

Definition 2.10. Let $\mathcal{O}$ be an operad in $\mathbf{E}$. The *envelope* $\mathrm{Env}(\mathcal{O})$ is a strict symmetric monoidal category whose objects are natural numbers and whose morphisms are defined by:

$$\mathrm{Env}(\mathcal{O})(m, n) := \bigotimes_{m_1 + \cdots + m_n = m} (\mathcal{O}(m_1) \otimes \cdots \otimes \mathcal{O}(m_n)) \underset{\Sigma_{m_1} \times \cdots \times \Sigma_{m_n}}{\otimes} \Sigma_m, \tag{2.5}$$

where the coproduct runs over all partitions of m into n parts. The Eq (2.5) is a coproduct of coequalisers where the $\Sigma_{m_1} \times \ldots \times \Sigma_{m_n}$-action on Σ_m is given by left multiplication on the image $\Sigma_{m_1} \times \ldots \times \Sigma_{m_n}$ in Σ_m and right Σ_{m_i}-action on $\mathcal{O}(m_i)$.[1]

A morphism in $\mathrm{Env}(\mathcal{O})(m, n)$ is thus represented by a pair

$$((f_1, \ldots, f_n), \sigma),$$

---

[1] The notation

$$X \underset{H}{\otimes} Y := \mathrm{coeq}\left(\coprod_{h \in H} X \otimes Y \underset{\mathrm{id} \otimes h}{\overset{h \otimes \mathrm{id}}{\rightrightarrows}} X \otimes Y \right)$$

denotes the coequalizer that enforces H-equivariance on the tensor product.

where each $f_i \in \mathcal{O}(m_i)$, the m_i sum to m, and $\sigma \in \Sigma_m$ acts by permuting the inputs. For any two morphisms $g \in \mathrm{Env}(\mathcal{O})(m;n)$ and $f \in \mathrm{Env}(\mathcal{O})(p;q)$, their monoidal product $g \boxtimes h \in \mathrm{Env}(\mathcal{O})(m+p;n+q)$ is defined by concatenation:

$$(f_1, \dots, f_n, g_1, \dots, g_q; \sigma \oplus \sigma') \in \mathrm{Env}(\mathcal{O})(m+p, n+q),$$

where $\sigma \oplus \sigma'$ is the image under the canonical embedding $\Sigma_m \times \Sigma_p \hookrightarrow \Sigma_{m+p}$. See, for example, [Phi22] for more details on this construction.

This construction defines the left adjoint in an adjunction between operads and props:

$$\begin{tikzcd} \mathbf{Op}(\mathbf{E}) \arrow[r, shift left, "\mathrm{Env}"] & \mathbf{Prop}(\mathbf{E}) \arrow[l, shift left, "u"] \end{tikzcd} \tag{2.6}$$

Here, the right adjoint u sends a prop to its underlying operad by extracting the operations with one output: $u(\mathcal{P})(n) := \mathcal{P}(n,1)$. See, for example, [HV02, 2.6.1] or [HR15, Proposition 11] for more details.

An *algebra over a prop* $\mathcal{P}$ in a symmetric monoidal category $\mathbf{E}$ is a symmetric monoidal functor $F : \mathcal{P} \to \mathbf{E}$ such that $F(1) = A$. The object A is then said to carry a $\mathcal{P}$-algebra structure. The category of such algebras is denoted $\mathrm{Alg}_{\mathbf{E}}(\mathcal{P})$. If the prop in question is generated by an operad, the notion of algebras coincide in the following sense.

Proposition 2.11. *Let $\mathcal{O}$ be an operad in $\mathbf{E}$, and let $\mathrm{Env}(\mathcal{O})$ be its envelope. Then there is a natural isomorphism of categories:*

$$\mathrm{Alg}_{\mathbf{E}}(\mathcal{O}) \cong \mathrm{Alg}_{\mathbf{E}}(\mathrm{Env}(\mathcal{O})).$$

2.4. Metric props and the envelope of a cyclic operad.

To encode invariant bilinear forms at the level of props, one can extend the envelope construction using data from a cyclic operad. The resulting object is called a *metric prop* in [HV02, 3.1.3].

Definition 2.12. Let $\mathcal{O}^{cyc}$ be a cyclic operad whose underlying operad is $\mathcal{O}$. The *metric prop* associated to $\mathcal{O}^{cyc}$, $\Pi(\mathcal{O}^{cyc})$ is the prop $\mathrm{Env}(\mathcal{O})$ together with two additional generating morphisms

$$d \in \Pi(\mathcal{O}^{cyc})(2,0) \quad \text{and} \quad b \in \Pi(\mathcal{O}^{cyc})(0,2)$$

for which the composites

$$1 \cong 0 \otimes 1 \xrightarrow{b \otimes \mathrm{id}_1} 1 \otimes 1 \otimes 1 \xrightarrow{\mathrm{id}_1 \otimes d} 1 \otimes 0 \cong 1 \tag{2.7}$$

and

$$1 \cong 1 \otimes 0 \xrightarrow{\mathrm{id}_1 \otimes b} 1 \otimes 1 \otimes 1 \xrightarrow{d \otimes \mathrm{id}_1} 0 \otimes 1 \cong 1 \tag{2.8}$$

compose to the identity map in $\Pi(\mathcal{O}^{cyc})(1,1)$. Moreover, we require that for any operation $f \in \mathcal{O}(n)$, the composite

$$n \cong 0 \otimes n \xrightarrow{b \otimes \mathrm{id}_n} 1 \otimes 1 \otimes n = 1 \otimes n \otimes 1 \xrightarrow{\mathrm{id}_1 \otimes f \otimes \mathrm{id}_1} 1 \otimes 1 \otimes 1 \xrightarrow{\mathrm{id}_1 \otimes d} 1 \otimes 0 \cong 1 \tag{2.9}$$

agrees with the cyclic action $z^*(f)$ in $\mathcal{O}^{cyc}(n)$.

An *algebra over the metric prop* $\Pi(\mathcal{O}^{\mathrm{cyc}})$ is a symmetric monoidal functor $F : \Pi(\mathcal{O}^{\mathrm{cyc}}) \to \mathbf{E}$ such that $F(1) = A$ and the images

$$F(d) : A \otimes A \to \mathbb{1} \quad \text{and} \quad F(b) : \mathbb{1} \to A \otimes A$$

define a non-degenerate symmetric bilinear form on A. The category of such algebras is denoted $\mathrm{Alg}_{\mathbf{E}}(\Pi(\mathcal{O}^{\mathrm{cyc}}))$.

Lemma 2.13 ([HV02], Lemma 3.1.4)**.** *Let $\mathcal{O}^{\mathrm{cyc}}$ be a cyclic operad in $\mathbf{E}$. Then there is an equivalence of categories:*

$$\mathrm{Alg}_{\mathbf{E}}(\Pi(\mathcal{O}^{\mathrm{cyc}})) \cong \mathrm{Alg}_{\mathbf{E}}(\mathcal{O}^{\mathrm{cyc}}).$$

2.5. **Completion of Operads and Props.** Let $\mathbb{K}$ be a field. For a group G, its prounipotent completion over $\mathbb{K}$ is a prounipotent group $\widehat{\mathsf{G}}_{\mathbb{K}}$ equipped with a group homomorphism $\mathsf{G} \to \widehat{\mathsf{G}}_{\mathbb{K}}$ such that for any prounipotent group $\widehat{\mathsf{H}}_{\mathbb{K}}$ and a homomorphism $\mathsf{G} \to \widehat{\mathsf{H}}_{\mathbb{K}}$, there exists a unique homomorphism $\widehat{\mathsf{G}}_{\mathbb{K}} \to \widehat{\mathsf{H}}_{\mathbb{K}}$ such that $\mathsf{G} \to \widehat{\mathsf{G}}_{\mathbb{K}} \to \widehat{\mathsf{H}}_{\mathbb{K}}$. The prounipotent group $\widehat{\mathsf{G}}_{\mathbb{K}}$ is defined by the grouplike elements of the completed group algebra $\widehat{\mathbb{K}[\mathsf{G}]}$. Here, the completion $\widehat{\mathbb{K}[\mathsf{G}]}$ is given by taking the inverse limit of the filtration $\{\mathbb{K}[\mathsf{G}]/I^n\}_{n\geq 0}$,

$$\widehat{\mathbb{K}[\mathsf{G}]} := \lim_{n} \mathbb{K}[\mathsf{G}]/I^n,$$

where $I = \mathsf{Ker}\ (\mathbb{K}[\mathsf{G}] \longrightarrow \mathbb{K})$ is the augmentation ideal of $\mathbb{K}[\mathsf{G}]$.

The prounipotent completion of groups is generalised to groupoids and operads in groupoids. Moreover, the completion operation $\widehat{(-)} : \mathbf{Op}(\mathbf{Gr}) \to \mathbf{Op}(\widehat{\mathbf{Gr}})$ from the category of operads in groupoids to the category of prounipotent operads in groupoids is a *symmetric monoidal functor* ; see [Fre17, Proposition 9.2.2] for details.

Definition 2.14. For an operad $\mathcal{O} = \{\mathcal{O}(n)\}_{n\geq 0}$ in groupoids, the prounipotent completion of $\mathcal{O}$ is given by

$$\widehat{\mathcal{O}}_{\mathbb{K}} = \{\widehat{\mathcal{O}}_{\mathbb{K}}(n)\}_{n\geq 0},$$

where $\widehat{\mathcal{O}}$ is obtained by applying the functor $\widehat{\mathbf{gr}}_{\mathbb{K}} : \mathbf{Gr} \to \widehat{\mathbf{Gr}}_{\mathbb{K}}$ in each arity and $\widehat{\mathcal{O}}_{\mathbb{K}}$ is the prounipotent groupoid.

Definition 2.15. For a cyclic operad $\mathcal{O}^{cyc} = \{\mathcal{O}^{cyc}(n)\}_{n\geq 0}$ in groupoids, the prounipotent completion of $\mathcal{O}^{cyc}$ is given by

$$\widehat{\mathcal{O}^{cyc}}_{\mathbb{K}} = \{\widehat{\mathcal{O}^{cyc}}_{\mathbb{K}}(n)\}_{n\geq 0} = \{\widehat{\mathcal{O}}^{cyc}_{\mathbb{K}}(n)\}_{n\geq 0} = \widehat{\mathcal{O}}^{cyc}_{\mathbb{K}},$$

where $\widehat{\mathcal{O}}$ is obtained by applying the functor $\widehat{\mathbf{gr}}_{\mathbb{K}} : \mathbf{Gr} \to \widehat{\mathbf{Gr}}_{\mathbb{K}}$ in each arity and $\widehat{\mathcal{O}}^{cyc}_{\mathbb{K}}$ is the prounipotent groupoid $\widehat{\mathcal{O}}_{\mathbb{K}}$ togetherwith the cyclic structure of $\mathcal{O}$.

Example 2.16. For the braid group B_n, its prounipotent completion is the inverse limit of the filtration $\{\mathbb{K}[\mathsf{B}_n]/I^n\}_{n\geq 0}$ of the group algebra $\mathbb{K}[\mathsf{B}_n]$ with respect to the two-sided ideal $I = \langle \beta_i - \beta_i^{-1}\rangle$ generated by $\beta_i - \beta_i^{-1}$ for $0 \leq i \leq n-1$, where β_i are the generators of B_n.

Example 2.17. Let F_2 be a free group of rank 2 generated by the variables x and y. Its prounipotent completion $\widehat{\mathsf{F}}_2$ over the field $\mathbb{K}$ is $\exp(\mathfrak{lie}_2)$, where $\mathfrak{lie}_2$ is the complete filtered Lie algebra over $\mathbb{K}$ generated by the set $\{\log x, \log y\}$.

Notation 2.18. We write $\widehat{\mathsf{G}}$ for the prounipotent completion of a group G over the field $\mathbb{K}$ if the underlying field $\mathbb{K}$ understood from the context.

The completion functor $\widehat{(-)} : \mathbf{Op}(\mathbf{Gr}) \to \mathbf{Op}(\widehat{\mathbf{Gr}})$ is symmetric monoidal implies the following proposition.

Proposition 2.19. *Prounipotent completion of the prop associated to an operad is isomorphic to the prop associated to the prounipotent completion of an operad, i.e.*

$$\widehat{\mathrm{Env}(\mathcal{O})} \cong \mathrm{Env}(\widehat{\mathcal{O}}).$$

Proof. The prounipotent completion of an operad is given by the application of the functor $\widehat{(-)} : \mathbf{Op}(\mathbf{Gr}) \to \mathbf{Op}(\widehat{\mathbf{Gr}})$ in each arity, that is, $\widehat{\mathcal{O}} = \{\widehat{\mathcal{O}}(n)\}$. Since the envelope construction is defined via iterated tensor products,

$$\mathrm{Env}(\mathcal{O})(m;n) := \bigotimes_{\Sigma m_i = m} \mathcal{O}(m_1) \otimes \ldots \otimes \mathcal{O}(m_n) \otimes_{\Sigma_{m_1}\times\ldots\times\Sigma_{m_n}} \Sigma_m,$$

it follows that

$$\widehat{\mathrm{Env}(\mathcal{O})} = \widehat{\bigotimes_{\Sigma m_i = m} \mathcal{O}(m_1) \otimes \ldots \otimes \mathcal{O}(m_n) \otimes_{\Sigma_{m_1}\times\ldots\times\Sigma_{m_n}} \Sigma_m} = \bigotimes_{\Sigma m_i = m} \widehat{\mathcal{O}}(m_1) \otimes \ldots \otimes \widehat{\mathcal{O}}(m_n) \otimes_{\Sigma_{m_1}\times\ldots\times\Sigma_{m_n}} \Sigma_m$$

and thus $\widehat{\mathrm{Env}(\mathcal{O})} \cong \mathrm{Env}(\widehat{\mathcal{O}})$ as required. □

The metric prop associated with the cyclic operad $\mathcal{O}^{cyc}$ is obtained from $\mathrm{Env}(\mathcal{O})$ by formally adjoining dualizing morphisms to the prop $\mathrm{Env}(\mathcal{O})$. The following proposition also follows from the fact that prounipotent completion is a symmetric monoidal functor.

Proposition 2.20. *The prounipotent completion of the metric prop associated with a cyclic operad is isomorphic to the metric prop associated to the prounipotent completion of a cyclic operad, i.e.*

$$\widehat{\Pi(\mathcal{O}^{cyc})} \cong \Pi(\widehat{\mathcal{O}}^{cyc}).$$

□

3. Cyclic operad of ribbon braids

The operad FD_2 of framed little disks is known to admit a cyclic structure [Bud08]. It was later shown by Müller and Woike [MW22] that its groupoid model $\pi(\mathsf{FD}_2) \simeq \mathsf{CoRB}$, the operad of ribbon braids, admits a cyclic structure. It was pointed out by Campos, Idrissi, and Willwacher [CIW19] that the operad of parenthesized ribbon operads PaRB also admits a cyclic structure. Specifically, cyclic actions [2] on the associativity, braiding, and twisting morphisms were defined. In this section, we explicitly prove that the operad of parenthesized ribbon braids is a cyclic operad using the structure described in [CIW19].

3.1. Braids and ribbon braids.

The braid group on n-strands, hereafter denoted by B_n, is the fundamental group of the space of unordered configurations of n points in the complex plane. This group has a preferred presentation

$$\mathsf{B}_n = \langle \beta_1, \dots, \beta_{n-1} \mid \beta_i\beta_j = \beta_j\beta_i \text{ when } |i-j| \geq 2;\ \beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1} \rangle .$$

For any $n \geq 1$, we have the short exact sequence

$$1 \longrightarrow \mathsf{PB}_n \longrightarrow \mathsf{B}_n \xrightarrow{\pi} \Sigma_n \longrightarrow 1$$

where the map π sends a braid generator β_i to the relevant transposition $(i\ i+1)$ in Σ_n. The kernel of π, PB_n, is called the *pure braid group* on n-strands. The pure braid groups are generated by elements

$$x_{ij} = \beta_{j-1}\dots\beta_{i+1}\beta_i^2\beta_{i+1}^{-1}\dots\beta_{j-1}^{-1}, \text{ for } 1 \leq i < j \leq n.$$

The *ribbon braid group* on n-strands, denoted RB_n, is the fundamental group of the space of unordered configurations in the plane where each point is equipped with a label in S^1. This group is generated by the braid generators $\beta_1, \dots, \beta_{n-1}$ as well as *twists* $\tau_1, \dots, \tau_n$ subject to the braid relations

$$\beta_i\beta_j = \beta_j\beta_i \text{ when } |i-j| \geq 2;\ \beta_i\beta_{i+1}\beta_i = \beta_{i+1}\beta_i\beta_{i+1}$$

and

$$\beta_i\tau_j = \tau_j\beta_i\ j \notin \{i, i+1\},\ \beta_i\tau_{i+1} = \tau_i\beta_i,\ \tau_i\tau_j = \tau_j\tau_i\ i \neq j.$$

The group of *pure ribbon braids*, denoted PRB_n, is the kernel of the projection $\mathsf{RB}_n \to \Sigma_n$.

3.2. The operad of parenthesized ribbon braids.

One can consider the ribbon braid groups as groupoids whose objects are elements of the underlying symmetric group.

Definition 3.1. For $n \geq 1$, we define a symmetric sequence of groupoids $\mathsf{CoRB}(n)$ with:

- $\mathrm{ob}(\mathsf{CoRB}(n)) = \Sigma_n$;
- morphisms $\mathrm{Hom}_{\mathsf{CoRB}(n)}(\sigma_1, \sigma_2)$ are elements of the ribbon braid group RB_n whose underlying permutation is $\sigma_2\sigma_1^{-1}$.

Categorical composition is given by the multiplication of ribbon braids. Using the group isomorphism, $\mathsf{RB}_n \cong \mathsf{B}_n \rtimes \mathbb{Z}^{\times n}$, we can write any morphism $r \in \mathrm{Hom}_{\mathsf{CoRB}(n)}(\sigma_1, \sigma_2)$ as a pair $r = (\gamma, [\tau_1, \dots, \tau_n])$ where γ is a braid on n-strands with underlying permutation $\pi(\gamma) = \sigma_2\sigma_1^{-1} \in \Sigma_n$. The tuple $[\tau_1, \dots, \tau_n] \in \mathbb{Z}^{\times n}$ is the data of the "twists", where $\tau_i \in \mathbb{Z}$ is the number of twists of the ith ribbon strand.

---

[2] cyclic action on braiding and twisting morphisms defined by [CIW19] coincide with the one given by [MW22]

The symmetric group Σ_n acts naturally on the objects of each groupoid CoRB_n and the collection $\mathsf{CoRB} = \{\mathsf{CoRB}_n\}_{n\geq 1}$ forms a symmetric operad in the category of groupoids. The operadic composition functors

$$\mathsf{CoRB}(n) \times \mathsf{CoRB}(m) \xrightarrow{\circ_i} \mathsf{CoRB}(n+m-1)$$

are defined at the level of objects, as in the associative operad Σ (eg [BdBHR19, Definition 6.1]). Informally, operadic composition of morphisms is given by "cabling", inserting a ribbon braid from RB_m into the kth strand of a ribbon braid in RB_n. To be more explicit, we first introduce some notion: for an integer $m \geq 0$, we write $\mathbf{R}_m$ for the mth full twist $(\beta_1 \cdots \beta_{m-1})^m$ in PB_m. Given morphisms $r_1 = (\gamma_1, [\tau_1, \dots, \tau_n]) \in \mathsf{CoRB}(n)$ and $r_2 = (\gamma_2, [\tau_1', \dots, \tau_m']) \in \mathsf{CoRB}(m)$ then

$$r_1 \circ_1 r_2 := (\omega, [\tau_1, \dots, \tau_{i-1}, \tau_i + \tau_1', \dots, \tau_i + \tau_m', \tau_{i+1}, \dots \tau_n]) \in \mathsf{CoRB}(n+m-1)$$

where $\omega := \gamma_1 \circ_i ((\mathbf{R}_m)^{\tau_i} \cdot \gamma_2)$ and $(\mathbf{R}_m)^{x_k}$ is the x_k-fold categorical composition of $\mathbf{R}_m$ considered as an element in the braid group B_n. See, for example, [BdBHR19, Definition 6.4] and [Wah01, Section 1.5] for more details.

Definition 3.2. The *magma operad* $\Omega = \{\Omega(n)\}_{n\geq 1}$ is the free operad in **Set** generated by a binary operation $\mu(x_1, x_2) = x_1 x_2 \in \Omega(2)$. As such, the set $\Omega(n)$ is the set of all planar, binary, rooted trees with n leaves, labelled by $\{1, \dots, n\}$. Here, the term binary means that every vertex in our rooted tree has two inputs and one output. Operadic composition

$$\Omega(n) \times \Omega(m) \xrightarrow{\circ_i} \Omega(n+m-1)$$

is given by grafting the root of a tree $T' \in \Omega(m)$ to the ith leaf of a tree $T \in \Omega(n)$.

Definition 3.3. For $n \geq 1$, we define a sequence of categories $\mathsf{PaRB}(n)$ with:

- $\mathrm{ob}(\mathsf{PaRB}(n)) = \Omega(n)$;
- morphisms $\mathrm{Hom}_{\mathsf{PaRB}(n)}(p_1, p_2) := \mathrm{Hom}_{\mathsf{CoRB}(n)}(u(p_1), u(p_2))$ are elements of the ribbon braid group RB_n whose underlying permutation is $u(p_2)u(p_1)^{-1}$.

The collection $\mathsf{PaRB} = \{\mathsf{PaRB}(n)\}_{n\geq 1}$ forms an operad in groupoids where composition functors

$$\mathsf{PaRB}(n) \times \mathsf{PaRB}(m) \xrightarrow{\circ_i} \mathsf{PaRB}(n+m-1)$$

are defined on objects as in the operad Ω. On morphisms, the operad composition is defined as in CoRB.

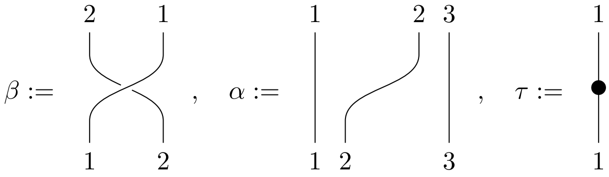


FIGURE 3. Generating morphisms braiding β, associator α and twist τ of PaRB

At the level of objects, the operad PaRB is the free operad generated by a single binary operation $\mu \in \mathrm{ob}(\mathsf{PaRB}(2))$ (Definition 3.2). Operations in the operad PaRB can all be presented as operadic and categorical compositions of the ribbon braid generators, $\beta \in \mathsf{B}_2$ and $\tau \in \mathsf{RB}_1 \cong \mathbb{Z}$ and an isomorphism $\alpha : \mu \circ_1 \mu \to \mu \circ_2 \mu$ called the *associator* [Fre17]. Here $\beta \in \mathsf{B}_2$ and $\tau \in \mathsf{RB}_1$ are seen as isomorphisms in $\mathrm{Hom}_{\mathsf{PaRB}(2)}(12, 21)$ and $\mathrm{Hom}_{\mathsf{PaRB}(1)}(1,1)$, respectively, and α is an identity element of RB_3. Thus, it is not present in the list of generators of RB_n, but shows up as a generating morphism of PaRB when viewed as isomorphisms in $\mathrm{Hom}_{\mathsf{PaRB}(3)}((12)3, 1(23))$ (see [BdBHR19]). As a consequence, maps out of the operad PaRB admit a particularly simple presentation, summarised in the following theorem.

Theorem 3.4 ([BdBHR19, Lemma 7.4],)**.** *Let $\mathcal{Q}$ be any operad in groupoids. There is a bijection between the set of operad maps $f : \mathsf{PaRB} \to \mathcal{Q}$ and the set of values*

$$f(\mu) = \mathtt{m} \in \mathrm{ob}(\mathcal{Q}(2)), \quad f(\tau) = \mathtt{t} \in \mathcal{Q}(1), \quad f(\beta) = \mathtt{b} \in \mathcal{Q}(2) \quad \textit{and} \quad f(\alpha) = \mathtt{a} \in \mathcal{Q}(3)$$

which satisfy the following equations:

$$(\mathrm{T}) \qquad (\mathtt{t} \circ_1 1_{\mathtt{m}}) = \mathtt{b} \cdot (21)^* \mathtt{b} \cdot (1_{\mathtt{m}} \circ (\mathtt{t}, \mathtt{t})) \in \mathcal{Q}(2).$$

$$(\mathrm{H1}) \qquad (1_{\mathtt{m}} \circ_1 \mathtt{b}) \cdot ((213)^* \mathtt{a}) \cdot (213)^* (1_{\mathtt{m}} \circ_2 \mathtt{b}) = \mathtt{a} \cdot (\mathtt{b} \circ_2 1_{\mathtt{m}}) \cdot (231)^* \mathtt{a} \in \mathcal{Q}(3).$$

$$(\mathrm{H2}) \qquad (1_{\mathtt{m}} \circ_2 \mathtt{b}) \cdot (132)^* \mathtt{a}^{-1} \cdot (132)^* (1_{\mathtt{m}} \circ_1 \mathtt{b}) = \mathtt{a}^{-1} \cdot (\mathtt{b} \circ_1 1_{\mathtt{m}}) \cdot (312)^* \mathtt{a}^{-1} \in \mathcal{Q}(3).$$

$$(\mathrm{P}) \qquad (\mathtt{a} \circ_1 1_{\mathtt{m}}) \cdot (\mathtt{a} \circ_3 1_{\mathtt{m}}) = (1_{\mathtt{m}} \circ_1 \mathtt{a}) \cdot (\mathtt{a} \circ_2 1_{\mathtt{m}}) \cdot (1_{\mathtt{m}} \circ_2 \mathtt{a}) \in \mathcal{Q}(4).$$

□

The operad PaRB can be equivalently defined as an operadic pullback of CoRB along the map $\omega : \Omega \to \mathrm{ob}(\mathsf{CoRB})$. In particular, $\mathsf{PaRB} = \omega^*(\mathsf{CoRB})$. It is isomorphic to the restriction $\pi(\mathsf{FD}_2)_{|\Omega}$ of the fundamental groupoid of FD_2 to the suboperad Ω via the disk centering map. Moreover, we have a zigzag of categorical equivalence.

$$\mathsf{CoRB} \xleftarrow{\sim} \mathsf{PaRB} \cong \pi_0(\mathsf{FD}_2)_{|\Omega} \xrightarrow{\sim} \pi_0(\mathsf{FD}_2).$$

3.3. **The cyclic operad of ribbon braids.** There is a natural forgetful functor $u : \mathbf{Cyc}(\mathbf{E}) \to \mathbf{Op}(\mathbf{E})$ from cyclic operads to operads, which admits both a left and a right adjoint:

$$\begin{tikzcd} \mathbf{Cyc}(\mathbf{E}) \arrow[r, "u"] & \mathbf{Op}(\mathbf{E}), \arrow[l, bend right, "u_!"'] \arrow[l, bend left, "u_*"] \end{tikzcd}$$

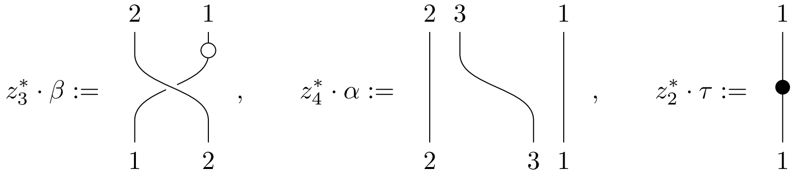
These adjunctions provide formal methods for equipping an operad with a cyclic structure, and could in principle be applied to the operad PaRB of parenthesized ribbon braids. However, the cyclic structure we consider is not the one produced by either of these universal constructions. Instead, our approach is more geometric and is motivated by an involutive symmetry of the configuration spaces underlying the framed little discs operad.

The operad PaRB provides a groupoid model for the operad of framed little 2-discs, in the sense that $\pi(\mathsf{FE}_2(n)) \simeq \mathsf{PaRB}(n)$. It is known that the framed little discs operad admits a cyclic structure, first observed in [Bud08], and that this structure can be transferred to the groupoid level, as discussed in [CIW19, Section 3.1] and [MW22]. While cyclic structures on PaRB have thus been previously constructed, we provide here a detailed verification of the cyclic operad axioms as we required some of these details for later constructions and they were not fully described in the literature.

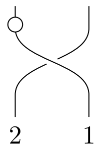
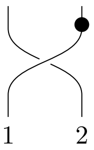
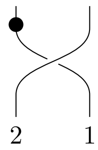
Definition 3.5. At the level of morphisms, it is sufficient to define the cyclic structure on the generating morphisms as

$$z_2^*(\tau) = \tau \in \mathrm{Hom}_{\mathsf{PaRB}(1)}(1,1), \quad z_3^*(\beta) = (\beta^{-1})(\mathrm{id}_2 \circ_2 \tau^{-1}) \in \mathrm{Hom}_{\mathsf{PaRB}(2)}(12, 21),$$
$$\text{and} \quad z_4^*(\alpha) = (231)^* \alpha^{-1} \in \mathrm{Hom}_{\mathsf{PaRB}(3)}((12)3, 1(23)).$$

The cyclic action z^* on the braiding β, associator α and twist τ, of the generating morphisms of PaRB can be translated as follows:



Moreover, the cyclic action on the braiding $\beta : \mu(x_1, x_2) \longrightarrow \mu(x_2, x_1)$ determines the action on β^{-1}, $(21)^*\beta$ and $(21)^*\beta^{-1}$ as in [MW22, Section 5].

$z_3^* \cdot (21)^*\beta :=$  , $z_3^* \cdot \beta^{-1} :=$  , $z_3^* \cdot (21)^*\beta^{-1} :=$ 

Here, the symbol ● on a ribbon strand represent the positive twist and ○ represents a negative twist.

Lemma 3.6. *The cyclic action z_4^* on the associator $\alpha : \mu \circ_1 \mu \longrightarrow \mu \circ_2 \mu$ is a Σ_4-action that extends the Σ_3-action.*

Proof. We rewrite the action $z_4^* \cdot \alpha = (231)^*\alpha^{-1}$ as $z_4^* \cdot \alpha_{1,2,3} = \alpha^{-1}_{2,3,1}$ to facilitate computation and notation. The action z_4^* on $\alpha_{1,2,3}$ can be understood by flipping the labels 0 and 1 on the boundary elements of the tree, whose root is always marked 0 as in Figure 4.

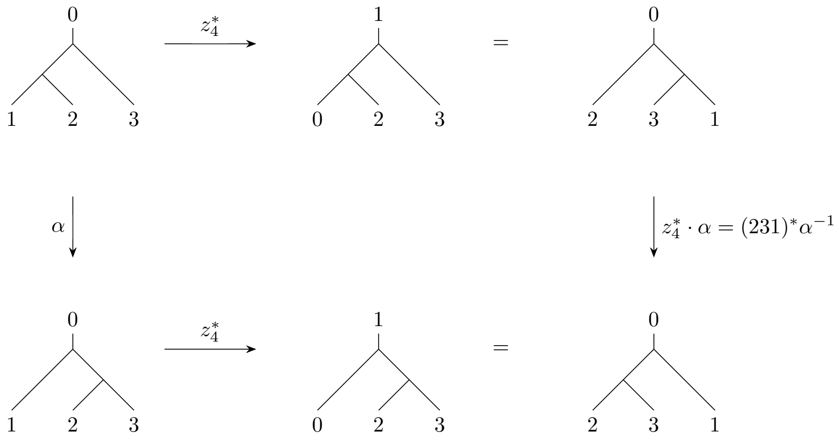


Figure 4. Cyclic action on associator $\alpha : \mu \circ_1 \mu \longrightarrow \mu \circ_2 \mu$

Similarly, we have $z_4^* \cdot \alpha^{-1} = (231)^*\alpha$ and $(z_4^*)^2 \cdot \alpha^{-1} = z_4^*((231)^*\alpha^{-1}) = \alpha^{-1}$, therefore $(z_4^*)^4 \cdot \alpha = \alpha$. We denote $\sigma_{2,1}$ for the transposition $(21)^*$. To check the cyclic action is compatible with the symmetric group action, we first check that the following relations in Σ_4 applied on α holds:

(1) $(z_4^*)^4 = 1$.

(2) $z_4^*\sigma_{2,1}z_4\sigma_{2,1} = \sigma_{2,1}(z_4^*)^3$.

(3) $\sigma_{2,1}(z_4^*)^2\sigma_{2,1}(z_4^*)^2 = (z_4^*)^2\sigma_{2,1}(z_4^*)^2\sigma_{2,1}$.

The relation (1) holds due to ${z_4^*}^2 \cdot \alpha = \alpha$. For (2), $z_4^*\sigma_{2,1}\alpha = z_4^*(\alpha_{2,1,3}) = \alpha_{3,1,2}$ and then,

$$z_4^*\sigma_{2,1}\alpha_{3,1,2} = z_4^*(\alpha_{3,2,1}) = \alpha^{-1}_{1,3,2}$$

and, $\sigma_{2,1}(z_4^*)^3(\alpha) = \sigma_{2,1}(z_4^*)(\alpha) = \sigma_{2,1}(\alpha^{-1}_{2,3,1}) = \alpha^{-1}_{1,3,2}$. For the last relation (3), we use the action $z_4^*(\alpha_{3,1,2}) = \alpha_{2,1,3}$. Now, one can easily see the action $\sigma_{2,1}(z_4^*)^2(\alpha) = \alpha_{2,1,3}$, and we finally have,

$$\sigma_{2,1}(z_4^*)^2(\alpha_{2,1,3}) = \sigma_{2,1}(z_4^*)(\alpha_{3,1,2}) = \sigma_{2,1}(\alpha_{2,1,3}) = \alpha_{1,2,3}$$
$$(z_4^*)^2\sigma_{2,1}(z_4^*)^2\sigma_{2,1}(\alpha) = (z_4^*)^2\sigma_{2,1}(z_4^*)^2(\alpha_{2,1,3}) = \sigma_{2,1}(\alpha_{2,1,3}) = \alpha_{1,2,3}$$

We need to check the relation $z_3^*\sigma_{2,1} = \sigma_{2,1}(z_3^*)^2$ from Σ_3 on α to claim that the action z_4^* is a Σ_4-action on associator that extends Σ_3-action:

$$z_3^*\sigma_{2,1}(\alpha) = z_3^*\alpha_{2,1,3} = \alpha_{3,2,1}.$$
$$\sigma_{2,1}(z_3^*)^2(\alpha) = \sigma_{2,1}(z_3^*)\alpha_{2,3,1} = \sigma_{2,1}\alpha_{3,1,2} = \alpha_{3,2,1}.$$

□

Lemma 3.7. *Let z_2^* denote the non-trivial element in Σ_1^+. The z^*-actions on the generating morphisms in Definition 3.5 define a cyclic structure on the operad* PaRB.

Proof. The cyclic action on generators extends to the cyclic structure of PaRB via the formula in(2.3). It is enough to check the compatibility with the pentagon Eq(P), the hexagons Eq(H1) and Eq(H2). The proof follows the strategy of [MW22, Proposition 5.2], with parenthesization. The pentagon equation is given by

$$(\alpha \circ_1 \mathrm{id}_2)\cdot(\alpha \circ_3 \mathrm{id}_2) = (\mathrm{id}_2 \circ_1 \alpha)\cdot(\alpha \circ_2 \mathrm{id}_2)\cdot(\mathrm{id}_2 \circ_2 \alpha) \in \mathrm{Hom}_{\mathsf{PaRB}}(((12)3)4, 1(2(34))). \tag{P}$$

We first apply the cyclic action z_4 on the morphism $(\alpha \circ_1 \mathrm{id}_2) : (\mu \circ_1 \mu)\circ_1 \mu \longrightarrow \mu \circ (\mu, \mu)$, which amounts to applying the action on $(\mu \circ_1 \mu)\circ_1 \mu$ and $\mu \circ (\mu, \mu)$ then rewrite the resulting morphisms in terms of operadic composition of morphism in PaRB. We obtain $z_4^* \cdot (\alpha \circ_1 \mathrm{id}_2) = (2341)^*(\mathrm{id}_2 \circ_2 \alpha^{-1})$, similar to Figure 4. Similarly, one can compute the following action on the components of Eq(P);

$$\begin{aligned} z_4^*\cdot(\alpha \circ_3 \mathrm{id}_2) &= (2341)^*(\alpha^{-1} \circ_2 \mathrm{id}_2), & z_4^*\cdot(\mathrm{id}_2 \circ_1 \alpha) &= (2341)^*(\alpha^{-1} \circ_3 \mathrm{id}_2),\\ z_4^*\cdot(\alpha \circ_2 \mathrm{id}_2) &= (2341)^*(\alpha^{-1} \circ_1 \mathrm{id}_2), & z_4^*\cdot(\mathrm{id}_2 \circ_2 \alpha) &= (2341)^*(\mathrm{id}_2 \circ_1 \alpha), \end{aligned}$$

It is immediate from Figure 5 that that action is preserved when applied on Eq(P). The hexagon equations are

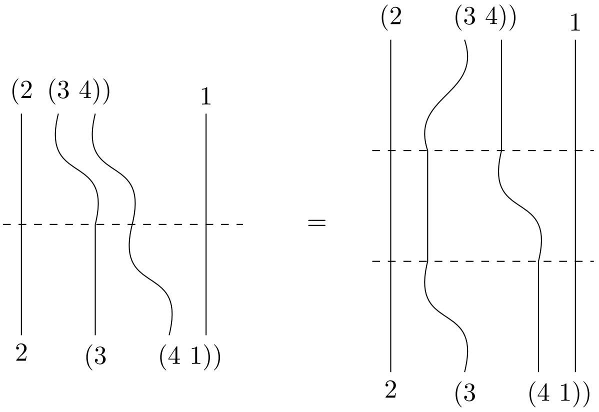


Figure 5. Cyclic action on Pentagon equation P

$$\alpha\cdot(\beta \circ_2 \mathrm{id}_2)\cdot(231)^*\alpha = (\mathrm{id}_2 \circ_1 \beta)\cdot(213)^*)\alpha\cdot(213)^*)(\mathrm{id}_2 \circ_2 \beta) \in \mathrm{Hom}_{\mathsf{PaRB}}((12)3, 2(31)). \tag{H1}$$

$$\alpha^{-1}\cdot(\beta \circ_1 \mathrm{id}_2)\cdot(312)^*\alpha^{-1} = (\mathrm{id}_2 \circ_2 \beta)\cdot(132)^*)\alpha^{-1}\cdot(132)^*)(\mathrm{id}_2 \circ_1 \beta) \in \mathrm{Hom}_{\mathsf{PaRB}}(1(23), (31)2). \tag{H2}$$

We use the relation $z^*(213)^* = (213)^*(z^*)^2$ to compute the action applied on the left side of Eq(H1) reduces to the following:

$$(z_4^*\cdot\alpha)\cdot(z_4^*\cdot(\beta \circ_2 \mathrm{id}_2))\cdot(z_4^*\cdot((231)^*\alpha)) = (231)^*(\alpha^{-1}\cdot(\beta^{-1}(\mathrm{id}_2 \circ_1 \tau^{-1})\circ_1 \mathrm{id}_2)\cdot\alpha)$$

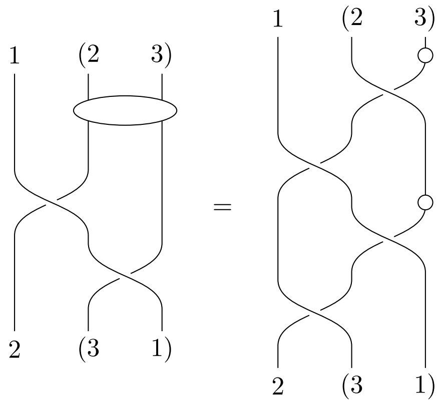
It can be written equivalent to $(231)^*\beta_2^{-1}\beta_1^{-1}\beta_2^{-1}\beta_2^{-1}\tau_2^{-1}\tau_3^{-1}$ as an element of $\mathrm{Hom}_{\mathsf{PaRB}}(2(31), 1(23))$. Moreover, the relation applied on $(213)^*((z_4^*)^2\cdot((\mathrm{id}_2 \circ_2 \beta))$ gives

$$((213)^* z_4^*\cdot((231)^*(\mathrm{id}_2 \circ_1 \beta)) = (312)^*\beta_1^{-1}\beta_2^{-1}\tau_3^{-1}. \tag{3.1}$$

Using the last equation, the right side of Eq(H1) is the straightforward, and it is given by

$$z_4^*\cdot((\mathrm{id}_2 \circ_1 \beta)\cdot(213)^*\alpha\cdot(213)^*(\mathrm{id}_2 \circ_2 \beta)) = (231)^*\beta_1^{-1}\beta_2^{-1}\tau_3^{-1}\beta_1^{-1}\beta_2^{-1}\tau_3^{-1} \in \mathrm{Hom}_{\mathsf{PaRB}}(2(31), 1(23)).$$

Finally, the relation Eq(H1) upon the cyclic action can be translated as below, identifying both sides.



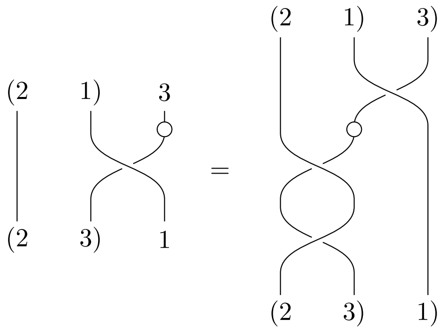
Similarly, we obtain the left side of Eq(H2) as follows

$$(z_4^* \cdot \alpha^{-1}) \cdot (z_4^* \cdot (\beta \circ_1 \mathrm{id}_2)) \cdot (z_4^* \cdot (312)^* \alpha^{-1}) = (231)^* \alpha \cdot (231)^* (\mathrm{id}_2 \circ_2 (\beta^{-1} (\mathrm{id}_2 \circ_2 \tau^{-1})) \cdot (213)^* \alpha^{-1}.$$

It is equivalent to $(231)^* \beta_2^{-1} \tau_2^{-1} \in \mathrm{Hom}_{\mathsf{PaRB}}((23)1, (21)3)$. Using the relation $z_4^*(132)^* = (213)^* z_4^*$, the right side of Eq(H2) reduces to

$$z_4^* \cdot ((\mathrm{id}_2 \circ_2 \beta) \cdot (132)^* \alpha^{-1} \cdot (132)^* (\mathrm{id}_2 \circ_1 \beta)) = (231)^* (\mathrm{id}_2 \circ_1 \beta) \cdot (321)^* \alpha \cdot (321)^* \left(\beta^{-1} (\mathrm{id}_2 \circ_2 \tau^{-1}) \circ_2 \mathrm{id}_2\right)$$

which, is equivalent to $(231)^* \beta_1 \beta_1^{-1} \tau_2^{-1} \beta_2^{-1} \in \mathrm{Hom}_{\mathsf{PaRB}}((23)1, (21)3)$. Finally, the relation Eq(H2) upon the cyclic action can be translated as below, identifying both sides.



At last, the cyclic compatibility with the ribbon relation $\tau \circ \mathrm{id}_2 = \beta \cdot (21)^* \beta \cdot (\mathrm{id}_2 \circ_1 \tau) \cdot (\mathrm{id}_2 \circ_2 \tau)$ in $\mathrm{Hom}_{\mathsf{PaRB}}((12),(12))$ follows immediate from [MW22, Lemma 5.1].

□

Remark 3.8. The cyclic action on the operad of colored braids CoRB and the operad of parenthesized braids PaRB differ by the action on the associator. In other words, the associator coincides with identity morphism on three strands in CoRB, which has trivial cyclic action. It is therefore required to check the compatibility with the pentagon equation in the case of PaRB, whereas the same is identity in CoRB. The last lemma lifts the cyclic action z^* of CoRB to its pullback $w^*\mathsf{CoRB} = \mathsf{PaRB}$ along the map $w^* : \Omega \longrightarrow \mathrm{ob}(\mathsf{CoRB})$, that is $w^*(z^* \cdot \mathsf{CoRB}) = z^* \cdot (w^* \mathsf{CoRB})$.

4. CYCLIC OPERAD OF CHORD DIAGRAMS

In [CIW19], the authors describe a cyclic structure on the operad of framed Drinfeld-Kohno lie algebra. In this section, we observe that the cyclic structure on the operad of ribbon chord diagrams lifts to the operad of parenthesised ribbon chord diagrams, similar to the lifting of the cyclic structure of CoRB to PaRB as in Section 3. This lifting was previously done by Willwacher [Wil24, Section 5.1].

4.1. **Drinfeld-Kohno Lie algebras.** There is a natural Lie algebra assigned to the prounipotent completion of a group G (see, for example, Section 8.3 of [Fre17]). As a well-known example, the Drinfeld-Kohno Lie algebras $\mathfrak{t}_n$ is the Lie algebras associated to the prounipotent pure braid groups PB_n.

Definition 4.1. The *framed Drinfeld-Kohno* Lie algebra $\mathfrak{ft}_n$ is a degree completed free Lie algebra generated by symbols $\{t_{ij} = t_{ji}, 1 \leq i, j \leq n\}$, with relations

$$\begin{aligned} [t_{ij}, t_{kl}] &= 0 \quad \text{for } \{i,j\} \cap \{k,l\} = \emptyset, \\ [t_{ij}, t_{ki} + t_{kj}] &= 0 \quad \text{for distinct } i,j,k, \\ [t_{ii}, t_{jk}] &= 0 \quad \text{for any } i,j,k. \end{aligned} \tag{4.1}$$

The framed Drinfeld-Kohno Lie algebra $\mathfrak{ft}_n$ is the Lie algebra associated to the prounipotent completion of the pure ribbon braid group PRB_n. The Lie algebras $\mathfrak{ft} = \{\mathfrak{ft}_n\}_{n\geq 0}$ form an operad in completed Lie algebras (cf. [Sev09, Section 1.3]) in which the operadic composition

$$\circ_k : \mathfrak{ft}_m \oplus \mathfrak{ft}_n \longrightarrow \mathfrak{ft}_{m+n-1}$$

is defined by:

$$0 \circ_k t_{ij} \mapsto t_{i+k-1j+k-1} \quad \text{for all } k,$$

$$t_{ij} \circ_k 0 \quad \mapsto \begin{cases} t_{i+n-1j+n-1} & \text{if } k < i < j, \\ \sum_{p=i}^{i+n-1} t_{pj+n-1} & \text{if } k = i < j, \\ t_{ij+n-1} & \text{if } i < k < j, \\ \sum_{q=j}^{j+n-1} t_{iq} & \text{if } i < k = j, \\ t_{ij} & \text{if } i < j < k. \end{cases} \quad \text{and} \quad t_{ii} \circ_k 0 \quad \mapsto \begin{cases} t_{i+n-1i+n-1} & \text{if } k < i, \\ \sum_{p=i}^{i+n-1} t_{pp} + \sum_{\{p,q\} \subset m} t_{pq} & \text{if } k = i, \\ t_{ii} & \text{if } i < k. \end{cases}.$$

One can assemble the Lie algebras $\mathfrak{ft}_n$ into a cyclic operad in Lie algebras ([Wil24, Section 5.1]). The universal enveloping algebra construction induces a functor

$$\mathbf{Op}(\mathfrak{lie}) \xrightarrow{\hat{U}(-)} \mathbf{Op}(\mathrm{Hopf})$$

taking (cyclic) operads in Lie algebras to (cyclic) operads in complete Hopf algebras.

Definition 4.2. For each $n \geq 1$, let $\widehat{\mathsf{RCD}}_{\mathbb{K}}(n)$ denote the groupoid with a single object, and

$$\mathrm{Hom}_{\widehat{\mathsf{RCD}}_{\mathbb{K}}(n)}(*, *) = \exp(\mathfrak{ft}_n).$$

The collection $\widehat{\mathsf{RCD}}_{\mathbb{K}} = \{\widehat{\mathsf{RCD}}_{\mathbb{K}}(n)\}$ forms an operad in complete Hopf algebras.

In complete Hopf algebras $\mathcal{H}$, the set of primitive elements is in bijection with the set of grouplike elements [Fre17, Proposition 8.1.5]. An element $g \in \mathcal{H}$ is group-like if it satisfies $\Delta(g) = g \otimes g$, and $\epsilon(g) = 1$, where $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$ is coproduct map and $\epsilon : \mathcal{H} \longrightarrow \mathbb{K}$ is counit map of $\mathcal{H}$. In particular, the elements $e^h \in \exp(\mathfrak{ft}_n)$ are in one-to-one correspondence with the elements of $h \in G(\hat{U}(\mathfrak{ft}_n))$. We briefly recall that elements of the completed Hopf algebra $\hat{U}(\mathfrak{ft}_n)$ are polynomials in the symbols t_{ij}, $1 \leq i, j \leq n$. The symmetric groups act on $\widehat{\mathsf{RCD}}_{\mathbb{K}}(n)$ by permuting the indices of the generators $t_{ij} \in \mathfrak{ft}_n$, e.g. for a monomial $t_{ij}t_{lk} \in \hat{U}(\mathfrak{ft}_n)$ and $\sigma \in \Sigma_n$ we define an action

$$t_{ij}t_{lk} \xrightarrow{\sigma^*} t_{\sigma(i)\sigma(j)}t_{\sigma(l)\sigma(k)}.$$

The cyclic structure on $\widehat{\mathsf{RCD}}_{\mathbb{K}}$ can be completely determined by defining the following action of the (01)-transposition on $\hat{U}(\mathfrak{ft}_n)$:

$$(01)^*(t_{ij}) = \begin{cases} t_{ij} \text{ if } i \neq 1 \text{ and } j \neq 1; \\ -\sum_{k=1}^{n} t_{kj} \text{ if } i = 1 \text{ and } j \neq 1; \\ \sum_{k,l=1}^{n} t_{kl} \text{ if } i = 1 \text{ and } j = 1. \end{cases} \tag{4.2}$$

Remark 4.3. Alternatively, one could define a cyclic structure on the Lie algebras $\mathfrak{ft}_n$ by formally adding generators t_{00}, $t_{0i} = t_{i0}$, $i = 1, \dots, n$ and rewriting the relations in a cyclically invariant form:

$$\begin{aligned} t_{ij} &= t_{ji} \\ [t_{ij}, t_{kl}] &= 0 \quad \text{for } \{i,j\} \cap \{k,l\} = \emptyset \\ \sum_{i=0}^{n} t_{ij} &= 0\,. \end{aligned}$$

One can check that this is equivalent to the cyclic structure described in (4.2) by noticing that one can use the relations to eliminate the generators t_{ii} for $i = 0, \dots, n$.

Definition 4.4. For each $n \geq 1$, we define a sequence of prounipotent groupoids $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}(n)$ with:

- objects $\mathrm{ob}(\widehat{\mathsf{PaRCD}}_{\mathbb{K}}(n)) = \Omega(n)$;
- morphisms defined by $\mathrm{Hom}_{\widehat{\mathsf{PaRCD}}_{\mathbb{K}}(n)}(p,q) = \exp(\mathfrak{ft}_n)$.

The categorical composition in $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}(n)$ is given by the multiplication in $\exp(\mathfrak{ft}_n)$.

The operad of (completed) parenthesized ribbon chord diagrams $\widehat{\mathsf{PaRCD}}_{\mathbb{K}} = \{\widehat{\mathsf{PaRCD}}_{\mathbb{K}}(n)\}_{n \geq 1}$ is an operad in the category of prounipotent groupoids where the operadic composition at the level of objects is the one of the operad Ω. On morphisms, it is induced by the operadic composition of $\mathfrak{ft} = \{\mathfrak{ft}_n\}$,

$$e^x \circ_i e^y = e^{x \circ_i y}.$$

All morphisms in the operad $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ can be written as (linear combinations) of categorical and operadic compositions of the following generating morphisms:

$$X_{1,2} \in \mathrm{Hom}_{\mathsf{PaRCD}(2)}((12),(21)), H_{1,2} \in \mathrm{Hom}_{\mathsf{PaRCD}(2)}((12),(12)), I_1 \in \mathrm{Hom}_{\mathsf{PaRCD}(1)}(1,1)$$

and $A_{1,2,3} \in \mathrm{Hom}_{\mathsf{PaRCD}(2)}((12)3, 1(23))$ depicted in Figure 6.

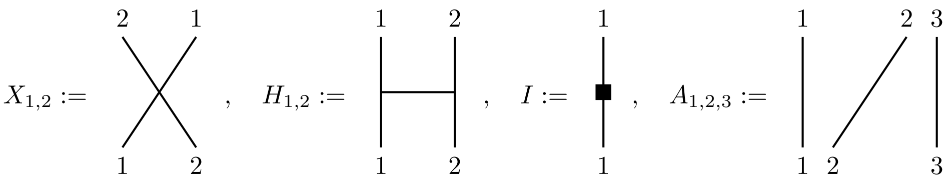


FIGURE 6. Generating morphisms of the operad PaRCD.

Definition 4.5. For any two operads $\mathcal{P}$ and $\mathcal{Q}$ in groupoids, A morphism of operad $\mathcal{P} \longrightarrow \mathcal{Q}$ is called *categorical equivalence* if it induces an equivalence of categories $\mathcal{P}(n) \longrightarrow \mathcal{Q}(n)$ in each arity.

The operad PaRCD can equivalently be defined as an operadic pullback along the map $w : \Omega \longrightarrow \mathrm{ob}\,\mathsf{RCD}$, that is, $w^*\mathsf{RCD} = \mathsf{PaRCD}$, where $\mathrm{Hom}_{\mathsf{PaRCD}(n)}(p,q) = \mathrm{Hom}_{\mathsf{RCD}(n)}(w(p), w(q))$, for all $p, q \in \Omega(n)$ (see [Fre17, Section 6.1.5] for a general pullback construction of operads in groupoids).

Proposition 4.6. *There is a categorical equivalence between the operads* $\widehat{\mathsf{RCD}}_{\mathbb{K}}$ *and* $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$.

Proof. We need to show that there is an equivalence of prounipotent groupoids $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}(n) \xrightarrow{\sim} \widehat{\mathsf{RCD}}_{\mathbb{K}}(n)$ for each n. We notice that the pullback along the map $w : \Omega(n) \longrightarrow *$ that sends every rooted tree in Ω to $*$ that induces the following identification

$$\mathrm{Hom}_{\mathsf{PaRCD}(n)}(p,q) = \mathrm{Hom}_{\mathsf{RCD}(n)}(w(p), w(q)) = \mathrm{Hom}_{\mathsf{RCD}(n)}(*,*) = \exp(\hat{U}(\mathfrak{t}_n)), \tag{4.3}$$

for any $p, q \in \Omega(n)$. The map $w : \Omega(n) \longrightarrow *$ is clearly surjective by definition and fully faithful by (4.3). □

The cyclic action z^* on the generating morphisms $H_{1,2}, I, X_{1,2}$ and $A_{1,2,3}$ of PaRCD is obtained from RCD as follows:

$$z_2^*(I) = I \in \mathrm{Hom}_{\widehat{\mathsf{PaRCD}_{\mathbb{K}}}(1)}(1,1), \quad z_3^*(H_{1,2}) = -H_{1,2} - I_2 \in \mathrm{Hom}_{\widehat{\mathsf{PaRCD}_{\mathbb{K}}}(2)}(12,12),$$
$$z_3^*(X_{1,2}) = X_{1,2} \in \mathrm{Hom}_{\widehat{\mathsf{PaRCD}_{\mathbb{K}}}(2)}(12,12) \quad \text{and} \quad z_4^*(A_{1,2,3}) = A_{2,3,1}^{-1} \in \mathrm{Hom}_{\widehat{\mathsf{PaRCD}_{\mathbb{K}}}(3)}((12)3, 1(23)).$$

The following result is immediate from Lemma 3.6 applied on $A_{1,2,3}$.

Corollary 4.7. *The cyclic action z_4^* on the associator $A_{1,2,3} : \mu \circ_1 \mu \longrightarrow \mu \circ_2 \mu$ is a Σ_4-action that extends the Σ_3-action.*

We make use of this cyclic action in Section 8 to establish its relationship with the Drinfeld associator and the graded Grothendieck-Teichmüller group.

5. Tangles and Chord diagrams

In this section, we exploit the metric prop construction of Section 2.3 to describe the category of tangles and chord diagrams from the cyclic operad of (parenthesized) ribbon braids and the cyclic operad of (parenthesized) chord diagrams.

5.1. **Tangles.** A *tangle* is the image of a proper embedding of finitely many copies of the interval $I = [0,1]$ and S^1 in the cube $[-1,1]^3$ such that the boundary points lie on the faces $[-1,1]\times\{0\}\times\{\pm1\}$. These endpoints are typically arranged in a prescribed order along these boundary faces. In [Tur89], Turaev introduced a strict tensor category of framed, oriented tangles $\mathbf{T}$ whose objects are finite sequences of symbols $+$ and $-$, and whose morphisms are isotopy classes of framed oriented tangles.

Let $\mathbb{M}(\{+,-\})$ denote the free monoid on the symbols $+$ and $-$. For example, the words $\emptyset$, $++-$, and $-+--$ are all elements of $\mathbb{M}(\{+,-\})$. The *source* of a tangle T is the word in $\mathbb{M}(\{+,-\})$ read along the oriented interval $[-1,1]\times\{0\}\times\{1\}$, where each boundary point is assigned the sign $+$ (respectively, $-$) if the orientation of the tangle at that point is downward (respectively, upward). The *target* is defined similarly along $[-1,1]\times\{0\}\times\{-1\}$.

Definition 5.1. We let $\mathbf{T}$ be the strict tensor category whose objects are words $w \in \mathbb{M}(\{+,-\})$ and morphisms $T \in \mathrm{Hom}_{\mathbf{T}}(w,w')$ are isotopy classes of framed oriented tangles with $\mathrm{source}(T) = w$ and $\mathrm{target}(T) = w'$.

Composition in the category $\mathbf{T}$,

$$\mathrm{Hom}_{\mathbf{T}}(w,w') \times \mathrm{Hom}_{\mathbf{T}}(w',w'') \longrightarrow \mathrm{Hom}_{\mathbf{T}}(w,w'')$$

is obtained by "stacking and rescaling" (cf. [KT$^+$95, Section 2.1]). The tensor product

$$\otimes : \mathbf{T}\times\mathbf{T} \longrightarrow \mathbf{T}$$

at the level of objects is given by concatenation of words in the alphabet $\{+,-\}$, and at the level of morphisms by placing one tangle next to the other. The unit object in $\mathbf{T}$ is the empty word $\emptyset$. The category $\mathbf{T}$ admits duals, where the dual of an object $w = (+-++-)$ is defined by reversing the word and flipping each sign: $w^* = (+--+-)$.

The category $\mathbf{T}$ can be equipped with a ribbon category structure in which the tensor product is given by horizontal juxtaposition of tangles, with the unit object the empty word $\emptyset$. The braiding

$$c_{X,Y} : X\otimes Y \to Y\otimes X$$

is represented by the over-crossing of the strands of X over those of Y. Duals are defined by reversing the orientation of strands and applying cups and caps as coevaluation and evaluation morphisms. The twist

$$\theta_X : X \to X$$

is given by inserting a full positive twist in each strand of X. These structural morphisms satisfy the axioms of a ribbon category up to isotopy, making $\mathbf{T}$ into a strict ribbon monoidal category [KT$^+$95].

Theorem 5.2 ([RT90][Shu94][Yet88])**.** *The category of framed oriented tangles* $\mathbf{T}$ *is the universal free strict ribbon category on one object, which means that for any ribbon category* $\mathbf{R}$ *and any object* $X \in \mathbf{R}$*, there is a unique strict monoidal functor* $F : \mathbf{T} \longrightarrow \mathbf{R}$ *such that* $F(+) = X$ *preserving the ribbon structure.*

We define the category $q\mathbf{T}$ of *q-tangles* (or *quantum tangles*) as a non-strict refinement of the strict ribbon category $\mathbf{T}$ (cf. [BN97], [LM96], [JM19]). The objects of $q\mathbf{T}$ are sequences of parenthesized words in the symbols $+$ and $-$; that is, $\mathrm{ob}(q\mathbf{T}) = \Omega(+,-)$, the free magma generated by $\{+,-\}$. This parenthesization reflects the non-associative nature of tensor products in general ribbon categories, where the associator is not necessarily the identity. The morphisms in $q\mathbf{T}$ are the same as those in $\mathbf{T}$, but the monoidal structure is now only associative up to a specified isomorphism. In particular, for any objects $s, t, u \in \Omega(+,-)$, there is a distinguished associator isomorphism

$$(s \otimes t) \otimes u \xrightarrow{\alpha_{s,t,u}} s \otimes (t \otimes u).$$

The following is immediate from Theorem 5.2.

Corollary 5.3. *The category of q-tangles* $q\mathbf{T}$ *is the free ribbon category (not necessarily strict) generated by one object.*

5.1.1. *The ribbon braid operads and balanced monoidal categories.* In this section we show that the categories of $\mathbf{T}$ and $q\mathbf{T}$ are closely related to the (cyclic) operads CoRB and PaRB.

In [JS93], Joyal and Street constructed a free strict braided monoidal category $\mathbf{B} = (\mathbf{B}, \oplus, 0)$ generated by one object, which is a universal strict braided monoidal category. That is, for any strict braided monoidal category $\mathbf{C}$ and any object $X \in \mathbf{C}$, there exists a unique strict braided monoidal functor $F : \mathbf{B} \to \mathbf{C}$ such that $F(1) = X$ and F preserves the braiding and monoidal structure Objects in this category $\mathbf{B}$ are the natural numbers $\mathbb{N}$ and morphisms are defined as

$$\mathrm{Hom}_{\mathbf{B}}(p,q) = \begin{cases} \mathsf{B}_p & \text{if } p = q \\ \emptyset & \text{if } p \neq q, \end{cases}$$

where B_p denotes the braid group on p strands (Section 3.1). Composition in $\mathbf{B}$ is given by the mulitplication of braids. The monoidal product on $\mathbf{B}$ is defined on objects as the sum of natural numbers and on morphisms by the direct sum $\oplus : \mathbf{B}_m \times \mathbf{B}_n \longrightarrow \mathbf{B}_{m+n}$, which visually stacks a braid on m strands next to a braid on n strands.

Similarly, we can construct the free strict balancedmonoidal category $\mathbf{R} = (\mathbf{R}, \oplus, 0)$ whose set of objects is the set of natural numbers $\mathbb{N}$ and whose morphisms are defined as

$$\mathrm{Hom}_{\mathbf{R}}(p,q) = \begin{cases} \mathsf{RB}_p & \text{if } p = q \\ \emptyset & \text{if } p \neq q, \end{cases}$$

where RB_p denotes the braid group on p strands (Section 3.1; [Shu94] [RT90]). It was shown in [Wah01, Theorem 1.4.7] that the operad CoRB plays a similar role in that there exists a one-to-one correspondence between CoRB-algebras,

$$\bar{\rho} : \mathsf{CoRB} \to \mathrm{End}(\mathbf{C})$$

and strict balanced monoidal[3] structures on $\mathbf{C}$. We include the following for completeness.

Proposition 5.4. *Let* $\widehat{\mathbf{C}}_{\mathbb{K}}$ *be a prounipotent completion of the strict* $\mathbb{K}$*-linear monoidal category* $\mathbf{C}$*. Then there exist maps of operads* $\rho : \widehat{\mathsf{CoRB}}_{\mathbb{K}} \to \mathrm{End}(\widehat{\mathbf{C}}_{\mathbb{K}})$ *and the strict balanced monoidal structures on* $\widehat{\mathbf{C}}_{\mathbb{K}}$*.*

Proof. Let $\bar{\rho} : \mathsf{CoRB} \to \mathrm{End}(\mathbf{C})$ be an operad map which equips $\mathbf{C}$ with a strict balanced monoidal structure. The $\mathbb{K}$-linearity of $\mathbf{C}$ extends the map $\bar{\rho} : \mathsf{CoRB} \to \mathrm{End}(\mathbf{C}_{\mathbb{K}})$. Since the morphisms of $\mathsf{CoRB}_{\mathbb{K}}$ are linear combinations of generating morphisms of CoRB, and the augmentation ideal $I = \mathsf{Ker}\ (\mathsf{CoRB}_{\mathbb{K}} \longrightarrow \mathbb{K})$ induces a projective system $\{\mathsf{CoRB}_{\mathbb{K}}/I^n\}_{m \geq 0}$, we note that the image $\bar{\rho}(I)$ gives a projective system $\mathbf{C}_{\mathbb{K}}/\mathbf{I}^n$ of categories, where the power $\mathbf{I}^n = \bar{\rho}(I^n)$ is the subideal of $\mathbf{I} = \bar{\rho}(I)$. Now taking the inverse limit gives the map $\rho : \widehat{\mathsf{CoRB}}_{\mathbb{K}} \to \mathrm{End}(\widehat{\mathbf{C}}_{\mathbb{K}})$ preserving the strict balanced monoidal structure on $\widehat{\mathbf{C}}_{\mathbb{K}}$. ☐

---

[3] Wahl uses the term "ribbon category" to mean balanced monoidal category here. For us, ribbon categories are balanced monoidal categories with dual objects, see Definition 1.3.

As we saw in Section 3.2, the operad PaRB of parenthesized ribbon braids is a cofibrant resolution of the operad CoRB in the category of operads in groupoids. That is, there is a natural weak equivalence of operads in groupoids

$$\mathsf{PaRB} \xrightarrow{\sim} \mathsf{CoRB}.$$

This resolution lifts the structure of the operad CoRB to a setting in which we now have a non-trivial associativity isomorphism. By Mac Lane's coherence theorem and its extensions [JS93, Shu94], every balanced monoidal category is equivalent to a strict balanced monoidal category in which associativity and unit constraints are strict identities.

Proposition 5.5. *Let $\widehat{\mathbf{C}}_{\mathbb{K}}$ be the prounipotent completion of a $\mathbb{K}$-linear category $\mathbf{C}$. Then there is a one-to-one correspondence between maps of operads*

$$\rho : \widehat{\mathsf{PaRB}}_{\mathbb{K}} \longrightarrow \mathrm{End}(\widehat{\mathbf{C}}_{\mathbb{K}})$$

and balanced monoidal structures on $\widehat{\mathbf{C}}_{\mathbb{K}}$.

Proof. This follows from Theorem 3.4 and by adapting the argument of Proposition 5.4, replacing CoRB with PaRB. □

5.1.2. *Envelopes of the ribbon braid operads and tangles.* The category of framed, oriented tangles $\mathbf{T}$ is obtained from $\mathbf{R}$ by formally adjoining duals; that is, $\mathbf{T}$ is the free strict ribbon category on one object. This construction appears explicitly in the work of Shum [Shu94], and gives a topological realization of the abstract process of extending a balanced monoidal category to a ribbon one by freely adding duals and the associated evaluation and coevaluation morphisms.

Definition 5.6. Let $\mathbf{T}'$ be the category of $\mathbf{T}$ of framed oriented tangles equipped with an additional strict self-duality relation $(+)^* = +$.

Topologically, imposing the relation $(+)^* = +$ in the category $\mathbf{T}'$ corresponds to identifying each oriented strand with its dual, effectively making orientation reversal invisible. This simplifies the duality structure: cups and caps become symmetric loops, and evaluation and coevaluation maps now go from $+ \otimes +$ to the unit object and back. In this setting, duality becomes involutive and self-dual at the level of objects. Such self-duality conditions appear in various contexts—for instance, in the study of Grothendieck-Verdier categories, in Verdier duality in which the intersection cohomology sheaf is self-dual [GM80], in (categorified) higher representation theory [DSPS20], in fusion categories and modular tensor functors where non-invertible objects in Tambara–Yamagami categories are self-dual [ENO05, TY98].

Recall that the prop associated to the operad CoRB, namely $\mathrm{Env}(\mathsf{CoRB})$, is given by

$$\coprod_{\Sigma m_i = m} \big(\mathsf{CoRB}(m_1) \times \ldots \times \mathsf{CoRB}(m_n)\big) \times_{\Sigma_{m_1} \times \ldots \times \Sigma_{m_n}} \Sigma_m.$$

This defines a strict monoidal category whose objects are natural numbers, $\mathrm{ob}(\mathrm{Env}(\mathsf{CoRB})) = \mathbb{N}$, and whose morphisms are generated by compositions and tensor products of ribbon braids. That is, for $p, q \in \mathbb{N}$, the morphism set $\mathrm{Hom}_{\mathrm{Env}(\mathsf{CoRB})}(p, q)$ is identified with a disjoint union of ribbon braids groups indexed over partitions of $m = q$. This category $\mathrm{Env}(\mathsf{CoRB})$ is isomorphism to $\mathbf{R}$, the free strict balanced monoidal category generated by one object (with object 1 interpreted as $+$).

Proposition 5.7. *The metric prop $\Pi(\mathsf{CoRB}^{\mathrm{cyc}})$ associated to the cyclic operad $\mathsf{CoRB}^{\mathrm{cyc}}$ is isomorphic, as a symmetric monoidal category, to the category $\mathbf{T}'$ of framed, oriented tangles with self-dual objects.*

Proof. We first define a symmetric monoidal functor $F : \mathrm{Env}(\mathsf{CoRB}) \to \mathbf{T}'$ by specifying its action on generating objects and morphisms:

$$F(1) = +, \quad F(\beta_{1,2}) = c_{+,+}, \quad F(\alpha_{1,2,3}) = \alpha_{+,+,+}, \quad F(\tau_1) = \theta_+, \tag{5.1}$$

where $c_{+,+}$, $\alpha_{+,+,+}$, and θ_+ denote the braiding, associator, and twist in $\mathbf{T}'$, respectively.

Since the enveloping construction Env is fully faithful, specifying a symmetric monoidal functor $F : \mathrm{Env}(\mathsf{CoRB}) \to \mathbf{T}'$ is equivalent to giving a map of operads $\rho : \mathsf{CoRB} \to \mathrm{End}(\mathbf{T}')$. The data in (5.1) defines such a map by assigning the generators of CoRB to the corresponding structure maps in $\mathbf{T}'$, and the coherence axioms for braiding, associativity, and balancing ensure that the relations in CoRB are preserved. In particular, the image of ρ defines a strict balanced monoidal structure on $\mathbf{T}'$.

Moreover, the morphisms in $\mathrm{Env}(\mathsf{CoRB})$ correspond to ribbon braids, and the relations imposed in the definition of $\mathrm{Env}(\mathsf{CoRB})$ match the equivalence relations for ribbon braid isotopy in $\mathbf{T}'$. Therefore, for any objects $p, q \in \mathrm{Env}(\mathsf{CoRB})$, the functor F induces a bijection

$$\mathrm{Hom}_{\mathrm{Env}(\mathsf{CoRB})}(p, q) \cong \mathrm{Hom}_{\mathbf{T}'}(F(p), F(q)). \tag{5.2}$$

Now consider the metric prop $\Pi(\mathsf{CoRB}^{\mathrm{cyc}})$, which consists of the symmetric monoidal category $\mathrm{Env}(\mathsf{CoRB})$ equipped with distinguished morphisms $b \in \mathrm{Hom}(0, 2)$ and $d \in \mathrm{Hom}(2, 0)$ satisfying the zig-zag identities and equivariance conditions of Definition 2.12. The functor F extends to a functor of props

$$F' : \Pi(\mathsf{CoRB}^{\mathrm{cyc}}) \to \mathbf{T}'$$

by setting $F'(b) = b_+$ and $F'(d) = d_+$, the standard cup and cap morphisms in $\mathbf{T}'$.

The zig-zag relations (2.7) and (2.8) are satisfied in $\mathbf{T}'$ as part of the duality structure, and the equivariance condition (2.9) corresponds to the naturality of the twist and braiding in the ribbon structure. Hence, F' respects all of the defining relations of the metric prop $\Pi(\mathsf{CoRB}^{\mathrm{cyc}})$.

Since F' is fully faithful by (5.2) and essentially surjective (both categories have the same objects), it follows that F' is an isomorphism of symmetric monoidal categories.

□

The envelope $\mathrm{Env}(\mathsf{PaRB})$ differs from $\mathrm{Env}(\mathsf{CoRB})$ only at the level of objects. The tensor product in $\mathrm{Env}(\mathsf{PaRB})$ is defined by substitution of trees: for $p, q \in \Omega$, the tensor product $p \otimes q$ is given by the binary operation $\mu(p, q)$ corresponding to grafting in the rooted planar binary tree structure. See Section 6.2.8 of [Fre17] for the relation between the envelope $\mathrm{Env}(\mathsf{PaB})$ and the free parenthesized braided categories defined by Bar-Natan in [BN98] through a free algebra construction. This leads us to the following Corollary of Proposition 5.7.

Corollary 5.8. *The metric prop* $\Pi(\mathsf{PaRB}^{\mathrm{cyc}})$ *associated to the cyclic operad* $\mathsf{PaRB}^{\mathrm{cyc}}$ *is isomorphic, as a symmetric monoidal category, to the category* $q\mathbf{T}'$ *of parenthesized framed tangles with self-dual objects.*

Proof. The isomorphism $H : \Pi(\mathsf{PaRB}^{\mathrm{cyc}}) \to q\mathbf{T}'$ is defined analogously to the functor F' in Proposition 5.7, but with an additional assignment

$$H(\mu) = (++),$$

which records the parenthesized tensor product structure at the level of objects. Since every object in $q\mathbf{T}'$ can be built by repeated applications of μ starting from the generator $+$, this ensures essential surjectivity. The verification of the cyclic structure, zig-zag identities, and duality relations follows from the same arguments as in the proof of Proposition 5.7, now applied to the parenthesized setting. □

Finally, Proposition 2.13 implies that the algebras over $\widehat{\mathsf{PaRB}_{\mathbb{K}}}^{\mathrm{cyc}}$ in a $\mathbb{K}$-linear category $\widehat{\mathbf{C}}_{\mathbb{K}}$ are precisely the algebras over the prop $\Pi(\widehat{\mathsf{PaRB}_{\mathbb{K}}}^{\mathrm{cyc}})$. Using Corollary 5.8 and Theorem 5.2, it follows that such algebras correspond to ribbon structures on $\widehat{\mathbf{C}}_{\mathbb{K}}$. That is,

$$\mathrm{Alg}_{\widehat{\mathbf{C}}_{\mathbb{K}}}(\Pi(\widehat{\mathsf{PaRB}_{\mathbb{K}}}^{\mathrm{cyc}})) \cong \mathrm{Alg}_{\widehat{\mathbf{C}}_{\mathbb{K}}}(\widehat{\mathsf{PaRB}_{\mathbb{K}}}^{\mathrm{cyc}})$$

is the category of $\mathbb{K}$-linear ribbon categories with self-dual generating objects.

Remark 5.9. Cyclic algebras can be studied in symmetric monoidal bicategory. In particular, a cyclic framed little 2 -disks algebra in symmetric monoidal bicategory **Lex** of finite linear categories characterises *ribbon Grothendieck–Verdier category*, see for more details [MW22].

5.2. **Chord diagrams.** The category $\mathbf{A}(\mathbb{K})$ is an infinitesimal symmetric category built out of chord diagrams. The objects of $\mathbf{A}(\mathbb{K})$ are monoids generated by the signed set $\{+,-\}$, the same as the objects in $\mathbf{T}$. For any two objects $p, q \in \mathbf{A}(\mathbb{K})$, the hom set $\mathrm{Hom}_{\mathbf{A}(\mathbb{K})}(p,q)$ is an $\mathbb{K}$- module generated by the chord diagrams between p and q modulo the $4T$ relation (Figure 7),

$$\mathrm{Hom}_{\mathbf{A}(\mathbb{K})}(p,q) = \mathrm{Span}_{\mathbb{K}}\{\text{chord diagrams between } p \text{ and } q\}/4T.$$

The composition of morphisms is given by vertical stacking of chord diagrams, while the tensor product is given on objects by concatenation and on morphisms by horizontal juxtaposition. The unit object is the empty word $\emptyset$, so $\mathbf{A}(\mathbb{K})$ is a strict monoidal category. An infinitesimal braiding $t^p_{i,j} \in \mathrm{Hom}_{\mathbf{A}(\mathbb{K})}(p,p)$ is defined by placing a single chord between the i^{th} and j^{th} strands in the identity diagram on p. Duality is implemented by morphisms

$$b_p : \emptyset \to p \otimes p^* \quad \text{and} \quad d_p : p \otimes p^* \to \emptyset,$$

where p^* is the reversal of p with all signs flipped, as in the tangle category. This structure makes $\mathbf{A}(\mathbb{K})$ into a strict infinitesimal symmetric monoidal category with duals; see [KT+95, Section 5.3] for further details.

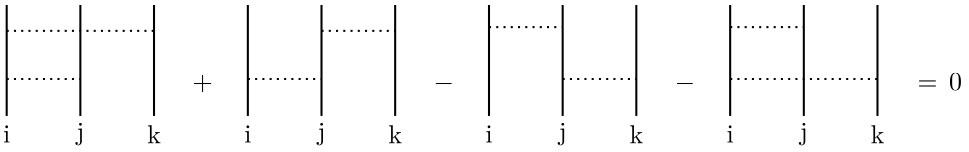


Figure 7. The 4T relation

Theorem 5.10 ([Car93, KT+95])**.** *The category $\mathbf{A}(\mathbb{K})$ is the universal free strict infinitesimal symmetric monoidal category with duals generated by a single object.* □

This means that for any infinitesimal symmetric category $\mathbf{S}$ with duals and any object $X \in \mathbf{S}$, there is a unique strict monoidal functor $G : \mathbf{A}(\mathbb{K}) \to \mathbf{S}$ such that $G(+) = X$ and which preserves the infinitesimal braiding and duality structure:

$$G(-) = X^*, \quad G(t_{+,+}) = t_{X,X}, \quad G(c_{+,+}) = c_{X,X}, \quad G(b_+) = b_X, \quad G(d_+) = d_X.$$

5.2.1. *The chord diagram operads and infinitesimal symmetric monoidal categories.* We now describe how the operads $\widehat{\mathsf{PaA}}_{\mathbb{K}}$ and $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ encode the structure of infinitesimal symmetric and ribbon monoidal categories. These operads are built out of chord diagrams and closely related to the category $\mathbf{A}(\mathbb{K})$ introduced in the previous subsection.

We begin by defining a framed veresion of the operad $\widehat{\mathsf{A}}_{\mathbb{K}}$ from [Fre17, Section 10.3.1], which encodes infinitesimal structures on symmetric monoidal categories.

Definition 5.11. For each $n \geq 1$, let $\widehat{\mathsf{A}}_{\mathbb{K}}(n)$ denote the groupoid with a single object and endomorphism set

$$\mathrm{Hom}_{\widehat{\mathsf{A}}_{\mathbb{K}}(n)}(*,*) = \hat{U}(\mathfrak{ft}_n),$$

the degree-complete universal enveloping algebra of the graded Drinfeld–Kohno Lie algebra $\mathfrak{ft}_n$. The collection $\widehat{\mathsf{A}}_{\mathbb{K}} = \{\widehat{\mathsf{A}}_{\mathbb{K}}(n)\}$ forms an operad in complete Hopf groupoids, with composition induced by the operadic composition in $\mathfrak{ft}_n$ (see Section 4.1 for details).

The operad $\widehat{\mathsf{PaA}}_{\mathbb{K}}$ is a parenthesized refinement of $\widehat{\mathsf{A}}_{\mathbb{K}}$ defined via pullback along the map $w : \Omega \to \mathrm{ob}\,\widehat{\mathsf{A}}_{\mathbb{K}}$, i.e.

$$\widehat{\mathsf{PaA}}_{\mathbb{K}} := w^* \widehat{\mathsf{A}}_{\mathbb{K}}.$$

The morphisms in $\widehat{\mathsf{PaA}}_{\mathbb{K}}(n)$ are given by

$$\mathrm{Hom}_{\widehat{\mathsf{PaA}}_{\mathbb{K}}(n)}(p,q) = \hat{U}(\mathfrak{ft}_n)$$

for all $p, q \in \Omega(n)$ and composition is defined via multiplication in $\hat{U}(\mathfrak{ft}_n)$. The operad $\widehat{\mathsf{PaA}}_{\mathbb{K}}$ is generated under operadic and categorical composition by five elements:

$$\mathsf{X}_{1,2} \in \mathrm{Hom}_{\mathsf{PaA}(2)}((12),(21)), \quad \mathsf{H}_{1,2} \in \mathrm{Hom}_{\mathsf{PaA}(2)}((12),(12)), \quad \mathsf{I}_1 \in \mathrm{Hom}_{\mathsf{PaA}(1)}(1,1), \quad \text{and}$$
$$\mathsf{A}_{1,2,3} \in \mathrm{Hom}_{\mathsf{PaA}(3)}(((12)3),(1(23))).$$

These generators correspond respectively to the symmetry, infinitesimal braiding, identity, and associator isomorphisms of infinitesimal symmetric monoidal categories. The following result is a universal property describing maps out of $\widehat{\mathsf{PaA}}_{\mathbb{K}}$ from [Fre17, Theorem 10.3.4].

Theorem 5.12. *Let $\mathcal{Q}$ be any operad in the complete Hopf groupoids. There is a bijection between the set of operad maps $f : \widehat{\mathsf{PaA}}_{\mathbb{K}} \to \mathcal{Q}$ and the set of values*

$$f(\mu) = \mathtt{m} \in \mathrm{ob}(\mathcal{Q}(2)), \quad f(I) = \mathtt{I} \in \mathcal{Q}(1), \quad f(H_{1,2}) = \mathtt{H}_{1,2} \in \mathcal{Q}(2) \quad f(X_{1,2}) = \mathtt{X}_{1,2} \in \mathrm{ob}(\mathcal{Q}(2))$$
$$\text{and} \quad f(A_{1,2,3}) = \mathtt{A}_{1,2,3} \in \mathcal{Q}(3)$$

which satisfy the following equations:

(SH) $$\mathtt{H}_{12,3} = \mathtt{A}^{-1}_{1,2,3}\mathtt{H}_{2,3}\mathtt{A}_{1,2,3} + \mathtt{X}^{-1}_{2,1}\mathtt{A}^{-1}_{2,1,3}\mathtt{H}_{1,3}\mathtt{A}_{2,1,3}\mathtt{X}_{2,1} \in \mathcal{Q}(3).$$

(Inv) $$\mathtt{H}_{1,2} = \mathtt{X}_{1,2}\mathtt{H}_{2,1}\mathtt{X}^{-1}_{1,2} \in \mathcal{Q}(2).$$

(P) $$(\mathtt{A} \circ_1 1_{\mathtt{m}}) \cdot (\mathtt{A} \circ_3 1_{\mathtt{m}}) = (1_{\mathtt{m}} \circ_1 \mathtt{A}) \cdot (\mathtt{A} \circ_2 1_{\mathtt{m}}) \cdot (1_{\mathtt{m}} \circ_2 \mathtt{A}) \in \mathcal{Q}(4).$$

(H') $$(1_{\mathtt{m}} \circ_1 \mathtt{X}_{1,2}) \cdot \mathtt{A}_{2,1,3} \cdot (213)^*(1_{\mathtt{m}} \circ_2 \mathtt{X}_{1,2}) = \mathtt{A}_{1,2,3} \cdot (\mathtt{X}_{1,2} \circ_2 1_{\mathtt{m}}) \cdot \mathtt{A}_{2,3,1} \in \mathcal{Q}(3).$$

□

Corollary 5.13. *Let $\mathbf{C}$ be a prounipotent $\mathbb{K}$-linear category. Then giving a map of operads $\rho : \widehat{\mathsf{PaA}}_{\mathbb{K}} \to$ End($\mathbf{C}$) is equivalent to equipping $\mathbf{C}$ with the structure of an infinitesimal symmetric monoidal category. In particular, the image under ρ of the generating morphisms I, X, H, A define the natural isomorphisms which define the unit, symmetry, infinitesimal braiding, and associator, respectively, satisfying the coherence relations* (SH)–(H').

Proof. This follows immediately from Theorem 5.12 by taking $\mathcal{Q} = \mathrm{End}(\mathbf{C})$. Since $\mathbf{C}$ is prounipotently complete and $\mathbb{K}$-linear, the endomorphism operad End($\mathbf{C}$) is an operad in complete Hopf groupoids, and thus the theorem applies. □

The morphisms A, I, and X in $\widehat{\mathsf{PaA}}_{\mathbb{K}}$ correspond directly to the associativity, unit, and symmetry generators in the operad $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$. However, the infinitesimal braiding generator $\mathsf{H}_{1,2}$ does not appear in $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$; instead, its exponential $e^{\mathsf{H}_{1,2}}$ belongs to the suboperad $\widehat{\mathsf{PaRCD}}_{\mathbb{K}} \subset \widehat{\mathsf{PaA}}_{\mathbb{K}}$ consisting of group-like elements. Thus, the operad $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ governs the formal integration of infinitesimal symmetric monoidal structures, allowing one to exponentiate the infinitesimal braidings to obtain formal symmetric (or ribbon) monoidal structures over $\mathbb{K}[[\hbar]]$.

Corollary 5.14. *Let $\mathbf{C}$ be a prounipotent $\mathbb{K}$-linear category. Then a morphism of operads $\rho : \widehat{\mathsf{PaRCD}}_{\mathbb{K}} \to$ End($\mathbf{C}[[\hbar]]$) equips $\mathbf{C}[[\hbar]]$ with the structure of a formal symmetric (or ribbon) monoidal category. Moreover, such a morphism arises by exponentiating the infinitesimal structure defined by a map $\widehat{\mathsf{PaA}}_{\mathbb{K}} \to \mathrm{End}(\mathbf{C})$.* □

As with the operads of parenthesized braids and ribbon braids, the natural map

$$\widehat{\mathsf{PaRCD}}_{\mathbb{K}} \longrightarrow \widehat{\mathsf{RCD}}_{\mathbb{K}},$$

which forgets parenthesization, defines an equivalence of operads in complete Hopf groupoids. In particular, $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ serves as a cofibrant resolution of $\widehat{\mathsf{RCD}}_{\mathbb{K}}$, and encodes associativity data explicitly at the level of objects.

We now turn to the corresponding categorical envelopes. The relationship between $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ and the chord diagram category $\mathbf{A}(\mathbb{K})$ mirrors the relationship between PaRB and the tangle category $\mathbf{T}$: both encode infinitesimal symmetric structures with duals and admit universal characterizations in terms of maps into completed linear categories.

To reflect the presence of parenthesized boundary data in $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$, we introduce a parenthesized version of the chord diagram category. Let $\mathbf{A}^{\mathrm{par}}(\mathbb{K})$ denote the $\mathbb{K}$-linear tensor category whose objects are elements of the free magma on the signed set $\{+,-\}$, and whose morphisms are chord diagrams with parenthesized boundary conditions. As with $\mathbf{A}(\mathbb{K})$, morphisms are linear combinations of chord diagrams modulo the $4T$ relation, with vertical stacking and horizontal juxtaposition defining categorical composition and tensor product. The duality and infinitesimal structure are given by the images of the generating morphisms $\mathsf{I}, \mathsf{X}, \mathsf{H}, \mathsf{A}$ in $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$.

The following lemma formalizes the relationship between the cyclic operad $\widehat{\mathsf{RCD}}_{\mathbb{K}}^{\mathrm{cyc}}$ and the category $\mathbf{A}(\mathbb{K})$:

Lemma 5.15. *The metric prop associated to the cyclic operad $\widehat{\mathsf{RCD}}_{\mathbb{K}}^{\mathrm{cyc}}$ is isomorphic, as a symmetric monoidal category, to the category $\mathbf{A}(\mathbb{K})$ of chord diagrams.*

Proof. The metric prop $\Pi(\widehat{\mathsf{RCD}}_{\mathbb{K}}^{\mathrm{cyc}})$ is constructed from the operad $\widehat{\mathsf{RCD}}_{\mathbb{K}}^{\mathrm{cyc}}$ by freely adjoining categorical composition and tensor product, along with duality morphisms b and d, and imposing the coherence relations encoded in the cyclic structure. Since the objects of $\Pi(\widehat{\mathsf{RCD}}_{\mathbb{K}}^{\mathrm{cyc}})$ are sequences of signs and the morphisms are generated by chord diagrams modulo the $4T$ relation, the resulting tensor category is precisely the chord diagram category $\mathbf{A}(\mathbb{K})$, with its strict symmetric monoidal structure and duals.

Moreover, the defining relations of $\widehat{\mathsf{RCD}}_{\mathbb{K}}^{\mathrm{cyc}}$ ensure that duality and symmetry structures descend appropriately to the prop. Therefore, we obtain an isomorphism of symmetric monoidal categories:

$$\Pi(\widehat{\mathsf{RCD}}_{\mathbb{K}}^{\mathrm{cyc}}) \cong \mathbf{A}(\mathbb{K}).$$

□

Corollary 5.16. *The metric prop $\Pi(\widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{\mathrm{cyc}})$ is isomorphic, as a tensor category, to the parenthesized chord diagram category $\mathbf{A}^{\mathrm{par}}(\mathbb{K})$.* □

6. Drinfeld's Associators

Let $\mathfrak{f}_2$ be the degree-completed free Lie algebra on two variables x and y and $\hat{U}(\mathfrak{f}_2) = \mathbb{K}\langle\langle t_{12}, t_{23}\rangle\rangle$ be a non-commutative formal power series ring in two variables x and y. The enveloping algebra, $\hat{U}(\mathfrak{f}_2)$, admits a natural Hopf algebra structure, and the group-like elements of $U(\mathfrak{f}_2)$ are of the form $\exp(-)$, that is, an element $s \in \hat{U}(\mathfrak{f}_2)$ is group-like if and only if it admits a form $s = \exp(x)$ for some $x \in \mathfrak{f}_2$. Drinfeld in [Dri90] defines an associator to be a particuler grouplike element of $\mathbb{K}\langle\langle t_{12}, t_{23}\rangle\rangle$ that satisfies some equations as follows.

Definition 6.1. Let $\mathbb{K}$ be a field which contains $\mathbb{Q}$. A *Drinfeld associator* is a pair $(\lambda, \Phi) \in \mathbb{K}^{\times} \times \exp(\mathfrak{f}_2)$ that satisfies the following equations.

$$\Phi(x_1, x_2)\Phi(x_2, x_1) = 1, \tag{I}$$

$$\Phi(t_{12}, t_{23})\Phi(t_{12}+t_{13}, t_{24}+t_{34})\Phi(t_{23}, t_{34}) = \Phi(t_{13}+t_{23}, t_{34})\Phi(t_{12}, t_{23}+t_{24}) \quad \text{in} \quad \exp(\mathfrak{t}_4), \tag{P}$$

$$\Phi(t_{12}, t_{23}) \cdot e^{\lambda t_{23}/2} \cdot \Phi(t_{23}, t_{31}) \cdot e^{\lambda t_{31}/2} \cdot \Phi(t_{13}, t_{12}) \cdot e^{\lambda t_{12}/2} = 1 \qquad \text{in} \quad \exp(\mathfrak{t}_3); \tag{H}$$

in the complete associative algebra $\mathbb{K}\langle\langle t_{12}, t_{23}\rangle\rangle/(t_{12}+t_{23}+t_{13})$.

We write $\mathsf{Assoc}_{\mathbb{K}}$ for the set of Drinfeld associators.

Remark 6.2. The field $\mathbb{K}$ contains $\mathbb{Q}$ because Drinfeld showed the existence of rational associators and associators with coefficients in $\mathbb{Z}$ do not exist, see [Dri90, Section 2, Section 5]. There are two known explicit examples of associators: one constructed by Drinfeld is $\Phi_{\mathbf{KZ}}$ over $\mathbb{C}$ using the monodromy of Knizhnik-Zamolodchikov equations and the other constructed by Alekseev-Torossian $\Phi_{\mathbf{AT}}$ in [AT10] using integration theory of singular differential forms on semialgebraic chains related to the solutions to the Kashiwara-Vergne problem.

The following proposition is similar to [Fre17, Proposition 10.2.6; Theorem 10.2.9] and [Wil24, Section 5].

Proposition 6.3. *A morphism of operads* $f : \widehat{\mathsf{PaRB}}_{\mathbb{K}} \to \widehat{\mathsf{RCD}}_{\mathbb{K}}$ *is uniquely determined by a scalar parameter* $\lambda \in \mathbb{k}$ *and a group-like element of the complete tensor algebra on two generators* $\Phi(x,y) \in T(x,y)$ *satisfying the unit, involution, hexagon and pentagon relations such that:*

$$f(\tau) = e^{\frac{\lambda}{2}t_{11}}, \quad f(\beta) = e^{\frac{\lambda}{2}t_{12}} \quad \textit{and} \quad f(\alpha) = \Phi(t_{12}, t_{23})$$

in RCD*. Here,* τ*,* β *and* α *are the relevant generating isomorphisms of* PaRB*.*

Lemma 3.2 of [CIW19] implies that the assignment of Proposition 6.3 determines a map of cyclic operads.

Lemma 6.4. *A morphism of operads* $f : \mathsf{PaRB} \to \widehat{\mathsf{RCD}}_{\mathbb{K}}$ *given by the pair* (λ, Φ) *uniquely lifts to a morphism of cyclic operads.*

Proof. It suffices to check the cyclic compatibility on the generators of PaRB.

For braiding: $z^* f(\beta_{1,2}) = z^*(e^{\frac{\lambda}{2}t_{12}}) = e^{\frac{\lambda}{2}z^* t_{12}} = e^{\frac{\lambda}{2}(-t_{12}-t_{22})} = f(\beta_{1,2}^{-1} \cdot \tau_2) = f(z^*\beta_{1,2})$.

For twist: $z^* f(\tau_1) = z^*(e^{\frac{\lambda}{2}t_{11}}) = e^{\frac{\lambda}{2}z^* t_{11}} = e^{\frac{\lambda}{2}t_{11}} = f(z^*\tau_1)$.

For associativity: $f(z^*\alpha_{1,2,3}) = f(\alpha_{2,3,1}^{-1}) = \Phi(t_{23}, t_{31})^{-1} = \Phi(t_{31}, t_{23})$, and

$$z^* f(\alpha_{1,2,3}) = z^*\Phi(t_{12}, t_{23}) = \Phi(-t_{12} - t_{22} - t_{32}, t_{23}) = \Phi(t_{31} - t_{22}, t_{23}) = \Phi(t_{31}, t_{23}).$$

where the second equality uses the cyclic action on t_{12} seen as an element in $\mathfrak{t}_3$, the third uses the central identity $t_{12} + t_{23} + t_{31} = 0$ and $t_{ij} = t_{ji}$ for all i, j. The last equality uses the fact that t_{22} is central and for any central element z, $\Phi(x + z, y) = \Phi(x, y)$. ☐

Lemma 6.5. *A morphism of operads* $f : \mathsf{PaRCD} \to \widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ *given by the pair* (λ, Φ)*, in particular*

$$f(I_1) = \lambda I_1, \quad f(H_{1,2}) = \lambda H_{1,2}, \quad f(X_{1,2}) = X_{1,2} \quad \textit{and} \quad f(A_{1,2,3}) = \Phi(t_{12}, t_{23}) \cdot A_{1,2,3},$$

is a morphism of cyclic operads.

Proof. We similarly need to check the cyclic compatibility as follows.

- $z_2^* \cdot f(I_1) = z_2^* \cdot \lambda I_1 = \lambda I_1 = f(z_2^* \cdot I_1)$.
- $z_2^* \cdot f(H_{1,2}) = z_2^* \cdot \lambda H_{1,2} = \lambda(-H_{1,2} - I_2) = f(-H_{1,2} - I_2) = f(z_2^* \cdot H_{1,2})$.

To check the cyclic action on the associator $A_{1,2,3}$, we first note that the generators t_{12} and t_{23} of $\mathfrak{t}_3$ are written operadically as

$$t_{12} = \mathrm{id}_2 \circ_1 H_{1,2} = H_{1,2}, \quad \text{and} \quad t_{23} = A_{1,2,3}(\mathrm{id}_2 \circ_2 H_{1,2})A_{1,2,3}^{-1} = A_{1,2,3}H_{2,3}A_{1,2,3}^{-1}.$$

Now we check that $z_4^* \cdot t_{12} = z_3^* \cdot (\mathrm{id}_2 \circ_1 H_{1,2}) = -(\mathrm{id}_2 \circ_1 H_{1,2}) - (\mathrm{id}_2 \circ_2 I) - (\mathrm{id}_2 \circ_2 H_{2,1})$ and $z_4^* \cdot t_{23} = A_{2,3,1}^{-1}(\mathrm{id}_2 \circ_2 H_{1,2})A_{2,3,1}$. Now we have the following

$$f(z_4^* \cdot A_{1,2,3}) = f(A_{2,3,1}^{-1}) = A_{2,3,1}^{-1} \cdot \Phi(t_{23}, t_{31})^{-1} = A_{2,3,1}^{-1} \cdot \Phi(t_{31}, t_{23}),$$

Since $z_4^* \cdot \Phi(t_{12}, t_{23}) = \Phi(t_{31}, t_{23})$, we get $z_4^* \cdot f(A_{1,2,3}) = \Phi(t_{31}, t_{23}) \cdot A_{2,3,1}^{-1}$, which reduces to

$$\Phi(t_{31}, A_{2,3,1}^{-1}t_{23}A_{2,3,1}) \cdot A_{2,3,1}^{-1} = A_{2,3,1}^{-1} \cdot \Phi(t_{31}, t_{23})^{-1} \cdot A_{2,3,1} \cdot A_{2,3,1}^{-1} = A_{2,3,1}^{-1} \cdot \Phi(t_{31}, t_{23}).$$

☐

Lemma 6.6. *A morphism of operads* $f : \mathsf{PaRB} \to \widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ *given by the pair* (λ, Φ)*, in particular*

$$f(\tau) = e^{\frac{\lambda}{2}t_{11}} \cdot I_1, \quad f(\beta) = e^{\frac{\lambda}{2}t_{12}} \cdot X_{1,2} \quad \textit{and} \quad f(\alpha) = \Phi(t_{12}, t_{23}) \cdot A_{1,2,3},$$

is a morphism of cyclic operads.

Proof. We need to check the cyclic compatibility of the map f. The action $z_2^* \cdot I_1 = I_1$ and $z_2^* \cdot t_{11} = t_{11}$, implies $z_2^* \cdot f(\tau) = e^{\frac{\lambda}{2}t_{11}} \cdot I_1 = f(z_2^* \cdot \tau)$. Similarly, $z_3^* \cdot f(\beta_{1,2}) = e^{\frac{\lambda}{2}(-t_{12}-t_{22})} \cdot X_{1,2}$ and $f(z_3^* \cdot \beta_{1,2}) = f(\beta_{1,2}^{-1}\tau_2^{-1}) = e^{\frac{\lambda}{2}(-t_{12}-t_{22})} \cdot X_{1,2}$. The action on the associator $\alpha_{1,2,3}$ is the same as in Lemma 6.5. ☐

Lemma 6.5 and Lemma 6.6 imply the following isomorphisms.

$$\mathbf{Cyc}^+(\widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{cyc}, \widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{cyc}) \cong \mathbf{Op}^+(\widehat{\mathsf{PaRCD}}_{\mathbb{K}}, \widehat{\mathsf{PaRCD}}_{\mathbb{K}}) \cong \mathbf{Op}^+(\mathsf{PaRCD}, \widehat{\mathsf{PaRCD}}_{\mathbb{K}}), \tag{6.1}$$

$$\mathbf{Cyc}^+(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc}, \widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{cyc}) \cong \mathbf{Op}^+(\widehat{\mathsf{PaRB}}_{\mathbb{K}}, \widehat{\mathsf{PaRCD}}_{\mathbb{K}}) \cong \mathbf{Op}^+(\mathsf{PaRB}, \widehat{\mathsf{PaRCD}}_{\mathbb{K}}). \tag{6.2}$$

Proposition 6.7. *There is a bijection between* $\mathsf{Assoc}_{\mathbb{K}}$ *and the set of all object-fixing operad isomorphisms* $\widehat{\mathsf{PaRB}}_{\mathbb{K}} \longrightarrow \widehat{\mathsf{PaRCD}}_{\mathbb{K}}$.

Proof. Universal properties of completion imply that every map $\mathsf{PaRB} \to \widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ is the unique extension of an operad map $\widehat{\mathsf{PaRB}}_{\mathbb{K}} \to \widehat{\mathsf{PaRCD}}_{\mathbb{K}}$. We can therefore apply [BdBHR19, Lemma 7.4] to deduce that an operad map $F : \widehat{\mathsf{PaRB}}_{\mathbb{K}} \longrightarrow \widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ defines an operad map $\bar{F} : \widehat{\mathsf{PaB}}_{\mathbb{K}} \to \widehat{\mathsf{PaCD}}_{\mathbb{K}}$ if we set $F(\tau) = 0$. The assignment $F \mapsto \bar{F}$ defines a map $\mathrm{Iso}_0(\widehat{\mathsf{PaRB}}_{\mathbb{K}}, \widehat{\mathsf{PaRCD}}_{\mathbb{K}}) \to \mathrm{Iso}_0(\widehat{\mathsf{PaB}}_{\mathbb{K}}, \widehat{\mathsf{PaCD}}_{\mathbb{K}})$ that admits a section. Indeed, an operad map $\bar{F} : \widehat{\mathsf{PaB}}_{\mathbb{K}} \longrightarrow \widehat{\mathsf{PaCD}}_{\mathbb{K}}$ can be extended to an operad map $F : \widehat{\mathsf{PaRB}}_{\mathbb{K}} \longrightarrow \widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ via the values

$$F(\beta_{1,2}) = \bar{F}(\beta_{1,2}) = e^{\mu t_{12}/2} X_{1,2}, \qquad F(\alpha_{1,2,3}) = \bar{F}(\alpha_{1,2,3}) = f(t_{12}, t_{23}) \mathsf{A}_{1,2,3}, \quad \text{and} \quad F(\tau) = e^{\lambda t_{11}/2} I,$$

for some $\lambda \in \mathbb{K}^{\times}$. Since we require that $F(\tau) \in \widehat{\mathsf{PaCD}}_{\mathbb{K}}(1)$ respects the ribbon twist axiom, we have that

$$\begin{aligned} F(\tau \circ_1 \mathrm{id}_{1,2}) = e^{\frac{\lambda t_{11}}{2}} \circ_1 e^0 = e^{\frac{\lambda(t_{11} \circ_1 0)}{2}} = e^{\frac{\lambda(t_{11}+t_{22}+2t_{12})}{2}} = \\ F(\beta_{1,2} \cdot \beta_{2,1} \cdot (\mathrm{id}_{1,2} \circ_1 \tau) \cdot (\mathrm{id}_{1,2} \circ_2 \tau)) = e^{\frac{\mu t_{12}}{2}} \cdot e^{\frac{\mu t_{12}}{2}} \cdot e^{\frac{\lambda(0 \circ_1 t_{11})}{2}} \cdot e^{\frac{\lambda(0 \circ_2 t_{22})}{2}} = e^{\mu t_{12} + \frac{\lambda(t_{11}+t_{22})}{2}}. \end{aligned} \tag{6.3}$$

Since the elements t_{ii} are central in $\mathfrak{ft}_2$, the map F will only satisfy the ribbon twist axiom if $\lambda = \mu$. It follows that every framed $\mathbb{K}$-associator $F : \widehat{\mathsf{PaRB}}_{\mathbb{K}} \longrightarrow \widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ reduces to the data of a pair $(\mu, f) \in \mathbb{K}^{\times} \times \exp(\mathfrak{t}_3)$ which satisfies the defining equations of the $\mathbb{K}$-associator $\bar{F} : \widehat{\mathsf{PaB}}_{\mathbb{K}} \longrightarrow \widehat{\mathsf{PaCD}}_{\mathbb{K}}$. ☐

Remark 6.8. The Proposition 6.7 differs from [Gon18, Proposition 5.3] in the presentation of the framed Drinfeld-Kohno Lie algebra. By replacing $t_{ii}/2$ with t_{ii}, one obtains Gonzalez's result on the bijection between framed $\mathbb{K}$-associators and $\mathbb{K}$-associators. This arises from the difference between our presentation of the framed Drinfeld-Kohno Lie algebra and the presentation by Severa [Sev09].

Proposition 6.9. *There is a bijection between the set of Drinfeld's associators and the set of object-fixing cyclic operad isomorphisms* $\mathbf{Cyc}^+(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc}, \widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{cyc})$.

Proof. It follows immediately from Proposition 6.7 and Lemma 6.5. ☐

Recall that, for an infinitesimal symmetric $\mathbb{K}$-linear category $\mathbf{S}$, one can define a category $\mathbf{S}[[\hbar]]$ of formal integration of $\mathbf{S}$ whose objects are the same as the objects of $\mathbf{S}$ and morphisms are given by

$$\mathrm{Hom}_{\mathbf{S}[[\hbar]]}(X, Y) = \mathrm{Hom}_{\mathbf{S}}(X, Y) \otimes_{\mathbb{K}} \mathbb{K}[[\hbar]],$$

for any object $X, Y \in \mathbf{S}$. The following theorem states that, given a choice of Drinfeld associator Φ, $\mathbf{A}(\mathbb{K})[[\hbar]]$ admits a ribbon structure.

Theorem 6.10 ([Car93, KT$^+$95])**.** *Given a choice of Drinfeld associator* Φ*, the category* $\mathbf{A}(\mathbb{K})[[\hbar]]$ *acquires a ribbon monoidal structure given by*

$$t_{+,+} = \hbar t_{X,X}, \quad c_{+,+} = c_{X,X} \exp(\hbar t_{X,X}/2), \quad b_+ = b_X, \quad \theta_+ = \exp(\hbar C_X/2),$$

where $C_X \in \mathrm{End}(X)$ *is a Casimir element given by* $-(b_X \otimes \mathrm{id}_X)(t_{X,X^*} \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes d_X)$*. The associativity isomorphism*

$$\alpha_{+,+,+} = \Phi(\hbar t_{X,X} \otimes \mathrm{id}_X, \hbar\, \mathrm{id}_X \otimes t_{X,X})$$

☐

Remark 6.11. It follows from Proposition 5.5 and Corollary 5.14 that given a PaRCD^{cyc} algebra structure on a $\mathbb{K}[[\hbar]]$- linear category $\widehat{\mathbf{C}}_{\mathbb{K}}$, the ribbon structure on $\widehat{\mathbf{C}}_{\mathbb{K}}$ implied by Theorem 6.10 is equivalent to the induced ribbon monoidal structure given by precomposition with isomorphism of cyclic operads $\Phi : \widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc} \to \widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{cyc}$.

$$\begin{tikzcd}
\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc} \arrow[r, "\Phi"] \arrow[rr, dashed, bend right] & \widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{cyc} \arrow[r, "\rho"] & \mathrm{End}(\widehat{\mathbf{C}}_{\mathbb{K}})
\end{tikzcd}$$

7. GROTHENDIECK-TEICHMÜLLER GROUP $\widehat{\mathsf{GT}}_{\mathbb{K}}$

In this section, we construct a natural action of the Grothendieck–Teichmüller group $\widehat{\mathsf{GT}}_{\mathbb{K}}$ on the completed cyclic operad of parenthesized ribbon braids $\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}$ by identifying $\widehat{\mathsf{GT}}_{\mathbb{K}}$ with the group of object-fixing automorphisms $\mathrm{Aut}_0(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}})$. This perspective also provides a new and streamlined proof of the formality of the cyclic framed little disks operad $\mathsf{FD}_2^{\mathrm{cyc}}$, deduced from the non-cyclic case FD_2 via the profinite $\widehat{\mathsf{GT}}$-action of [BdBHR19].

Building on the results of Section 3.3, we further describe how the $\widehat{\mathsf{GT}}_{\mathbb{K}}$-action lifts from braids to tangles, extending the familiar operadic action to a setting compatible with duality and cyclic symmetry. This construction recovers, from a new viewpoint, the $\widehat{\mathsf{GT}}_{\mathbb{K}}$-action on tangles briefly outlined in Appendix D of Kassel–Turaev [KT+95].

7.1. **Maps out of the cyclic** PaRB**.** Let $\mathbf{Op}^{+}(\mathcal{O},\mathcal{P})$ denote the set of operad morphisms $f:\mathcal{O}\to\mathcal{P}$ that fix objects, and let $\mathbf{Cyc}^{+}(\mathcal{O},\mathcal{P})$ denote the corresponding set of cyclic operad morphisms that also fix objects.

Given a cyclic operad morphism $f:\mathsf{PaRB}^{\mathrm{cyc}}\to\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}$, we can restrict f to obtain an underlying map of operads $\bar{f}:\mathsf{PaRB}\to\widehat{\mathsf{PaRB}}_{\mathbb{K}}$. This defines a natural map:

$$\mathbf{Cyc}^{+}(\mathsf{PaRB}^{\mathrm{cyc}},\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}})\xrightarrow{u}\mathbf{Op}^{+}(\mathsf{PaRB},\widehat{\mathsf{PaRB}}_{\mathbb{K}}).$$

Lemma 7.1. *The forgetful map*

$$\mathbf{Cyc}^{+}(\mathsf{PaRB}^{\mathrm{cyc}},\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}})\xrightarrow{u}\mathbf{Op}^{+}(\mathsf{PaRB},\widehat{\mathsf{PaRB}}_{\mathbb{K}})$$

is an isomorphism. That is, any object-fixing operad map $\mathsf{PaRB}\to\widehat{\mathsf{PaRB}}_{\mathbb{K}}$ *uniquely extends to an object-fixing map of cyclic operads.*

Proof. A map $\bar{f}:\mathsf{PaRB}\to\widehat{\mathsf{PaRB}}_{\mathbb{K}}$ is determined by a scalar parameter $\lambda=1+2\mu$ in $\mathbb{K}$ and an element of the prounipotent completion of the free group on two generators, $g(x_{12},x_{23})\in\mathbb{F}_2$. More precisely, by applying Theorem 3.4, $\bar{f}$ is equivalent to the assignment:

$$\bar{f}(\mu)=\mu,\quad \bar{f}(\tau)=\tau^{\lambda},\quad \bar{f}(\beta)=\beta^{\lambda},\quad \text{and}\quad \bar{f}(\alpha)=g(x_{12},x_{13})\cdot\alpha.$$

To lift $\bar{f}$ to a map $f\in\mathbf{Cyc}^{+}(\mathsf{PaRB}^{\mathrm{cyc}},\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}})$, we must check that the map defined by $\bar{f}$ commutes with the cyclic action.

Let's start with the twist generator τ. Since $z_2^{*}\cdot\tau=\tau$, we have

$$\bar{f}(z_2^{*}\cdot\tau)=\bar{f}(\tau)=\tau^{\lambda}. \tag{7.1}$$

To compute the effect of the cyclic action $z_2^{*}\cdot\tau^{\lambda}$, note that the extended symmetric group action on $\mathsf{PaRB}(1)$ is a groupoid automorphism $\sigma^{*}:\mathsf{PaRB}(1)\to\mathsf{PaRB}(1)$. Thus:

$$\sigma^{*}(\tau^{\lambda})=\underbrace{\sigma^{*}(\tau)\sigma^{*}(\tau)\cdots\sigma^{*}(\tau)}_{\lambda}=(\sigma\cdot\tau)^{\lambda},$$

for any $\sigma\in\Sigma_1^{+}$. It follows that

$$\bar{f}(z_2^{*}\cdot\tau)=z_2^{*}\cdot\tau. \tag{7.2}$$

A similar argument holds for the generator $\beta\in\mathsf{PaRB}(2)$, as

$$\bar{f}(z_3^{*}\cdot\beta)=\bar{f}((\beta^{-1})(\mathrm{id}_2\circ_2\tau^{-1}))=\bar{f}(\beta)^{-1}\bar{f}(\mathrm{id}_2\circ_2\tau^{-1})=(\beta^{-\lambda})(\mathrm{id}_2\circ_2\tau^{-\lambda}) \tag{7.3}$$

and

$$z_3^* \cdot \bar{f}(\beta) = z_3^* \cdot \beta^\lambda = (z_3^* \cdot \beta)^\lambda = ((\beta^{-1})(\mathrm{id}_2 \circ_2 \tau^{-1}))^\lambda = (\beta^{-\lambda})(\mathrm{id}_2 \circ_2 \tau^{-\lambda}). \tag{7.4}$$

To look at the cyclic action on the associator α, we recall that the pure braid group $\widehat{\mathsf{PB}}_3$ is generated by $x_{12} = \beta_1^2$, $x_{23} = \beta_2^2$ and $x_{13} = \beta_2\beta_1^2\beta_2^{-1}$ where β_1 and β_2 are the standard generators of the braid group B_3. We can write these generators as operadic composites of the generating morphisms $\mathrm{id}_2, \beta \in \mathsf{PaRB}(2)$, i.e.

$$\beta_1 = \mathrm{id}_2 \circ_1 \beta \quad \text{and} \quad \beta_2 = \mathrm{id}_2 \circ_2 \beta.$$

We first compute the cyclic action on x_{12} and x_{23}, we have

$$z_4^* \cdot x_{12} = z_4^* \cdot (\mathrm{id}_2 \circ_1 \beta^2) = z_4^* \cdot ((\mathrm{id}_2 \circ_1 \beta) \cdot (\mathrm{id}_2 \circ_1 (21)^* \beta))$$

Using the axioms of a cyclic structure on its tree presentation, we have that

$$z_4^* \cdot (\mathrm{id}_2 \circ_1 \beta) = (z_3^* \cdot \beta) \circ_2 (z_3^* \cdot \mathrm{id}_2) = (231)^* \left(\beta^{-1}(\mathrm{id}_2 \circ_2 \tau^{-1})\right) \circ_2 \mathrm{id}_2 . \tag{7.5}$$

$$z_4^* \cdot (\mathrm{id}_2 \circ_1 (21)^*\beta) = (z_3^* \cdot (21)^*\beta) \circ_2 (z_3^* \cdot \mathrm{id}_2) = (312)^* \left(((21)^*\beta^{-1})(\mathrm{id}_2 \circ_2 \tau^{-1})\right) \circ_2 (\mathrm{id}_2). \tag{7.6}$$

Note that $z_4^* \cdot (\mathrm{id}_2 \circ_1 \beta) \in \mathrm{Hom}(2(31), 3(12))$, and $z_4^* \cdot (\mathrm{id}_2 \circ_1 (12)^*\beta) \in \mathrm{Hom}((31)2, 2(31))$. Upon substituting Eq(7.5) and Eq(7.6) in $z_4^* \cdot x_{12}$, we now have

$$z_4^* \cdot x_{12} = (231)^* \tau_1^{-1} \beta_1^{-1} \beta_2^{-1} (312)^* \beta_2^{-1} \beta_1^{-1} \tau_1^{-1} = (231)^* \tau_1^{-2} (\beta_1 \beta_2^2 \beta_1)^{-1}. \tag{7.7}$$

Noting that the $z_4^* \cdot x_{12} \in \mathrm{Hom}(2(31), 2(31))$, we can also write Eq(7.7):

$$z_4^* \cdot x_{12} = (231)^* \alpha^{-1} \tau_1^{-2} (\beta_1 \beta_2^2 \beta_1)^{-1} \alpha$$

In this way, the pure braid term $\tau_1^{-2}(\beta_1\beta_2^2\beta_1)^{-1}$ is in $\mathrm{Hom}((12)3, (12)3)$. It follows that we can use relations in PB_3 to further rededuce the morphism. In particular, since element $w = x_{12}x_{13}x_{23} = (\beta_1\beta_2)^3$ generates the center of PB_3, and $wx_{23}^{-1} = x_{12}x_{13} = \beta_1^2\beta_2\beta_1^2\beta_2^{-1} = \beta_1\beta_2^2\beta_1$ we can rewrite $z_4^* \cdot x_{12}$ as:

$$z_4^* \cdot x_{12} = (231)^* \alpha^{-1} \tau_1^{-2} (x_{23} w^{-1}) \alpha. \tag{7.8}$$

Similarly, we can compute

$$z_4^* \cdot x_{23} = z_4^* \cdot \left(\alpha(\mathrm{id}_2 \circ_2 \beta^2)\alpha^{-1}\right) = (z_4^* \cdot \alpha)(z_4^* \cdot (\mathrm{id}_2 \circ_2 \beta^2))(z_4^* \cdot \alpha^{-1}). \tag{7.9}$$

Since $z_4^* \cdot (\mathrm{id}_2 \circ_2 \beta^2) = (231)^* x_{12}$ and $z_4^* \cdot \alpha^{-1} = (231)^*\alpha$, Eq(7.9) reduces to

$$z_4^* \cdot x_{23} = (231)^* \left(\alpha^{-1} x_{12} \alpha\right). \tag{7.10}$$

We can now put this all together to show $\bar{f}(z_4^* \cdot \alpha) = z_4^* \cdot \bar{f}(\alpha)$. In particular, the left-hand side becomes:

$$\bar{f}(z_4^* \cdot \alpha) = \bar{f}((231)^* \alpha^{-1}) = (231)^* \left(\alpha^{-1} \cdot g(x_{12}, x_{23})^{-1}\right) = (231)^* \left(\alpha^{-1} \cdot g(x_{23}, x_{12})\right).$$

The right-hand side, $z_4^* \cdot \bar{f}(\alpha)$, becomes:

$$z_4^* \cdot \bar{f}(\alpha) = z_4^* \cdot (g(x_{12}, x_{23})) \cdot (z_4^* \cdot \alpha) = g(z_4^* \cdot x_{12}, z_4^* \cdot x_{23}) \cdot \left((231)^* \alpha^{-1}\right).$$

Substituting from Equations (7.8) and (7.10) and using the identity

$$g(h^{-1} x_{23} h, h^{-1} x_{12} h) = h^{-1} g(x_{23}, x_{12}) h$$

for any $h \in \mathsf{B}_3$, we get:

$$(231)^* \left(\alpha^{-1} \cdot g((\tau_1^{-2} w^{-1}) x_{23}, x_{12})\right).$$

Now, we know that for any p, q and r in $\widehat{\mathsf{RB}}_3$ with $rp = pr$ and $rq = qr$, we have $g(p, q) = g(rp, q) = g(p, rq)$. Moreover, since twists always commute with pure braids and w^{-1} is central, we know that $(\tau_1^{-2} w^{-1})$ commutes with x_{23} and x_{12}, and thus the factor of $\tau_1^{-2} w^{-1}$ disappears from the last equation, and we obtain

$$g((\tau_1^{-2} w^{-1}) x_{23}, x_{12}) = g(x_{23}, x_{12}),$$

as required. □

7.1.1. *The Grothendieck-Teichmüller group.* Let $\widehat{\mathsf{F}}_2$ denote the prounipotent completion of the free group F_2 on two generators x and y over a field $\mathbb{K}$ of characteristic zero. Given an element $f \in \widehat{\mathsf{F}}_2$ and a group homomorphism $\gamma : \widehat{\mathsf{F}}_2 \to \widehat{\mathsf{G}}_{\mathbb{K}}$ into a prounipotent group G, we write $f(a,b)$ for the image $\gamma(f)$, where $\gamma(x) = a$ and $\gamma(y) = b$.

Definition 7.2 ([Dri90])**.** The $\mathbb{K}$-prounipotent Grothendieck–Teichmüller monoid $\underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}$ consists of pairs $(\lambda, f) \in \mathbb{K} \times \widehat{\mathsf{F}}_2$ satisfying:

(1) $f(x,y) = f(y,x)$,

(2) $x^{\mu} f(x,y) y^{\mu} f(y,z) z^{\mu} f(z,x) = 1$ in $\widehat{\mathsf{F}}_2$, where $xyz = 1$ and $\lambda = 2\mu + 1$,

(3) $f(x_{12}, x_{23}) f(x_{12}x_{13}, x_{24}x_{34}) f(x_{23}, x_{34}) = f(x_{13}x_{23}, x_{23}) f(x_{12}, x_{23}x_{24})$ in $\widehat{\mathsf{PB}}_4$.

The group $\widehat{\mathsf{GT}}_{\mathbb{K}}$ is defined as the set of invertible elements in $\underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}$, with group law:

$$(\lambda_1, f_1) * (\lambda_2, f_2) = \left(\lambda_1\lambda_2,\ f_1\left(f_2(x,y)^{-1} x^{\lambda_2} f_2(x,y),\ y^{\lambda_2}\right) \cdot f_2(x,y)\right).$$

The defining relations of $\underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}$ mirror the coherence axioms for braided monoidal categories. The first two conditions correspond to the two hexagon identities, while the third encodes the pentagon identity for associativity. These are precisely the coherence constraints satisfied by any braided monoidal structure. The twisting axiom of a balanced braided monoidal category imposes no additional relation in $\widehat{\mathsf{GT}}_{\mathbb{K}}$, as it is automatically preserved under the structure defined by (λ, f). The following identification, combining results of [Fre17, Theorem 11.1.7] and [BdBHR19, Proposition 7.3], makes this precise:

Proposition 7.3. *Let* $\mathrm{End}^{+}_{\mathbf{Op}}(-)$ *denote the submonoid of operad endomorphisms that fix objects. There is an isomorphism of prounipotent monoids:*

$$\mathrm{End}^{+}_{\mathbf{Op}}(\widehat{\mathsf{PaB}}_{\mathbb{K}}) \cong \mathrm{End}^{+}_{\mathbf{Op}}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}) \cong \underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}.$$

That is, each object-fixing endomorphism of the operad $\widehat{\mathsf{PaRB}}_{\mathbb{K}}$ is uniquely determined by a pair (λ, f), with $\lambda \in \mathbb{K}$ and $f \in \widehat{\mathsf{F}}_2$, by specifying the image of the generating tuple (β, τ, α) as

$$(\beta, \tau, \alpha) \mapsto (\beta^{\lambda}, \tau^{\lambda}, f(x_{12}, x_{23}) \cdot \alpha_{1,2,3}),$$

and vice versa.

We now upgrade this result to the cyclic setting. The key point is that the cyclic structure on $\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}$ does not introduce any new constraints on object-fixing automorphisms, see Lemma 7.1.

Theorem 7.4. *Let* $\mathrm{End}^{+}_{\mathbf{Cyc}}(-)$ *denote the monoid of endomorphisms of a cyclic operad that fix the object set. There is an isomorphism of monoids:*

$$\mathrm{End}^{+}_{\mathbf{Cyc}}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}) \cong \underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}.$$

Proof. Since $\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}$ is the prounipotent completion of the cyclic operad $\mathsf{PaRB}^{\mathrm{cyc}}$, any map $f : \mathsf{PaRB}^{\mathrm{cyc}} \to \widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}$ factors uniquely through the universal property of the completion:

$$\begin{tikzcd}
\mathsf{PaRB}^{\mathrm{cyc}} \arrow[r, "i"] \arrow[rr, bend right, "f"'] & \widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}} \arrow[r, "\widetilde{f}"] & \widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}.
\end{tikzcd}$$

This yields an injective map

$$\epsilon : \mathbf{Cyc}^{+}(\mathsf{PaRB}^{\mathrm{cyc}}, \widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}) \hookrightarrow \mathrm{End}^{+}_{\mathbf{Cyc}}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}).$$

To see that ϵ is surjective, note that any endomorphism $\widetilde{f} \in \mathrm{End}^{+}_{\mathbf{Cyc}}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}})$ restricts to a morphism

$$f = \widetilde{f} \circ i : \mathsf{PaRB}^{\mathrm{cyc}} \to \widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}},$$

so that $\epsilon(f) = \widetilde{f}$. Hence, we have a bijection

$$\mathbf{Cyc}^{+}(\mathsf{PaRB}^{\mathrm{cyc}}, \widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}) \cong \mathrm{End}^{+}_{\mathbf{Cyc}}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}).$$

By Lemma 7.1, any object-fixing operad map $\bar{f} : \mathsf{PaRB} \to \widehat{\mathsf{PaRB}}_{\mathbb{K}}$ uniquely lifts to a map of cyclic operads $f : \mathsf{PaRB}^{\mathrm{cyc}} \to \widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}$, giving an isomorphism

$$\mathbf{Cyc}^{+}(\mathsf{PaRB}^{\mathrm{cyc}}, \widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}}) \cong \mathbf{Op}^{+}(\mathsf{PaRB}, \widehat{\mathsf{PaRB}}_{\mathbb{K}}).$$

As in the previous proposition, the right-hand side is identified with $\mathrm{End}^{+}_{\mathbf{Op}}(\widehat{\mathsf{PaRB}}_{\mathbb{K}})$, and hence with $\widehat{\underline{\mathsf{GT}}}_{\mathbb{K}}$, via [BdBHR19, Proposition 7.7] and [Fre17, Theorem 11.1.7]. ☐

7.2. **Rational formality of the cyclic framed little discs operad.** In this section, we prove the rational formality of the cyclic operad $\mathsf{FD}_2^{\mathrm{cyc}}$ of framed little disks using the explicit cyclic structure on PaRB (Section 3.3) and the action of the prounipotent Grothendieck–Teichmüller group $\widehat{\mathsf{GT}}_{\mathbb{Q}}$.

An operad $\mathcal{O}$ is said to be *formal* if there exists a zig-zag of quasi-isomorphisms of dg-operads connecting its chain operad $\mathcal{C}_*(\mathcal{O})$ to its homology operad $H_*(\mathcal{O})$. Rational formality of certain moduli space operads has been previously established—for example, Giansiracusa and Salvatore [GS12] proved the formality of the cyclic operad of genus-zero moduli spaces. Campos, Idrissi, and Willwacher [CIW19, Theorem 3.1] later showed that $\mathsf{FD}_2^{\mathrm{cyc}}$ is also formal using cyclic graph complex techniques.

We provide an alternative proof of this result using the action of $\widehat{\mathsf{GT}}_{\mathbb{Q}}$, following a strategy inspired by Boavida–Horel–Robertson [BdBHR19] and formalized through Petersen's formality criterion [Pet14]. Specifically, we show that the morphism $\mathsf{PaRB} \to \widehat{\mathsf{PaRB}}_{\mathbb{Q}}$ lifts compatibly to the cyclic setting, yielding a map $\mathsf{PaRB}^{\mathrm{cyc}} \to \widehat{\mathsf{PaRB}}_{\mathbb{Q}}^{\mathrm{cyc}}$ that respects the $\widehat{\mathsf{GT}}_{\mathbb{Q}}$-action. Applying Petersen's criterion gives the desired formality.

Proposition 7.5 ([Pet14])**.** *Let $\mathcal{O}$ be a dg operad over a field $\mathbb{K}$ of characteristic zero. If a grading automorphism of the homology operad $H_*(\mathcal{O})$ lifts to an automorphism of $\mathcal{O}$, then $\mathcal{O}$ is formal.*

The following result can be seen as a generalisation of the formality of operad FD_2 from [BdBHR19, Theorem 9.1] to its cyclic version in the prounipotent case.

Lemma 7.6. *The cyclic operads of framed little disks FD_2^{cyc}, equivalently $\mathcal{M}^{cyc}$, is rationally formal.*

Proof. This is by definition that $\widehat{\mathsf{GT}}_{\mathbb{Q}}$ acts on PaRB^{cyc}. We know that, by taking the realization of the classifying space $|B(\mathsf{PaRB}^{cyc})|$ we have a topological operad which is homotopy equivalent to the operad FD_2^{cyc}. As such, it acts on cyclic dg-operad of singular chains $\mathcal{C}_{\bullet}(\mathsf{FD}_2^{cyc}, \mathbb{Q})$. Recall that the homology operad $H_*(\mathsf{FD}_2^{cyc})$ is the operad of Batalin-Vilkovisky algebras, denoted by BV. Moreover, BV admits cyclic structure induced by the cyclic structure of the topological operad $\mathcal{M}$, the genus zero surface operad. Since BV is generated by the commutative algebra product $[-,-]$ in arity 2 and the operator Δ in arity 1, it is enough to check the $\widehat{\mathsf{GT}}_{\mathbb{Q}}$ action on the homology groups $H_0(\mathsf{FD}_2^{cyc}(2))$ and $H_1(\mathsf{FD}_2^{cyc}(1))$. $\widehat{\mathsf{GT}}_{\mathbb{Q}}$ action is trivial on $H_0(\mathsf{FD}_2(2))$ due to the assignment $\beta \longrightarrow \beta^{\lambda}$. The action extends to $H_0(\mathsf{FD}_2^{cyc}(2))$, since $z_3 \cdot \beta^{\lambda} = (z_3 \cdot \beta)^{\lambda}$, See Lemma 7.1. There is a map

$$\mathbb{Q}^{\times} \longrightarrow \mathrm{Aut}(H_*(\mathsf{FD}_2^{cyc}))$$

that determines an automorphism $\gamma_{\lambda} \in \mathrm{Aut}(H_n(FD))$ given by multiplication λ^n, for any $\lambda \in \mathbb{Q}^{\times}$. The prounipotent completion of $\mathbb{Z}$ over the field $\mathbb{K}$ of characteristic 0 is $\mathbb{K}$, therefore we have $\widehat{\mathsf{PRB}(1)}_{\mathbb{Q}} \cong \widehat{\mathbb{Z}}_{\mathbb{Q}} \cong \mathbb{Q}$. Since the cyclic action on $\mathsf{PRB}(1)$ is trivial, it follows that $\widehat{\mathsf{GT}}_{\mathbb{Q}}$ acts on $H_1(\mathsf{FD}_2^{cyc}(1))$ via $\widehat{\mathsf{GT}}_{\mathbb{Q}} \longrightarrow \mathbb{Q}^{\times} \longrightarrow \mathrm{Aut}(H_1(\mathsf{FD}_2^{cyc}(1)))$, where the first map sends the pair (λ, f) to λ. Now, using the fact that the cyclotomic character $\widehat{\mathsf{GT}}_{\mathbb{Q}} \longrightarrow \mathbb{Q}^{\times}$ is surjective, and Peterson's formality criteria in [Pet14] proves the claim. ☐

7.3. **Homotopy Automorphisms of PaRB^{cyc}.** We denote I to be the groupoid completion of $\{0 < 1\}$. Let $f, g : \mathcal{P} \to \mathcal{Q}$ be operad maps in groupoids. A homotopy between f and g is defined by the existence of a path object $\mathcal{Q}^I$ and a map $H : \mathcal{P} \to \mathcal{Q}^I$ such that the composition with the evaluation maps $d_0, d_1 : \mathcal{Q}^I \to \mathcal{Q}$ gives f and g, in particular

$$d_0 \circ H = f, \quad d_1 \circ H = g,$$

Since the forgetful functor $u : \mathbf{Cyc}(\mathbf{Gr}) \to \mathbf{Op}(\mathbf{Gr})$ admits both right and left adjoints, the path object Q^I in $\mathbf{Op}(\mathbf{Gr})$ lifts a path object in $u_!(Q^I) = E^I$ in $\mathbf{Cyc}$. Given two maps in f' and g' in $\mathbf{Cyc}(\mathbf{Gr})$ such that $u(f') = f$ and $u(g') = g$ and a homotopy $H : f \simeq g$, one obtains H', a homotopy lifting of H, by applying the left adjoint $u_!$. Write $\mathrm{HoEnd}(\mathcal{P})$ for the monoid of homotopy endomorphisms of a (cyclic) operad in groupoids $\mathcal{P}$.

Proposition 7.7. *The composition*

$$\mathrm{End}^+_{\mathbf{Op}}(\mathsf{PaRB}^{cyc}) \longrightarrow \mathrm{End}_{\mathbf{Op}}(\mathsf{PaRB}^{cyc}) \longrightarrow \mathrm{HoEnd}_{\mathbf{Op}}(\mathsf{PaRB}^{cyc})$$

is an isomorphism.

Proof. The only morphism from PaB to T is the trivial one, here we write T for the operad of twists, see [BdBHR19, Definition 7.5, Lemma 7.6]. The operad map $\mathsf{PaB} \longrightarrow \mathsf{T}$ factors through PaRB, where PaB injects into PaRB as a suboperad and the map $\mathsf{PaRB} \longrightarrow \mathsf{T}$ only remembers the twists. This assembles these operads in a short exact sequence

$$0 \longrightarrow \mathsf{PaB} \longrightarrow \mathsf{PaRB} \longrightarrow \mathsf{T} \longrightarrow 0$$

Furthermore, the adjoint to the forgetful functor from cyclic operads to operads provides a map from PaRB to PaRB^{cyc} and the map $\mathsf{PaRB}^{cyc} \longrightarrow \mathsf{T}$ sends a morphism to its number of twists. Thus, any morphism in $\mathrm{End}^+_{\mathbf{Op}}(\mathsf{PaRB}^{cyc})$ preserves $\mathsf{PaB} \subset \mathsf{PaRB}^{cyc}$.

Now, we want to show that $\mathrm{End}^+_{\mathbf{Op}}(\mathsf{PaB})$ is isomorphic to $\mathrm{End}^+_{\mathbf{Op}}(\mathsf{PaRB}^{cyc})$. The argument is verbatim to [BdBHR19, Proposition 7.7 (2)], which shows $\mathrm{End}^+_{\mathbf{Op}}(\mathsf{PaB}) \cong \mathrm{End}^+_{\mathbf{Op}}(\mathsf{PaRB})$. It is enough to check the compatibility with cyclic action of Definition 3.5 on the relation

$$\tau \circ \mathrm{id}_2 = \beta \cdot (21)^* \beta \cdot (\mathrm{id}_2 \circ_1 \tau) \cdot (\mathrm{id}_2 \circ_2 \tau)$$

in $\mathrm{Hom}_{\mathsf{PaRB}}(12, 12)$. The left side is

$$z_3^* \cdot (\tau \circ_1 \mathrm{id}_2) = z_3^* \cdot \mathrm{id}_2 \circ_2 z_2 \cdot \tau = (0, 0, \tau).$$

One computes $z_3^* \cdot (\mathrm{id}_2 \circ_2 \tau) = (2\tau, \tau, \tau)$, and $z_3^* \cdot (\mathrm{id}_2 \circ_2 \tau) = (0, \tau, 0)$, then the right side reduces to

$$z_3^* \cdot (\beta \cdot (21)^* \beta \cdot (\mathrm{id}_2 \circ_1 \tau) \cdot (\mathrm{id}_2 \circ_2 \tau)) = (-2\beta_1, -2\tau, 0) \cdot (2\tau, \tau, \tau) \cdot (0, \tau, 0) = (0, 0, \tau).$$

The last equality uses $\tau = \beta_1$ as pointed out in [BdBHR19, Proposition 7.7 (2)].Therefore, we deduce

$$\mathrm{End}^+_{\mathbf{Op}}(\mathsf{PaB}) \cong \mathrm{End}^+_{\mathbf{Op}}(\mathsf{PaRB}) \cong \mathrm{End}^+_{\mathbf{Op}}(\mathsf{PaRB}^{cyc}).$$

There is a commutative diagram

$$\begin{tikzcd}
\mathrm{End}^+_{\mathbf{Op}}(\mathsf{PaB}) \arrow[r] & \mathrm{HoEnd}^+_{\mathbf{Op}}(\mathsf{PaB}) \arrow[r, "f"] & \mathrm{HoEnd}^+_{\mathbf{Op}}(\mathsf{PB}(3)) \\
\mathrm{End}^+_{\mathbf{Op}}(\mathsf{PaRB}) \arrow[r] \arrow[u, "\cong"'] & \mathrm{HoEnd}^+_{\mathbf{Op}}(\mathsf{PaRB}) \arrow[r, "g"] & \mathrm{HoEnd}^+_{\mathbf{Op}}(\mathsf{PRB}(3)) \arrow[u, "k"] \arrow[d, equal] \\
\mathrm{End}^+_{\mathbf{Op}}(\mathsf{PaRB}^{cyc}) \arrow[r] \arrow[u, "\cong"'] & \mathrm{HoEnd}^+_{\mathbf{Op}}(\mathsf{PaRB}^{cyc}) \arrow[r, "h"] & \mathrm{HoEnd}^+_{\mathbf{Op}}(\mathsf{PRB}(3))
\end{tikzcd}$$

The top horizontal composition is injective due to [Hor17, Proposition 7.7]. By [BdBHR19, Proposition 7.8], the top square is commutative and the middle horizontal composite is injective. Note that for any operad $\mathcal{O}$ in groupoids, an endomorphism of $\mathcal{O}$ gives an endomorphism of $\mathcal{O}(n)$ by restriction in arity n that can further be restricted to an endomorphism of $\mathrm{ob}(\mathcal{O}(n))$ of objects $\mathcal{O}(n)$. Therefore, the map

$p : \mathrm{End}^{+}_{\mathbf{Op}}(\mathsf{PaRB}^{cyc}) \longrightarrow \mathrm{End}^{+}(\mathsf{PRB}(3))$ factors through $\mathrm{End}(\mathsf{PaRB}^{cyc}(3))$ by restriction maps, similar to f and g. We obtain a map $h : \mathrm{HoEnd}^{+}_{\mathbf{Op}}(\mathsf{PaRB}^{cyc}) \longrightarrow \mathrm{HoEnd}^{+}(\mathsf{PRB}(3))$, and the bottom square is commutative.

Using the isomorphism $\mathrm{End}^{+}_{\mathbf{Op}}(\mathsf{PaRB}) \cong \mathrm{End}^{+}_{\mathbf{Op}}(\mathsf{PaRB}^{cyc})$ and the injectivity of the map

$$\mathrm{End}^{+}_{\mathbf{Op}}(\mathsf{PaRB}) \longrightarrow \mathrm{HoEnd}^{+}_{\mathbf{Op}}(\mathsf{PaRB}),$$

we deduce that the map $\mathrm{End}^{+}_{\mathbf{Op}}(\mathsf{PaRB}^{cyc}) \longrightarrow \mathrm{HoEnd}^{+}_{\mathbf{Op}}(\mathsf{PaRB}^{cyc})$ is injective. Finally the composition

$$\mathrm{End}^{+}_{\mathbf{Op}}(\mathsf{PaRB}^{cyc}) \longrightarrow \mathrm{HoEnd}^{+}_{\mathbf{Op}}(\mathsf{PaRB}^{cyc}) \longrightarrow \mathrm{HoEnd}_{\mathbf{Op}}(\mathsf{PaRB}^{cyc})$$

is injective. Since the objects of PaB and PaRB^{cyc} are the same, the surjectivity of the composition map follows immediately from [Hor17, Theorem 7.8]. ☐

Corollary 7.8. *There exist isomorphisms of prounipotent monoids*

$$\mathrm{End}^{+}_{\mathbf{Op}}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc}) \cong \mathrm{HoEnd}_{\mathbf{Op}}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc}) \cong \underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}.$$

By taking invertible elements, we have an isomorphism of prounipotent groups

$$\mathrm{HoAut}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc}) \cong \widehat{\mathsf{GT}}_{\mathbb{K}}.$$

7.4. **GT action on Tangles.** We saw in Proposition 5.7 that there is a close relationship between Turaev's tangle category [Tur89] (Definition 5.1) and the metric prop associated with the cyclic operad $\mathsf{PaRB}^{\mathrm{cyc}}$ (Definition 2.12). In this section, we explain how the action of the Grothendieck–Teichmüller group $\widehat{\mathsf{GT}}_{\mathbb{K}}$ on the cyclic operad $\mathsf{PaRB}^{\mathrm{cyc}}$ lifts to the associated metric prop via the envelope construction. As a consequence, we obtain a $\widehat{\mathsf{GT}}_{\mathbb{K}}$-action on the category of q-tangles, extending the known $\widehat{\mathsf{GT}}_{\mathbb{K}}$-action on parenthesized braids. This construction is informed by the Galois actions on tangles described in [KT+95, Appendix D].

Recall that the envelope construction $\mathrm{Env} : \mathbf{Op}(\mathbf{E}) \to \mathbf{Prop}(\mathbf{E})$ assigns to an operad $\mathcal{O}$ its associated prop $\mathrm{Env}(\mathcal{O})$. This functor is left adjoint to the forgetful functor $u : \mathbf{Prop}(\mathbf{E}) \to \mathbf{Op}(\mathbf{E})$ which, in particular, defines a natural bijection

$$\mathbf{Op}(\mathsf{PaRB}, u(\mathrm{Env}(\mathsf{PaRB}))) \cong \mathbf{Prop}(\mathrm{Env}(\mathsf{PaRB}), \mathrm{Env}(\mathsf{PaRB})). \tag{7.11}$$

An object-fixing morphism $f : \mathrm{Env}(\mathsf{PaRB}) \to \mathrm{Env}(\mathsf{PaRB})$ is a strict symmetric monoidal functor such that the underlying function on object sequences is the identity. Let $\mathrm{End}^{+}_{\mathbf{Prop}}(-)$ denote the monoid of endomorphisms of a prop that fix the object set.

Lemma 7.9. *There is an isomorphism of prounipotent monoids:*

$$\underline{\widehat{\mathsf{GT}}}_{\mathbb{K}} \cong \mathrm{End}^{+}_{\mathbf{Prop}}(\mathrm{Env}(\widehat{\mathsf{PaRB}}_{\mathbb{K}})).$$

Proof. The corollary follows from the following sequence of isomorphisms

$$\mathrm{End}^{+}_{\mathbf{Op}}(\widehat{\mathsf{PaRB}}_{\mathbb{K}})) \cong \mathbf{Op}^{+}(\mathsf{PaRB}, u(\mathrm{Env}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}))) \cong \mathbf{Prop}^{+}(\mathrm{Env}(\mathsf{PaRB}), \mathrm{Env}(\widehat{\mathsf{PaRB}}_{\mathbb{K}})) \cong \mathrm{End}^{+}_{\mathbf{Prop}}(\mathrm{Env}(\widehat{\mathsf{PaRB}}_{\mathbb{K}})).$$

☐

The last corollary implies an isomorphism of prounipotent groups

$$\widehat{\mathsf{GT}}_{\mathbb{K}} \cong \mathrm{Aut}^{+}_{\mathbf{Prop}}\left(\mathrm{Env}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\mathrm{cyc}})\right),$$

where $\mathrm{Aut}^{+}_{\mathbf{Prop}}$ denotes the group of object-fixing automorphisms of props. To connect this action to tangles, we consider the metric prop generated by the cyclic operad $\mathsf{PaRB}^{\mathrm{cyc}}$.

The construction of the metric prop $\Pi(\mathsf{PaRB}^{\mathrm{cyc}})$ formally adjoins dualizing morphisms to $\mathrm{Env}(\mathsf{PaRB})$: a cap$d \in \Pi(\mathsf{PaRB}^{\mathrm{cyc}})(2,0)$ and a cup $b \in \Pi(\mathsf{PaRB}^{\mathrm{cyc}})(0,2)$, satisfying the zig-zag (triangle) identities. In the presence of a nontrivial associator $\alpha \in \mathsf{PaRB}(3)$, care must be taken when defining these dual morphisms to ensure compatibility with associativity. For example, the following composites must recover the identity morphism in $\Pi(\mathsf{PaRB}^{\mathrm{cyc}})(1,1)$:

$$1 \cong 0 \otimes 1 \xrightarrow{b \otimes \mathrm{id}_1} (1 \otimes 1) \otimes 1 \xrightarrow{\alpha} 1 \otimes (1 \otimes 1) \xrightarrow{\mathrm{id}_1 \otimes d} 1 \otimes 0 \cong 1 \tag{7.12}$$

and

$$1 \cong 1 \otimes 0 \xrightarrow{\mathrm{id}_1 \otimes b} 1 \otimes (1 \otimes 1) \xrightarrow{\alpha^{-1}} (1 \otimes 1) \otimes 1 \xrightarrow{d \otimes \mathrm{id}_1} 0 \otimes 1 \cong 1. \tag{7.13}$$

These coherence relations will not necessarily compose to the identity map in $\Pi(\mathsf{PaRB}^{cyc})(1,1)$ unless we carefully define d and b. We saw in Section 4 that each choice of Drinfeld associator Φ gives an isomorphism of cyclic operads $\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc} \cong \widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{cyc}$.

Proposition 7.10. *The action described above defines a* $\widehat{\mathsf{GT}}$*-action on* $\widehat{\Pi(\mathsf{PaRB}^{cyc})}$.

Proof. The Grothendieck-Teichmüller group $\underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}$ acts on $\widehat{\mathsf{PaRB}}_{\mathbb{K}}$ by sending the pair $(\lambda, f) \in \underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}$ to a unique isomorphism $F : \widehat{\mathsf{PaRB}}_{\mathbb{K}} \longrightarrow \widehat{\mathsf{PaRB}}_{\mathbb{K}}$ defined by the association

$$F(\mu) = \mu, \quad F(\beta) = \beta^{\lambda}, \quad F(\tau) = \tau^{\lambda}, \quad F(\alpha) = f(x_{12}, x_{23}) \cdot \alpha.$$

F induces an isomorphism of prop $\mathrm{Env}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}) \longrightarrow \mathrm{Env}(\widehat{\mathsf{PaRB}}_{\mathbb{K}})$, which as a result gives a $\underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}$ action on the prop $\mathrm{Env}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}) \cong \mathrm{Env}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc}) \cong \mathrm{Env}(\widehat{\mathsf{PaRB}^{cyc}})$ through F.

We define an isomorphism $F' : \widehat{(\Pi(\mathsf{PaRB}^{cyc}))} \longrightarrow \widehat{(\Pi(\mathsf{PaRB}^{cyc}))}$ by extending the map F to the duality pairing (d, b), using the assignment as in [KT+95, Appendix D], as follows

$$F'(b) = b, \quad F(d) = (\mathrm{id} \otimes \nu) d,$$

where $\nu = ((b \otimes \mathrm{id}) f(x_{12}, x_{23})(\mathrm{id} \otimes d))^{-1}$. The image of the pairing (d, b) must satisfy the relations

$$(\mathrm{id} \otimes b) \alpha (d \otimes \mathrm{id}) \cong \mathrm{id}\,. \tag{7.14}$$

Applying F' on the left side of (7.14) computes as follows

$$(\mathrm{id} \otimes b) f(x_{12}, x_{23}) \alpha (((\mathrm{id} \otimes \nu) d) \otimes \mathrm{id})$$

Now substituting ν gives $(\mathrm{id} \otimes b) f(x_{12}, x_{23}) \alpha (((\mathrm{id} \otimes ((b \otimes \mathrm{id}) f(x_{12}, x_{23})(\mathrm{id} \otimes d))^{-1}) d) \otimes \mathrm{id})$, after cancelling the term $f(x_{12}, x_{23})$ and using the relation $(b \otimes \mathrm{id})(\mathrm{id} \otimes d) = \mathrm{id}$, the equation simplifies to $(\mathrm{id} \otimes b) \alpha (d \otimes \mathrm{id})$, which is isomorphic to id. The image of the right side of (7.14) is id because $\widehat{\mathsf{GT}}_{\mathbb{K}}$ action on single-string tangle is trivial, therefore the image of the relation (7.14) holds under F'.

□

Remark 7.11. Kassel-Turaev in [KT+95, Appendix D] also gives an equivalent assignment on duality pairing (d, b), We note that the same assignemt on the prop map F' on the duality pairing (d, b), that is, as follows

$$F'(b) = b, \quad F'(d) = (\rho \otimes \mathrm{id}) d,$$

where $\rho = \big((\mathrm{id} \otimes b) f(x_{12}, x_{23})^{-1} (d \otimes \mathrm{id})\big)^{-1}$, gives a $\underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}$- action. In this case, the duality pairing must satisfy the relation $(b \otimes \mathrm{id}) \alpha^{-1} (\mathrm{id} \otimes d) \cong \mathrm{id}$ under the $\underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}$ action. Applying F' on $(b \otimes \mathrm{id}) \alpha^{-1} (\mathrm{id} \otimes d)$ gives

$$(b \otimes \mathrm{id}) \alpha^{-1} f(x_{12}, x_{23})^{-1} (\mathrm{id} \otimes ((\rho \otimes \mathrm{id}) d))$$

That simplifies to ω as follows

$$
\begin{aligned}
&(b\otimes \mathrm{id})\alpha^{-1}f(x_{12},x_{23})^{-1}(\mathrm{id}\otimes((\rho\otimes \mathrm{id})d))\\
&=(b\otimes \mathrm{id})\alpha^{-1}f(x_{12},x_{23})^{-1}(\mathrm{id}\otimes(\left(\left((\mathrm{id}\otimes b)f(x_{12},x_{23})^{-1}(d\otimes \mathrm{id})\right)^{-1}\otimes \mathrm{id}\right)d))\\
&=(b\otimes \mathrm{id})\alpha^{-1}(\mathrm{id}\otimes(\left(((\mathrm{id}\otimes b)(d\otimes \mathrm{id})\right)^{-1}\otimes \mathrm{id})d))\\
&=(b\otimes \mathrm{id})\alpha^{-1}(\mathrm{id}\otimes((\mathrm{id}^{-1}\otimes \mathrm{id})d))\\
&=(b\otimes \mathrm{id})\alpha^{-1}(\mathrm{id}\otimes d))\\
&=\mathrm{id}
\end{aligned}
\tag{7.15}
$$

The second equality substitutes ρ, third one obtain by cancelling $f(x_{12},x_{23})$ term, the fourth uses the identity $(\mathrm{id}\otimes b)(d\otimes \mathrm{id})=\mathrm{id}$. The desired relation holds under F'.

8. Graded Grothendieck-Teichmüller group $\widehat{\mathsf{GRT}}_{\mathbb{K}}$

8.1. **Graded Grothendieck-Teichmüller group - GRT.** For $\mathfrak{f}_2$ and any complete filtered algebra M with elements $a,b\in M$, an algebra morphism $\gamma:\hat{U}(\mathfrak{f}_2)\longrightarrow M$ from the universal enveloping algebra $\hat{U}(\mathfrak{f}_2)$ of $\mathfrak{f}_2$ such that $\gamma(x)=a$ and $\gamma(y)=b$. We write $f(a,b)$ for the image of $\gamma(f)$ of f under γ.

Definition 8.1 ([Dri90])**.** The graded Grothendieck- Teichmüller group $\widehat{\mathsf{GRT}}_{\mathbb{K}}$ is the semi-direct product $\widehat{\mathsf{GRT}}_1\rtimes\mathbb{K}^\times$, where the set $\widehat{\mathsf{GRT}}_1$ consists of elements $\Phi\in\exp(\mathfrak{f}_2)\subset\exp(\mathfrak{t}_3)$ satisfying the following relations,

$$\Phi(x,y)=\Phi(y,x)^{-1},\ \textit{in}\ \exp(\mathfrak{t}_3), \tag{I}$$

$$\Phi(x,y)\Phi(y,z)\Phi(z,x)=1,\ \text{whenever}\ x+y+z=0,\ \textit{in}\ \exp(\mathfrak{t}_3), \tag{H}$$

$$\Phi(t_{12},t_{23})\Phi(t_{12}+t_{13},t_{24}+t_{34})\Phi(t_{23},t_{34})=\Phi(t_{13}+t_{23},t_{34})\Phi(t_{12},t_{23}+t_{24}),\ \textit{in}\ \exp(\mathfrak{t}_4). \tag{P}$$

The group structure is given by, for any two element p_1 and p_2, by

$$(\Phi_1*\Phi_2)(x,y)=\Phi_1(\Phi_2(x,y)^{-1}x\Phi_2(x,y),y)\Phi_2(x,y). \tag{8.1}$$

The action of $\mathbb{K}^\times$ on $\widehat{\mathsf{GRT}}_1$, given by $\Phi(\lambda^{-1}x,\lambda^{-1}y)$ for $\lambda\in\mathbb{K}^\times$, induces a semidirect product $\widehat{\mathsf{GRT}}_1\rtimes\mathbb{K}^\times$.

The following Theorem characterizes the graded Grothendieck-Teichmüller group as an object-fixing automorphism group of the cyclic operad $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$. As a consequence, this defines the action of $\widehat{\mathsf{GRT}}_{\mathbb{K}}$ on the cyclic chord diagrams as pointed out in [Wil24, Proposition 5.2].

Theorem 8.2. $\widehat{\mathsf{GRT}}_{\mathbb{K}}\cong\mathrm{Aut}^+_{\mathbf{Cyc}}(\widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{\,cyc})$.

Proof. Since the operad morhism $\widehat{\mathsf{RCD}}_{\mathbb{K}}(n)\xrightarrow{\sim}\widehat{\mathsf{PaRCD}}_{\mathbb{K}}(n)$ is a categorical equivalence, Proposition 4.6. We can use [Fre17, Proposition 6.1.10] that gives a unique lifting of the operad map $\widehat{\mathsf{PaRB}}_{\mathbb{K}}\to\widehat{\mathsf{RCD}}_{\mathbb{K}}$ to $\widehat{\mathsf{PaRB}}_{\mathbb{K}}\to\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$, therefore we have $\mathrm{Iso}^+(\widehat{\mathsf{PaRB}}_{\mathbb{K}},\widehat{\mathsf{RCD}}_{\mathbb{K}})=\mathrm{Iso}^+(\widehat{\mathsf{PaRB}}_{\mathbb{K}},\widehat{\mathsf{PaRCD}}_{\mathbb{K}})$. From Lemma 6.3 and Lemma 6.6, these maps are isomorphisms of cyclic operads, so we have

$$\mathrm{Iso}^+(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\,cyc},\widehat{\mathsf{RCD}}_{\mathbb{K}}^{\,cyc})=\mathrm{Iso}^+(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\,cyc},\widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{\,cyc}).$$

We already know from Corollary 7.8 that $\widehat{\mathsf{GT}}_{\mathbb{K}}\cong\mathrm{Aut}^+_{\mathbf{Cyc}}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{\,cyc})$, and $\widehat{\mathsf{GRT}}_{\mathbb{K}}\cong\mathrm{Aut}^+_{\mathbf{Op}}(\widehat{\mathsf{PaCD}}_{\mathbb{K}})\cong\mathrm{Aut}^+_{\mathbf{Op}}(\widehat{\mathsf{PaRCD}}_{\mathbb{K}})$, see, for example, [Fre17, Theorem 10.3.10] and [BN98]. The second isomorphism follows from the fact that the framing on the chord diagrams do not add any new relation. The result now follows from $\mathrm{Aut}^+_{\mathbf{Op}}(\widehat{\mathsf{PaRCD}}_{\mathbb{K}})=\mathrm{Aut}^+_{\mathbf{Cyc}}(\widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{\,cyc})$ using Lemma 6.5.

□

Unlike PaB and PaRB, there is no known explicit presentation of PaCD, and hence PaRCD. Therefore, we next show that the defining relations of $\widehat{\mathsf{GRT}}_{\mathbb{K}}$ are preserved under the cyclic action z^*.

Proposition 8.3. *The cyclic action z^* preserves the relations involution* (I) *and the hexagon* (H) *relations.*

Proof. For the involution relation (I), it is

$$z^*\Phi(t_{12},t_{23}) = \Phi(-t_{12}-t_{22}-t_{32},t_{23}) = \Phi(-t_{12}-t_{32},t_{23}) = \Phi(t_{31},t_{23}) = \Phi(t_{23},t_{31})^{-1} = z^*\Phi(t_{23},t_{12})^{-1},$$

where we use the fact that t_{ii} and $t_{12}+t_{23}+t_{31}$ are central elements of $\mathfrak{ft}_3$. For the hexagon relation (H), we first note the following:

- $z^*\Phi(t_{12},t_{23}) = \Phi(t_{31},t_{23})$.
- $z^*\Phi(t_{13},t_{12}) = \Phi(-t_{13}-t_{23}-t_{33},t_{31}) = \Phi(-t_{13}-t_{23},t_{31}) = \Phi(t_{12},t_{31})$.
- $z^*\Phi(t_{23},t_{13}) = \Phi(t_{23},-t_{13}-t_{23}-t_{33}) = \Phi(t_{23},t_{12})$.

Now, applying the action z^* on

$$\Phi(t_{31},t_{12})\Phi(t_{23},t_{31})\Phi(t_{12},t_{23}) = 1 \tag{8.2}$$

We get

$$\begin{aligned} z^*\left(\Phi(t_{31},t_{12})\Phi(t_{23},t_{31})\Phi(t_{12},t_{23})\right) &= \Phi(t_{12},t_{13})\Phi(t_{23},t_{12})\Phi(t_{31},t_{23}) \\ &= \Phi(t_{13},t_{12})^{-1}\Phi(t_{12},t_{23})^{-1}\Phi(t_{23},t_{31})^{-1} \\ &= \Phi(t_{13},t_{12})^{-1}\left(\Phi(t_{23},t_{31})\Phi(t_{12},t_{23})\right)^{-1} \\ &= \Phi(t_{13},t_{12})^{-1}\Phi(t_{31},t_{12}) \\ &= 1 = z^*(1). \end{aligned}$$

Here the second equality uses (I) and the fourth uses (8.2). □

Definition 8.4. The *framed sphere braid* Lie algebra $\mathfrak{f}\mathfrak{B}_n$ is a degree completed free Lie algebra generated by symbols $\{X_{ij} = X_{ji}, 1 \le i \le j \le n\}$, with relations

$$\begin{aligned} [X_{ij}, X_{kl}] &= 0 \quad \text{for } \{i,j\}\cap\{k,l\} = \emptyset, \\ [X_{ij}, X_{ki}+X_{kj}] &= 0 \quad \text{for distinct } i,j,k, \\ [X_{ij}, X_{kk}] &= 0 \quad \text{for any } i,j, \text{ and } k, \\ \sum_{j=1}^{n} X_{ij} &= 0 \quad \text{for } 1 \le i \le j \le n. \end{aligned} \tag{8.3}$$

The Lie algebra $\mathfrak{f}\mathfrak{B}_n$ is a framed version of $\mathfrak{B}_n$ that appeared in Ihara [Iha94, Section 5.3]. There is a natural surjection $\mathfrak{ft}_n \to \mathfrak{f}\mathfrak{B}_{n+1}$ that sends $t_{ij} \mapsto X_{ij}$ for all $1 \le i \le j \le n$. The surjection map induces a morphism $U(\mathfrak{ft}_n) \to U(\mathfrak{f}\mathfrak{B}_{n+1})$.

Proposition 8.5. *The cyclic action z^* preserves the pentagon relation* (P).

Proof. Using [Fur10, Lemma 5], the pentagon equation (P) is equivalent to the following in $\hat{U}(\mathfrak{f}\mathfrak{B}_5)$:

$$\Phi(X_{12},X_{23})\cdot\Phi(X_{34},X_{45})\cdot\Phi(X_{51},X_{12})\cdot\Phi(X_{23},X_{34})\cdot\Phi(X_{45},X_{51}) = 1 \tag{8.4}$$

Equivalently,

$$\Phi(X_{12},X_{51})\cdot\Phi(X_{45},X_{34}) = \Phi(X_{23},X_{34})\cdot\Phi(X_{45},X_{51})\cdot\Phi(X_{12},X_{23}) \tag{8.5}$$

Applying the cyclic action on the left side of (P), we get

$$\Phi(-t_{12}-t_{22}-t_{32}-t_{42}, t_{23}+t_{24})\cdot\Phi(-t_{13}-t_{23}-t_{33}-t_{43}+t_{23}, t_{34}),$$

which is equivalent to

$$\Phi(-X_{12}-X_{22}-X_{32}-X_{42}, X_{23}+X_{24})\cdot\Phi(-X_{13}-X_{23}-X_{33}-X_{43}+X_{23}, X_{34})$$

in $\mathfrak{f}\mathfrak{B}_5$. Now, using the relations $\sum_{j=1}^{5} X_{ij} = 0$ gives the following

$$\Phi(X_{52}, X_{23}+X_{24})\cdot\Phi(X_{23}+X_{53}, X_{34}). \tag{8.6}$$

We note that the first term of (8.6) is

$$\Phi(X_{52}, X_{23}+X_{24}) = \Phi(X_{52}, -X_{21}-X_{25}-X_{22}) = \Phi(X_{52}, X_{15})$$

because $[X_{52}, X_{51} + X_{52} + X_{12}] = 0 = [X_{15}, X_{15} + X_{25} + X_{12}]$ and $[X_{15}, X_{22}] = 0 = [X_{52}, X_{22}]$.

Similarly, the second term of (8.6) is

$$\Phi(X_{23} + X_{53}, X_{34}) = \Phi(-X_{13} - X_{33} - X_{43}, X_{34}) = \Phi(X_{41}, X_{34})$$

because $[X_{41}, X_{13} + X_{33} + X_{34} + X_{14}] = 0 = [X_{41}, X_{13} + X_{33} + X_{34} + X_{14}]$.

Finally, the cyclic action on the left side of (P) reduces to

$$\Phi(X_{52}, X_{15}) \cdot \Phi(X_{41}, X_{34}). \tag{8.7}$$

Now, we apply the cyclic action on the right side of (P). We get

$$\Phi(t_{23}, t_{34}) \cdot \Phi(-t_{12} - t_{22} - t_{32} - t_{42} - t_{13} - t_{23} - t_{33} - t_{43}, t_{24} + t_{34}) \cdot \Phi(-t_{12} - t_{22} - t_{32} - t_{42}, t_{23})$$

Similarly, taking the image of the last equation in $\mathfrak{f}\mathfrak{B}_5$ and using the relation $\sum_{j=1}^{5} X_{ij} = 0$ gives

$$\Phi(X_{23}, X_{34}) \cdot \Phi(X_{52} + X_{53}, X_{24} + X_{34}) \cdot \Phi(X_{52}, X_{23}). \tag{8.8}$$

We now observe that the middle term of (8.8) reduces to

$$\begin{aligned} \Phi(X_{52} + X_{53}, X_{24} + X_{34}) &\overset{(1)}{=} \Phi(-X_{51} - X_{54} - X_{55}, -X_{14} - X_{44} - X_{54}) \\ &\overset{(2)}{=} \Phi(-X_{51} - X_{54} - X_{55}, X_{51}) \\ &\overset{(3)}{=} \Phi(X_{41}, X_{15}). \end{aligned}$$

Here equality (1) uses the relation from $\mathfrak{f}\mathfrak{B}_5$, (2) follows from $[X_{51}, X_{15} + X_{41} + X_{54} + X_{44}] = 0 = [X_{54} + X_{15}, X_{15} + X_{41} + X_{54} + +X_{44}]$ and for the last equality (3), we use the vanishing Lie brackets $[X_{15}, X_{41} + X_{54} + X_{15} + X_{55}] = 0 = [X_{41}, X_{41} + X_{54} + X_{15} + X_{55}]$

Finally, the cyclic action on the right hand side of (P) is

$$\Phi(X_{23}, X_{34}) \cdot \Phi(X_{41}, X_{15}) \cdot \Phi(X_{52}, X_{23}). \tag{8.9}$$

Using the following permuted pentagon obtained by applying permutation $(1\,5)$ on (8.5),

$$\Phi(X_{52}, X_{15}) \cdot \Phi(X_{41}, X_{34}) = \Phi(X_{23}, X_{34}) \cdot \Phi(X_{41}, X_{15}) \cdot \Phi(X_{52}, X_{23}) \tag{8.10}$$

We claim that the equations (8.7) and (8.9) are equal. □

Recall that there are free and transitive actions of $\widehat{\mathsf{GT}}_{\mathbb{K}}$ from left $\widehat{\mathsf{GRT}}_{\mathbb{K}}$ from right on the set of Drinfeld associators $\mathsf{Assoc}_{\mathbb{K}}$that commute with each other (see [Dri90]), thus the triple $(\widehat{\mathsf{GT}}_{\mathbb{K}}, \mathsf{Assoc}_{\mathbb{K}}, \widehat{\mathsf{GRT}}_{\mathbb{K}})$ forms a bitorsor.

Proposition 8.6. *The triple*

$$(\mathrm{Aut}^{+}(\widehat{\mathsf{PaRB}_{\mathbb{K}}^{cyc}}), \mathrm{Iso}^{+}(\widehat{\mathsf{PaRB}_{\mathbb{K}}^{cyc}} \longrightarrow \widehat{\mathsf{PaRCD}_{\mathbb{K}}^{cyc}}), \mathrm{Aut}^{+}(\widehat{\mathsf{PaRCD}_{\mathbb{K}}^{cyc}}))$$

forms a bitorsor.

Proof. The operadic identifications of the triple $(\widehat{\mathsf{GT}}_{\mathbb{K}}, \mathsf{Assoc}_{\mathbb{K}}, \widehat{\mathsf{GRT}}_{\mathbb{K}})$, as in [Fre17], also give a bitorsor triple

$$(\mathrm{Aut}^{+}(\widehat{\mathsf{PaRB}}_{\mathbb{K}}), \mathrm{Iso}^{+}(\widehat{\mathsf{PaRB}}_{\mathbb{K}} \longrightarrow \widehat{\mathsf{PaRCD}}_{\mathbb{K}}), \mathrm{Aut}^{+}(\widehat{\mathsf{PaRCD}}_{\mathbb{K}})). \tag{8.11}$$

Proposition 6.9, Corollary 7.8, and Theorem 8.2 implies that the bitorsor triple (8.11) uniquely corresponds to the bitorsor triple

$$(\mathrm{Aut}^{+}(\widehat{\mathsf{PaRB}_{\mathbb{K}}^{cyc}}), \mathrm{Iso}^{+}(\widehat{\mathsf{PaRB}_{\mathbb{K}}^{cyc}} \longrightarrow \widehat{\mathsf{PaRCD}_{\mathbb{K}}^{cyc}}), \mathrm{Aut}^{+}(\widehat{\mathsf{PaRCD}_{\mathbb{K}}^{cyc}})). \tag{8.12}$$

□

Moreover, this identification lifts to a bitorsor bijection

$$(\widehat{\mathsf{GT}}_{\mathbb{K}}, \mathsf{Assoc}_{\mathbb{K}}, \widehat{\mathsf{GRT}}_{\mathbb{K}}) \leftrightarrow (\mathrm{Aut}^{+}(\widehat{\mathsf{PaRB}_{\mathbb{K}}^{cyc}}), \mathrm{Iso}^{+}(\widehat{\mathsf{PaRB}_{\mathbb{K}}^{cyc}} \longrightarrow \widehat{\mathsf{PaRCD}_{\mathbb{K}}^{cyc}}), \mathrm{Aut}^{+}(\widehat{\mathsf{PaRCD}_{\mathbb{K}}^{cyc}})).$$

8.2. **GRT action on Chord diagrams.** We saw in Section 7.4 that there is an action of $\widehat{\mathsf{GT}}$ on the metric prop generated by the cyclic operad PaRB^{cyc} and from Section 4 we have cyclic operad isomorphisms $\widehat{\mathsf{PaRB}}_{\mathbb{K}}^{cyc} \longrightarrow \widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{cyc}$ are in bijection with Drinfeld associators. With these two results, we construct a graded Grothendieck-Teichmüller group $\widehat{\mathsf{GRT}}_{\mathbb{K}}$ action on the chord diagrams.

The following results are straightforward applications of Proposition 7.9, and Proposition 2.20 on $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$.

Corollary 8.7. *There is an isomorphism of prounipotent groups*

$$\widehat{\mathsf{GRT}}_{\mathbb{K}} \cong \mathrm{Aut}^{+}_{\mathbf{Prop}}(\mathrm{Env}(\widehat{\mathsf{PaRCD}}_{\mathbb{K}})) \cong \mathrm{Aut}^{+}_{\mathbf{Prop}}(\mathrm{Env}\widehat{(\mathsf{PaRCD})}).$$

□

Corollary 8.8. *There is an isomorphism of prounipotent metric prop associated to the cyclic operad* PaRCD^{cyc}

$$\widehat{(\Pi(\mathsf{PaRCD}^{cyc}))}_{\mathbb{K}} \cong \Pi(\widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{cyc}).$$

□

A Drinfeld associator Φ is a group-like element of the non-commutative formal power series in the free associative algebra $\hat{U}(\mathfrak{f}_2) = \mathbb{K}\langle\langle X, Y\rangle\rangle$ generated by two variables X and Y. In the category $\mathbf{A}(\mathbb{K})$, one can define the associative constraint $A' : (X \otimes Y) \otimes Z \longrightarrow X \otimes (Y \otimes Z)$ defined through the isomorphism A of $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$. This does not satisfy the zigzag identity with the associative constraint, however, it satisfies the following relation

$$(\mathrm{id} \otimes b)A(d \otimes \mathrm{id}) = \omega^{-1}, \tag{8.13}$$

where ω is a special case of the Kontsevich integral of the unknot. It is independent of the choice of Drinfeld associators similar to [LM96, Theorem 8].

In the category of parenthesized framed tangles, the composition of the cap $\cap : (+-) \to \varnothing$ and cup $\cup : \varnothing \to (+-)$ morphisms is identified with the 0–framed unknot $\cap \circ \cup := U$. As we saw in Theorem 6.10, a given Drinfeld associator Φ we can define an isomorphism between the completed category of q-tangles and the category of chord diagrams

$$Z_{\Phi} : q\mathbf{T} \to \mathbf{A}(\mathbb{K})[[\hbar]]$$

which has the property that

$$Z_{\Phi}(U) = Z_{\Phi}(\cap) \circ Z_{\Phi}(\cup) =: \omega,$$

where the distinguished group–like series ω is the Kontesevich Integral of the unknot ([BN97], [LM96], [KT^{+}95]). This value, also known as the wheeling element in ([BNLT03]; [CDM12]), is known to be independent of the choice of the associator [LM96, Theorem 8]. Under the isomorphism $\Pi(\mathsf{PaRB}^{\mathrm{cyc}}) \to q\mathbf{T}$ the cap is mapped to the evaluation map $d \in \Pi(\mathsf{PaRB}^{\mathrm{cyc}})(2,0)$ and the cup is mapped to the coevaluation map $b \in \Pi(\mathsf{PaRB}^{\mathrm{cyc}})(0,2)$. We can put this together to show that, for every Drinfeld associator Φ, we have the following commutative diagram of isomorphisms

$$\begin{tikzcd}
q\mathbf{T} \arrow[r, "Z_{\Phi}"] \arrow[d, "\cong"] & \mathbf{A}(\mathbb{K}) \arrow[d, "\cong"] \\
\Pi(\mathsf{PaRB}^{\mathrm{cyc}}) \arrow[r, "\varphi_{\Phi}"] & \Pi(\mathsf{PaRCD}^{cyc}),
\end{tikzcd}$$

with $\varphi_{\Phi}(d \circ b) = Z_{\Phi}(U) = \omega$ and this value is independent of the choice of associator.

Proposition 8.9. *The action described above defines a* $\widehat{\mathsf{GRT}}$*-action on* $\widehat{\Pi(\mathsf{PaRCD}^{cyc})}$.

Proof. The graded Grothendieck-Teichmüller group $\widehat{\mathsf{GRT}}_{\mathbb{K}}$ acts on $\widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ by sending the pair $(\lambda, f) \in \underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}$ to a unique isomorphism $G : \widehat{\mathsf{PaRCD}}_{\mathbb{K}} \to \widehat{\mathsf{PaRCD}}_{\mathbb{K}}$ defined by the assignment described in Lemma 6.5. G induces an isomorphism of the prop $\mathrm{Env}(\widehat{\mathsf{PaRCD}}_{\mathbb{K}}) \longrightarrow \mathrm{Env}(\widehat{\mathsf{PaRCD}}_{\mathbb{K}})$, which as a result gives a $\widehat{\mathsf{GRT}}_{\mathbb{K}}$ action on the prop $\mathrm{Env}(\widehat{\mathsf{PaRCD}}_{\mathbb{K}}) \cong \mathrm{Env}(\widehat{\mathsf{PaRCD}}_{\mathbb{K}}^{cyc}) \cong \mathrm{Env}(\widehat{\mathsf{PaRCD}^{cyc}})$ through G, Corollary 8.7.

We define an isomorphism $G' : \widehat{(\Pi(\mathsf{PaRCD}^{cyc}))} \longrightarrow \widehat{(\Pi(\mathsf{PaRCD}^{cyc}))}$ by extending the map F to the duality pairing (d, b), using the similar assignment as described in Proposition 7.10 as follows

$$G'(b) = b, \quad G'(d) = (\mathrm{id} \otimes \nu)d,$$

where $\nu = ((b \otimes \mathrm{id})\Phi(t_{12}, t_{23})(\mathrm{id} \otimes d))^{-1}$. The image of the pairing (d, b) must satisfy the relations (8.13). Applying F' on the left side of (8.13) computes as follows

$$(\mathrm{id} \otimes b)\Phi(t_{12}, t_{23}) \cdot A(((\mathrm{id} \otimes \nu)d) \otimes \mathrm{id})$$

Now substituting ν gives $(\mathrm{id} \otimes b)\Phi(t_{12}, t_{23}) \cdot A(((\mathrm{id} \otimes ((b \otimes \mathrm{id})\Phi(t_{12}, t_{23})(\mathrm{id} \otimes d))^{-1})d) \otimes \mathrm{id})$, after cancelling the term $\Phi(t_{12}, t_{23})$ and using the relation $(b \otimes \mathrm{id})(\mathrm{id} \otimes d) = \mathrm{id}$, the equation simplifies to $(\mathrm{id} \otimes b)A(d \otimes \mathrm{id})$, which equals to ω^{-1}. The image on the right side of (8.13) is also ω^{-1}, since $\underline{\widehat{\mathsf{GT}}}_{\mathbb{K}}$-action on unknot is trivial (this is a consequence of [Kon99, Theorem 8] and equivalently follows from [Fur20, Theorem A]), we arrive at a trivial $\widehat{\mathsf{GRT}}_{\mathbb{K}}$ action on ω^{-1}. Thus, the image of the relation (8.13) holds under G'.

□

Similar to Remark 7.11, the prop map G' on the duality pairing (d, b), that is, as follows

$$G'(b) = b, \quad G'(d) = (\rho \otimes \mathrm{id})d,$$

where $\rho = \big((\mathrm{id} \otimes b)\Phi(x_{12}, x_{23})^{-1}(d \otimes \mathrm{id})\big)^{-1}$, gives a $\widehat{\mathsf{GRT}}_{\mathbb{K}}$- action. In this case, the duality pairing must satisfy the relation $(b \otimes \mathrm{id})A^{-1}(\mathrm{id} \otimes d) = \omega$ under G'. Applying G' on $(b \otimes \mathrm{id})A^{-1}(\mathrm{id} \otimes d)$ gives

$$(b \otimes \mathrm{id})A^{-1}\Phi(x_{12}, x_{23})^{-1}(\mathrm{id} \otimes ((\rho \otimes \mathrm{id})d))$$

That simplifies to ω as follows

$$\begin{aligned}
&(b \otimes \mathrm{id})A^{-1}\Phi(x_{12}, x_{23})^{-1}(\mathrm{id} \otimes ((\rho \otimes \mathrm{id})d)) \\
&= (b \otimes \mathrm{id})A^{-1}\Phi(x_{12}, x_{23})^{-1}(\mathrm{id} \otimes ((((\mathrm{id} \otimes b)\Phi(x_{12}, x_{23})^{-1}(d \otimes \mathrm{id}))^{-1} \otimes \mathrm{id})d)) \\
&= (b \otimes \mathrm{id})A^{-1}(\mathrm{id} \otimes ((((\mathrm{id} \otimes b)(d \otimes \mathrm{id}))^{-1} \otimes \mathrm{id})d)) \\
&= (b \otimes \mathrm{id})A^{-1}(\mathrm{id} \otimes ((\mathrm{id}^{-1} \otimes \mathrm{id})d)) \\
&= (b \otimes \mathrm{id})A^{-1}(\mathrm{id} \otimes d)) \\
&= \omega
\end{aligned} \tag{8.14}$$

The desired relation holds under G' using the same arguments of Proposition 8.9.

Appendix A. Adjunctions between operads and cyclic operads

Operads and cyclic operads are themselves algebras over a coloured operad (e.g. [BM07, Example 1.56], [BBCL+22, Definition 2.9], [Luk10, Section 1.6.3], [DCH22, Appendix A]). This point of view allows us to use the considerable amount of pre-existing homotopical machinery developed to study algebras over operads in our context. For completeness, given any non-empty set $\mathfrak{C}$, a $\mathfrak{C}$-*colored symmetric sequence* is a family of objects $\mathcal{P} := \{\mathcal{P}(c; c_1, \ldots, c_k)\}_{k \geq 0}$ in $\mathbf{E}$, where $(c; c_1, \ldots, c_k)$ ranges over every list of colors in $\mathfrak{C}$ together with a map $\sigma^* : \mathcal{P}(c; c_1, \ldots, c_k) \to \mathcal{P}(c; c_{\sigma(1)}, \ldots, c_{\sigma(k)})$ for each $\sigma \in \Sigma_k$. A $\mathfrak{C}$-*colored operad* is a $\mathfrak{C}$-colored symmetric sequence $\mathcal{P}$ together with a family of partial composition maps

$$\circ_i : \mathcal{P}(c; c_1, \ldots, c_k) \times \mathcal{P}(d; d_1, \ldots, d_j) \to \mathcal{P}(c; c_1, \ldots, c_{i-1}, d_1, \ldots, d_j, c_{i+1} \ldots, c_k)$$

defined only when $c_i = d$, together with an element $\iota_c \in \mathcal{P}(c; c)$ for each $c \in \mathfrak{C}$, which satisfies unit, equivariance and associativity conditions. For more details see, for example, [BM07, Definition 1.1].

Remark A.1. When the color set is $\mathfrak{C} = \{*\}$, a $\mathfrak{C}$–colored operad is a one-colored operad. In this paper we will refer to both operads and colored operads as "operads", only mentioning the color set when necessary.

An *algebra* over a ($\mathfrak{C}$–colored) operad $\mathcal{P}$ is a collection of objects $\{X(c)\}_{c\in\mathfrak{C}}$ in $\mathbf{E}$ together with evaluation maps

$$\rho\colon \mathcal{P}(c; c_1, \dots, c_k) \times X(c_1) \times \cdots \times X(c_k) \longrightarrow X(c)$$

satisfying appropriate associativity, unit and equivariance conditions, see e.g. [BM07, Definition 1.2]. The category of $\mathcal{P}$-algebras in $\mathbf{E}$ is denoted $\mathrm{Alg}_{\mathbf{E}}(\mathcal{P})$.

A.1. **Operad for cyclic operads.** The operad for cyclic operads is an $\mathbb{N}$-colored operad $\mathcal{C}$ which we briefly describe below.

Definition A.2. Let (T, λ, ℓ) be an ordered, labeled tree. For a vertex $v \in V(T)$ with arity $|\operatorname{nb}(v)| = m$ and an ordered, labeled tree T' with $|\partial(T')| = m$, the *substitution* $T \bullet_v T'$ is obtained by removing the vertex v from T and identifying the neighborhood v with the boundary of T'.

Unless otherwise specified, we will require that all of our tree substitutions respect the labeling. To make this precise requires introducing a bit of notion, but one can find this explained in full detail (with examples) for rooted trees in [BBCL$^+$22]. The following definition is adapted from [Luk10, Section 1.6.3], but see also [DCH22, Appendix A].

Definition A.3. The *operad of cyclic operads* $\mathcal{C}$ is the $\mathbb{N}$–colored operad, for which

$$\mathcal{C}(n; m_1, \dots, m_k)$$

is the discrete space whose elements are strict isomorphism classes of *labeled*, ordered trees (T, λ, ℓ) where T is a tree with $|V(T)| = k$ vertices and $|\partial(T)| = n$ together with bijections

$$\lambda\colon \{1, \dots, k\} \to V(T) \quad \text{and} \quad \ell\colon \{0, 1, \dots, n-1\} \to \partial(T),$$

such that the vertex $\lambda(i)$ has arity m_i for each $1 \le i \le k$. The composition operation

$$\begin{aligned} \mathcal{C}(n; m_1, \dots, m_k) \times \mathcal{C}(m_i; b_1, \dots, b_l) &\xrightarrow{\circ_i} \mathcal{C}(n; m_1, \dots, b_1, \dots, b_l, \dots, m_k) \\ ((T, \lambda, \ell), (T', \lambda', \ell')) &\longmapsto (T, \lambda, \ell) \circ_i (T', \lambda', \ell') \end{aligned}$$

is induced by tree substitution that is compatible with the labeling (8). The unit for this composition, for the color n, is the element of $\mathcal{C}(n; n)$ represented by the n-star $\bigstar_n$ equipped with the canonical left-right labeling. The symmetric group Σ_k acts on $(T, \lambda, \ell) \in \mathcal{C}(n; m_1, \dots, m_k)$ by precomposition on the labeling λ of the vertices $V(T)$.

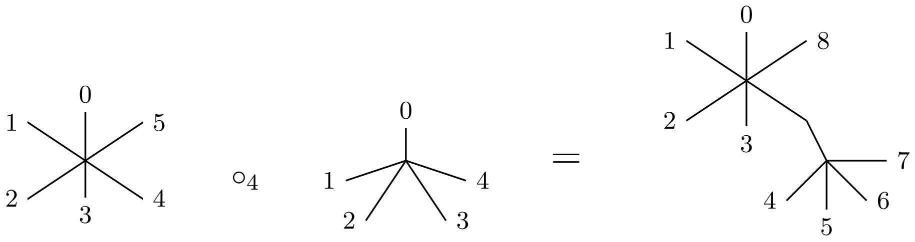


Figure 8. For trees (T, λ, ℓ) with $\lambda : \{1\} \to V(T)$, $\ell : \{0, 1, \dots, 5\} \to \partial(T)$ and (T', λ', ℓ') with $\lambda' : \{1\} \to V(T')$, $\ell' : \{0, 1, \dots, 4\} \to \partial(T')$, their composition in arity 4 is given by $(T, \lambda, \ell) \circ_4 (T', \lambda', \ell') = (T \circ_4 T', \lambda'', \ell'')$, with $\lambda'' : \{1, 2\} \to V(T \circ_4 T')$, $\ell'' : \{0, 1, \dots, 8\} \to \partial(T \circ_4 T')$.

Lemma A.4. *The category of algebras over the coloured operad $\mathcal{C}$ is isomorphic to the category of cyclic operads, i.e.*

$$\mathrm{Alg}_{\mathbf{E}}(\mathcal{C}) \cong \mathbf{Cyc}(\mathbf{E}).$$

A.2. **Operads and cyclic operads.** One can define the operad for operads, $\mathcal{O}$ (e.g. [BM07, Example 1.56], [BBCL$^+$22, Definition 2.9]), as a sub-operad of $\mathcal{C}$. Suppose that T is a genus zero ordered graph with at least one boundary element. Then, by virtue of our trees being planar, we can define a unique edge flow in the direction of the first element of $\partial(T)$, and call such an element the *root* of the tree T. This allows us to define a partial order on the edges of T in which the root is the minimal element. Using this order, we define the root of a vertex as the element of $\mathrm{nb}(v)$ closest to the root of the tree T. We call such a graph a *rooted tree* when, for each v, the root of v is also the minimal element of $\mathrm{nb}(v)$.

Lemma A.5. *Let $\mathcal{O}$ be the $\mathbb{N}$-coloured sub-operad of $\mathcal{C}$ whose operations are spanned by the rooted trees. Then the category of $\mathcal{O}$-algebras in $\mathbf{E}$ is isomorphic to the category of operads in $\mathbf{E}$,*

$$\mathrm{Alg}_{\mathbf{E}}(\mathcal{O}) \cong \mathbf{Op}(\mathbf{E}).$$

The inclusion of operads $u : \mathcal{O} \to \mathcal{C}$ induces a Quillen adjunction

$$\begin{tikzcd} \mathbf{Op}(\mathbf{E}) \arrow[r, shift left, "u_!"] & \mathbf{Cyc}(\mathbf{E}). \arrow[l, shift left, "u^*"] \end{tikzcd} \tag{A.1}$$

The left adjoint $u_! : \mathbf{Op}(\mathbf{E}) \to \mathbf{Cyc}(\mathbf{E})$ is the "cyclic envelope" of an operad. Full details on the Quillen adjunction (in this form) can be found in [DCH22]. An explicit description of the left adjoint can be found in [DCH21, Section 3.1] or [War19, Section 9].

References

[AT10] Anton Alekseev and Charles Torossian, *Kontsevich deformation quantization and flat connections*, Comm. Math. Phys. **300** (2010), no. 1, 47–64. MR 2725182

[BBCL$^+$22] L. Basualdo Bonatto, S. Chettih, A. Linton, S. Raynor, M. Robertson, and N. Wahl, *An infinity operad of normalized cacti*, Topology Appl. **316** (2022), Paper No. 108107, 54. MR 4438952

[BdBHR19] P. Boavida de Brito, G. Horel, and M. Robertson, *Operads of genus zero curves and the grothendieck–teichmüller group*, Geometry & Topology **23** (2019), 299–346.

[Bel79] G. V. Belyĭ, *Galois extensions of a maximal cyclotomic field*, Izv. Akad. Nauk SSSR Ser. Mat. **43** (1979), no. 2, 267–276, 479. MR 534593

[BM07] C. Berger and I. Moerdijk, *Resolution of coloured operads and rectification of homotopy algebras*, Categories in algebra, geometry and mathematical physics, Contemp. Math., vol. 431, Amer. Math. Soc., Providence, RI, 2007, pp. 31–58. MR 2342815

[BN97] Dror Bar-Natan, *Non-associative tangles*, Geometric topology (Athens, GA, 1993), AMS/IP Stud. Adv. Math., vol. 2.1, Amer. Math. Soc., Providence, RI, 1997, pp. 139–183. MR 1470726

[BN98] D. Bar-Natan, *On associators and the Grothendieck-Teichmuller group. I*, Selecta Math. (N.S.) **4** (1998), no. 2, 183–212. MR 1669949

[BNLT03] Dror Bar-Natan, Thang T. Q. Le, and Dylan P. Thurston, *Two applications of elementary knot theory to Lie algebras and Vassiliev invariants*, Geom. Topol. **7** (2003), 1–31. MR 1988280

[Bro12] Francis Brown, *Mixed Tate motives over $\mathbb{Z}$*, Ann. of Math. (2) **175** (2012), no. 2, 949–976. MR 2993755

[Bud08] R. Budney, *The operad of framed discs is cyclic*, J. Pure Appl. Algebra **212** (2008), no. 1, 193–196. MR 2355044

[Car93] Pierre Cartier, *Construction combinatoire des invariants de Vassiliev-Kontsevich des nœuds*, R.C.P. 25, Vol. 45 (French) (Strasbourg, 1992–1993), Prépubl. Inst. Rech. Math. Av., vol. 1993/42, Univ. Louis Pasteur, Strasbourg, 1993, pp. 1–10. MR 1331627

[CDM12] S. Chmutov, S. Duzhin, and J. Mostovoy, *Introduction to Vassiliev knot invariants*, Cambridge University Press, Cambridge, 2012. MR 2962302

[CiL24] Damien Calaque and Victor Roca i Lucio, *Associators from an operadic point of view*, 2024.

[CIW19] R. Campos, N. Idrissi, and T. Willwacher, *Configuration spaces of surfaces*, arXiv preprint arXiv:1911.12281 (2019), –.

[DCH21] G. C. Drummond-Cole and P. Hackney, *Dwyer-Kan homotopy theory for cyclic operads*, Proc. Edinb. Math. Soc. (2) **64** (2021), no. 1, 29–58. MR 4249838

[DCH22] ———, *Coextension of scalars in operad theory*, Math. Z. **301** (2022), no. 1, 275–314. MR 4405651

[Dri90] V. G. Drinfel'd, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Algebra i Analiz **2** (1990), no. 4, 149–181. MR 1080203

[DSPS20] Christopher Douglas, Christopher Schommer-Pries, and Noah Snyder, *Dualizable tensor categories*, vol. 268, American Mathematical Society, 2020.

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik, *Tensor categories*, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, Providence, RI, 2015. MR 3242743

[ENO05] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik, *On fusion categories*, Ann. of Math. (2) **162** (2005), no. 2, 581–642. MR 2183279

[Fre17] B. Fresse, *Homotopy of operads and Grothendieck-Teichmüller groups. Part 1*, Mathematical Surveys and Monographs, vol. 217, American Mathematical Society, Providence, RI, 2017, The algebraic theory and its topological background. MR 3643404

[Fur10] Hidekazu Furusho, *Pentagon and hexagon equations*, Ann. of Math. (2) **171** (2010), no. 1, 545–556. MR 2630048

[Fur17] ———, *Galois action on knots I: Action of the absolute Galois group*, Quantum Topol. **8** (2017), no. 2, 295–360. MR 3659492

[Fur20] ———, *Galois action on knots II: Proalgebraic string links and knots*, Periods in quantum field theory and arithmetic, Springer Proc. Math. Stat., vol. 314, Springer, Cham, [2020] ©2020, pp. 541–591. MR 4100690

[Get94] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. **159** (1994), no. 2, 265–285. MR 1256989

[GK95] E. Getzler and M. M. Kapranov, *Cyclic operads and cyclic homology*, Geometry, topology, & physics, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995, pp. 167–201. MR 1358617

[GM80] Mark Goresky and Robert MacPherson, *Intersection homology theory*, Topology **19** (1980), no. 2, 135–162. MR 572580

[Gon18] Martin Gonzalez, *Contributions to the theory of kzb associators*, Ph.D. thesis, Sorbonne Université, 2018.

[Gro97] Alexandre Grothendieck, *Esquisse d'un programme*, Geometric Galois actions, 1, London Math. Soc. Lecture Note Ser., vol. 242, Cambridge Univ. Press, Cambridge, 1997, With an English translation on pp. 243–283, pp. 5–48. MR 1483107

[GS12] Jeffrey Giansiracusa and Paolo Salvatore, *Cyclic operad formality for compactified moduli spaces of genus zero surfaces*, Trans. Amer. Math. Soc. **364** (2012), no. 11, 5881–5911. MR 2946936

[Hor17] G. Horel, *Profinite completion of operads and the Grothendieck-Teichmüller group*, Adv. Math. **321** (2017), 326–390. MR 3715714

[HR15] Philip Hackney and Marcy Robertson, *On the category of props*, Appl. Categ. Structures **23** (2015), no. 4, 543–573. MR 3367131

[HV02] V. Hinich and A. Vaintrob, *Cyclic operads and algebra of chord diagrams*, Selecta Math. (N.S.) **8** (2002), no. 2, 237–282. MR 1913297

[Iha94] Y. Ihara, *On the embedding of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *into* $\widehat{\mathrm{GT}}$, The Grothendieck theory of dessins d'enfants (Luminy, 1993), London Math. Soc. Lecture Note Ser., vol. 200, Cambridge Univ. Press, Cambridge, 1994, With an appendix: the action of the absolute Galois group on the moduli space of spheres with four marked points by Michel Emsalem and Pierre Lochak, pp. 289–321. MR 1305402

[JM19] David M. Jackson and Iain Moffatt, *An introduction to quantum and Vassiliev knot invariants*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, Cham, 2019. MR 3931694

[JS93] André Joyal and Ross Street, *Braided tensor categories*, Adv. Math. **102** (1993), no. 1, 20–78. MR 1250465

[Kas12] Christian Kassel, *Quantum groups*, vol. 155, Springer Science & Business Media, 2012.

[Kon99] Maxim Kontsevich, *Operads and motives in deformation quantization*, vol. 48, 1999, Moshé Flato (1937–1998), pp. 35–72. MR 1718044

[Kon17] ———, *Derived grothendieck–teichmüller group and graph complexes [after t. willwacher]*, Séminaire Bourbaki **1126** (2017), –.

[KT$^+$95] Christian Kassel, Vladimir Turaev, et al., *Chord diagram invariants of tangles and graphs*, Tech. report, Citeseer, 1995.

[LM96] Tu Quoc Thang Le and Jun Murakami, *The universal Vassiliev-Kontsevich invariant for framed oriented links*, Compositio Math. **102** (1996), no. 1, 41–64. MR 1394520

[Luk10] A. Lukács, *Cyclic operads, dendroidal structures, higher categories*, Utrecht University, 2010.

[Mac71] Saunders MacLane, *Categories for the working mathematician*, Graduate Texts in Mathematics, vol. Vol. 5, Springer-Verlag, New York-Berlin, 1971. MR 354798

[ML63] Saunders Mac Lane, *Natural associativity and commutativity*, Rice Univ. Stud. **49** (1963), no. 4, 28–46. MR 170925

[MSS02] M. Markl, S. Shnider, and J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002. MR 1898414

[MW22] L. Müller and L. Woike, *Cyclic Framed Little Disks Algebras, Grothendieck–Verdier Duality And Handlebody Group Representations*, The Quarterly Journal of Mathematics (2022), –, haac015.

[Pet14] Dan Petersen, *Minimal models, GT-action and formality of the little disk operad*, Selecta Math. (N.S.) **20** (2014), no. 3, 817–822. MR 3217461

[Phi22] Emma Phillips, *Operads, symmetric monoidal categories, and localizations*, Ph.D. thesis, University of New Hampshire, 2022.

[RT90] N. Yu. Reshetikhin and V. G. Turaev, *Ribbon graphs and their invariants derived from quantum groups*, Comm. Math. Phys. **127** (1990), no. 1, 1–26. MR 1036112

[Sev09] Pavol Severa, *Formality of the chain operad of framed little disks*, arXiv preprint arXiv:0902.3576 (2009), –.

[Shu94] Mei Chee Shum, *Tortile tensor categories*, Journal of Pure and Applied Algebra **93** (1994), no. 1, 57–110 (en).

[SW03] Paolo Salvatore and Nathalie Wahl, *Framed discs operads and Batalin-Vilkovisky algebras*, Q. J. Math. **54** (2003), no. 2, 213–231. MR 1989873

[Tur89] V. G. Turaev, *Operator invariants of tangles, and R-matrices*, Izv. Akad. Nauk SSSR Ser. Mat. **53** (1989), no. 5, 1073–1107, 1135. MR 1024455

[TY98] Daisuke Tambara and Shigeru Yamagami, *Tensor categories with fusion rules of self-duality for finite abelian groups*, J. Algebra **209** (1998), no. 2, 692–707. MR 1659954

[Wah01] N. Wahl, *Ribbon braids and related operads*, Ph.D. thesis, Oxford University, 2001.

Bibliography

[AET10] A. Alekseev, B. Enriquez, and C. Torossian, *Drinfeld associators, braid groups and explicit solutions of the Kashiwara-Vergne equations*, Publ. Math. Inst. Hautes Études Sci. **-** (2010), no. 112, 143–189. MR 2737979

[AKKN17] Anton Alekseev, Nariya Kawazumi, Yusuke Kuno, and Florian Naef, *Higher genus Kashiwara-Vergne problems and the Goldman-Turaev Lie bialgebra*, C. R. Math. Acad. Sci. Paris **355** (2017), no. 2, 123–127. MR 3612698

[AKKN18] ———, *The Goldman-Turaev Lie bialgebra in genus zero and the Kashiwara-Vergne problem*, Adv. Math. **326** (2018), 1–53. MR 3758425

[AM06] A. Alekseev and E. Meinrenken, *On the Kashiwara-Vergne conjecture*, Invent. Math. **164** (2006), no. 3, 615–634. MR 2221133

[AT12] Anton Alekseev and Charles Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators*, Ann. of Math. (2) **175** (2012), no. 2, 415–463. MR 2877064

[BdBHR19] P. Boavida de Brito, G. Horel, and M. Robertson, *Operads of genus zero curves and the grothendieck–teichmüller group*, Geometry & Topology **23** (2019), no. 1, 299–346.

[BFL$^+$99] Xavier Buff, Jérôme Fehrenbach, Pierre Lochak, Leila Schneps, and Pierre Vogel, *Espaces de modules des courbes, groupes modulaires et théorie des champs*, Panoramas et Synthèses [Panoramas and Syntheses], vol. 7, Société Mathématique de France, Paris, 1999. MR 1691792

[BN98] D. Bar-Natan, *On associators and the Grothendieck-Teichmuller group. I*, Selecta Math. (N.S.) **4** (1998), no. 2, 183–212. MR 1669949

[BND17] Dror Bar-Natan and Zsuzsanna Dancso, *Finite type invariants of w-knotted objects II: tangles, foams and the Kashiwara–Vergne problem*, Mathematische Annalen **367** (2017), no. 3-4, 1517–1586 (en).

[Bro13] Adrien Brochier, *Cyclotomic associators and finite type invariants for tangles in the solid torus*, Algebraic and Geometric Topology **13** (2013), no. 6, 3365–3409.

[Bud08] R. Budney, *The operad of framed discs is cyclic*, J. Pure Appl. Algebra **212** (2008), no. 1, 193–196. MR 2355044

[BV73] J. M. Boardman and R. M. Vogt, *Homotopy invariant algebraic structures on topological spaces*, Lecture Notes in Mathematics, vol. Vol. 347, Springer-Verlag, Berlin-New York, 1973. MR 420609

[CG20] Damien Calaque and Martin Gonzalez, *A moperadic approach to cyclotomic associators*, 2020.

[CiL24] Damien Calaque and Victor Roca i Lucio, *Associators from an operadic point of view*, 2024.

[CIW19] R. Campos, N. Idrissi, and T. Willwacher, *Configuration spaces of surfaces*, arXiv preprint arXiv:1911.12281 (2019), –.

[CV05] Ralph L Cohen and Alexander A Voronov, *Notes on string topology*, arXiv preprint math/0503625 (2005), –.

[DHLA$^+$25] Zsuzsanna Dancso, Iva Halacheva, Guillaume Laplante-Anfossi, Marcy Robertson, and Chandan Singh, *Genus zero kashiwara-vergne solutions from braids*, arXiv preprint arXiv:2507.16243 (2025), –.

[DHLAR23] Zsuzsanna Dancso, Iva Halacheva, Guillaume Laplante-Anfossi, and Marcy Robertson, *Kashiwara-vergne solutions degree by degree*, arXiv preprint arXiv:2310.20420 (2023), –.

[DHR21] Zsuzsanna Dancso, Iva Halacheva, and Marcy Robertson, *Circuit algebras are wheeled props*, Journal of Pure and Applied Algebra **225** (2021), no. 12, 106767 (en).

[DHR22] Zsuzsanna Dancso, Tamara Hogan, and Marcy Robertson, *A knot-theoretic approach to comparing the grothendieck-teichm\"{u} ller and kashiwara-vergne groups*, arXiv preprint arXiv:2211.11370 (2022), –.

[DHR23] Zsuzsanna Dancso, Iva Halacheva, and Marcy Robertson, *A topological characterisation of the Kashiwara-Vergne groups*, Trans. Amer. Math. Soc. **376** (2023), no. 5, 3265–3317. MR 4577332

[Dri90] V. G. Drinfel'd, *On quasitriangular quasi-Hopf algebras and on a group that is closely connected with* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, Algebra i Analiz **2** (1990), no. 4, 149–181. MR 1080203

[Duf77] Michel Duflo, *Opérateurs différentiels bi-invariants sur un groupe de Lie*, Ann. Sci. École Norm. Sup. (4) **10** (1977), no. 2, 265–288. MR 444841

[Enr07] Benjamin Enriquez, *Quasi-reflection algebras and cyclotomic associators*, Selecta Math. (N.S.) **13** (2007), no. 3, 391–463. MR 2383601

[Fre09] Benoit Fresse, *Modules over operads and functors*, Lecture Notes in Mathematics, vol. 1967, Springer-Verlag, Berlin, 2009. MR 2494775

[Fre17a] B. Fresse, *Homotopy of operads and Grothendieck-Teichmüller groups. Part 1*, Mathematical Surveys and Monographs, vol. 217, American Mathematical Society, Providence, RI, 2017, The algebraic theory and its topological background. MR 3643404

[Fre17b] Benoit Fresse, *Homotopy of operads and grothendieck-teichmüller groups 2: The applications of (rational) homotopy theory methods*, American Mathematical Society, 2017.

[Fur17] Hidekazu Furusho, *Galois action on knots I: Action of the absolute Galois group*, Quantum Topol. **8** (2017), no. 2, 295–360. MR 3659492

[Fur20] ______, *Galois action on knots II: Proalgebraic string links and knots*, Periods in quantum field theory and arithmetic, Springer Proc. Math. Stat., vol. 314, Springer, Cham, [2020] ©2020, pp. 541–591. MR 4100690

[Get94] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys. **159** (1994), no. 2, 265–285. MR 1256989

[Gro97] Alexandre Grothendieck, *Esquisse d'un programme*, Geometric Galois actions, 1, London Math. Soc. Lecture Note Ser., vol. 242, Cambridge Univ. Press, Cambridge, 1997, With an English translation on pp. 243–283, pp. 5–48. MR 1483107

[GS12] Jeffrey Giansiracusa and Paolo Salvatore, *Cyclic operad formality for compactified moduli spaces of genus zero surfaces*, Trans. Amer. Math. Soc. **364** (2012), no. 11, 5881–5911. MR 2946936

[HV02] V. Hinich and A. Vaintrob, *Cyclic operads and algebra of chord diagrams*, Selecta Math. (N.S.) **8** (2002), no. 2, 237–282. MR 1913297

[Iha94] Y. Ihara, *On the embedding of* $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ *into* $\widehat{\mathrm{GT}}$, The Grothendieck theory of dessins d'enfants (Luminy, 1993), London Math. Soc. Lecture Note Ser., vol. 200, Cambridge Univ. Press, Cambridge, 1994, With an appendix: the action of the absolute Galois group on the moduli space of spheres with four marked points by Michel Emsalem and Pierre Lochak, pp. 289–321. MR 1305402

[Kon17] Maxim Kontsevich, *Derived grothendieck–teichmüller group and graph complexes [after t. willwacher]*, Séminaire Bourbaki **1126** (2017), –.

[KSV95] Takashi Kimura, Jim Stasheff, and Alexander A. Voronov, *On operad structures of moduli spaces and string theory*, Comm. Math. Phys. **171** (1995), no. 1, 1–25. MR 1341693

[KT98] Christian Kassel and Vladimir Turaev, *Chord diagram invariants of tangles and graphs*, Duke Math. J. **92** (1998), no. 3, 497–552. MR 1620522

[KV78] Masaki Kashiwara and Michèle Vergne, *The Campbell-Hausdorff formula and invariant hyperfunctions*, Invent. Math. **47** (1978), no. 3, 249–272. MR 492078

[Lei04] T. Leinster, *Higher operads, higher categories*, London Mathematical Society Lecture Note Series, vol. 298, Cambridge University Press, Cambridge, 2004. MR 2094071

[LS97] Pierre Lochak and Leila Schneps, *The universal Ptolemy-Teichmüller groupoid*, Geometric Galois actions, 2, London Math. Soc. Lecture Note Ser., vol. 243, Cambridge Univ. Press, Cambridge, 1997, pp. 325–347. MR 1653018

[LV12] Jean-Louis Loday and Bruno Vallette, *Algebraic operads*, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 346, Springer, Heidelberg, 2012. MR 2954392

[LW23] Erik Lindell and Thomas Willwacher, *A(nother) model for the framed little disks operad*, Homology Homotopy Appl. **25** (2023), no. 1, 265–285. MR 4590326

[May72] J. P. May, *The geometry of iterated loop spaces*, Lecture Notes in Mathematics, vol. Vol. 271, Springer-Verlag, Berlin-New York, 1972. MR 420610

[MSS02] Martin Markl, Steve Shnider, and Jim Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, vol. 96, American Mathematical Society, Providence, RI, 2002. MR 1898414

[MW22] L. Müller and L. Woike, *Cyclic Framed Little Disks Algebras, Grothendieck–Verdier Duality And Handlebody Group Representations*, The Quarterly Journal of Mathematics (2022), –, haac015.

[PT04] Michaël Pevzner and Ch Torossian, *Duflo isomorphism and tangential cohomology*, Journal of geometry and Physics **51** (2004), no. 4, 486–505.

[Rou81] François Rouvière, *Démonstration de la conjecture de Kashiwara-Vergne pour l'algèbre* sl(2), C. R. Acad. Sci. Paris Sér. I Math. **292** (1981), no. 14, 657–660. MR 618878

[RS25] Marcy Robertson and Chandan Singh, *Grothendieck–teichm\" uller symmetries of cyclic operads and tangles*, arXiv preprint arXiv:2511.05911 (2025).

[Sch97] Leila Schneps, *The Grothendieck-Teichmüller group* $\widehat{\mathrm{GT}}$*: a survey*, Geometric Galois actions, 1, London Math. Soc. Lecture Note Ser., vol. 242, Cambridge Univ. Press, Cambridge, 1997, pp. 183–203. MR 1483118

[Sho00] Boris Shoikhet, *Tsygan formality and duflo formula*, arXiv preprint math/0012066 (2000), –.

[Sin26] Chandan Singh, *Cyclic symmetries of chord diagrams*, arXiv preprint arXiv:2604.04688 (2026).

[SW03] P. Salvatore and N. Wahl, *Framed discs operads and Batalin-Vilkovisky algebras*, Q. J. Math. **54** (2003), no. 2, 213–231. MR 1989873

[Ver99] Michèle Vergne, *Le centre de l'algèbre enveloppante et la formule de Campbell-Hausdorff*, C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), no. 9, 767–772. MR 1724537

[Wah01] N. Wahl, *Ribbon braids and related operads*, Ph.D. thesis, Oxford University, 2001.

[Wil16] Thomas Willwacher, *The homotopy braces formality morphism*, Duke Math. J. **165** (2016), no. 10, 1815–1964. MR 3522653

[Wil24] ———, *Cyclic operads through modules*, arXiv preprint arXiv:2403.18399 (2024), –.