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Integrable Stochastic Processes in Half-space

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Abstract

This thesis explores the dynamics and mathematical properties of stochastic processes defined in terms of the six-vertex model in half-space. The aim of this work is to, firstly, provide explicit constructions of two distinct families of rational symmetric functions in terms of commuting double-row semi-infinite transfer matrices. The first of which may be interpreted as a Markovian discrete-time interacting particle system, while the second is its dual. Central to our work is the explicit proof of a skew-Cauchy type summation identity between the two families, which yields several useful degenerations.

A partition function on a triangular geometry, which is a natural half-space analogue of the well-studied partition function with domain-wall boundary conditions, plays an important role in the dynamics of the first family. We demonstrate a connection between a Pfaffian formula for the triangular function and the celebrated determinantal one for the domain-wall function due to Izergin. Both partition functions are further analysed using techniques of shuffle products whereby both formulas are expressed as integer shuffle powers.

In the case of upper-triangular boundary conditions, the Cauchy identity has a natural interpretation as an eigenvector relation of the half-space Markov process. Here, the dual family plays the role of eigenvector and we provide an explicit coordinate Bethe ansatz type expression for it. We prove that the dual family satisfies an orthogonality relation, which is used to obtain a coordinate Bethe ansatz expression for the Markovian symmetric function for upper-triangular boundary conditions.

The remainder of the text is devoted to the asymmetric simple exclusion process in half-space, whose dynamics are obtained via a continuous-time limit of the Markovian symmetric function. We provide an explicit formula for its transition probability for generic initial state. In the limit of total asymmetry, the transition probability reduces to the form of a single Pfaffian for a large class of initial conditions, which is shown to generalise a celebrated result due to Schütz. This single Pfaffian is used to define a Pfaffian point process supported on Gelfand–Tsetlin patterns. Ultimately, this is used to construct Fredholm Pfaffian formulas for the joint particle current distribution conditioned on the number of particles in the system. The explicit Pfaffian kernel depends on a family of implicitly defined skew-orthogonal polynomials.

Declaration

I declare that:

- i. the thesis comprises only my original work towards the Doctor of Philosophy except where indicated in the Preface.
- ii. due acknowledgement has been made in the text to all other material used.
- iii. the thesis is no longer than the maximum word limit in length, exclusive of tables, maps, bibliographies, and appendices.
- iv. the thesis comprises 100% dissertation and 0% creative outputs.

Preface

- i. This thesis contains work from the following publications which the author (WM) was actively involved as an equal co-author:
 - (a) [P1]: A. Garbali, J. de Gier, W. Mead, and M. Wheeler. Symmetric Functions from the Six-Vertex Model in Half-Space. *Ann. Henri Poincaré*, 26(7):2557–2624, 2025. arXiv:2312.14348 (Published by Annales Henri Poincaré on 4 October 2024)
 - (b) [P2]: J. de Gier, W. Mead, D. Remenik, and M. Wheeler. Pfaffian Transition Probability and Correlation Kernel of TASEP in Half-space, 2025. arXiv:2512.02286 (Preprint; soon to be submitted for publication in *Communications in Mathematical Physics*)
- ii. The following parts of the thesis are based on the text [P1]: Sections 2.1, 2.3, 2.4.1, 2.4.2, 2.6, Chapters 3, 4 and Section 6.2. Section 5.3 is devoted to the proof of Theorem 5.12, which was posed as a conjecture in [P1]. The following parts of the thesis are based on the text [P2]: Every section in Chapter 6 except for 6.2 and the whole of Chapter 7, as well as Appendices A, B and C.
- iii. No work in this thesis has been submitted for other qualifications.
- iv. The following work was completed prior to and published during the candidature of WM, however, no part of it is contained within this thesis:
 - (a) [P3]: J. de Gier, W. Mead, and M. Wheeler. Transition probability and total crossing events in the multi-species asymmetric exclusion process. *J. Phys. A: Math. Theor.*, 56(25):255204, 2023. arXiv:2109.14232
- v. Neither digital editorial assistance nor generative artificial intelligence tools were used in the preparation of this thesis.
- vi. This thesis was prepared with the assistance of the WM’s supervisors: Jan de Gier and Michael Wheeler. Sections of the thesis based on [P1] were assisted through collaboration with Alexandr Garbali while sections based on [P2] were assisted through collaboration

with Daniel Remenik. All parties assisting the author are knowledgeable in the academic discipline of this thesis.

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Chapter 1

Introduction

This thesis is concerned with quantum integrable systems in the Kardar–Parisi–Zhang universality class in the semi-infinite geometry known as *half-space* with open boundary conditions. This introductory chapter provides an overview of related topics in integrable probability and symmetric functions. This serves primarily as a literature review for topics which are actively perused in this text, however, some effort is made to demonstrate related and complementary methods.

1.1 Yang–Baxter integrability

1.1.1 Heisenberg quantum spin chain

A central problem in the study of finite-dimensional quantum systems is the diagonalisation of the system’s Hamiltonian. Perhaps the earliest study of an explicitly quantum integrable system is that of the Heisenberg quantum spin chain [Hei28], a special case of which possesses the following Hamiltonian for a periodic chain of length L :

$$\mathcal{H}_{\text{XXZ}} = \frac{-J}{2} \sum_{i=1}^L (\sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z), \quad (1.1)$$

where J is the coupling constant while Δ is referred to as the anisotropy parameter. The system consists of L sites which have either up or down spins upon measurement, so that the Hamiltonian acts on some vector space isomorphic to $(\mathbb{C}^2)^{\otimes L}$. The matrices $\sigma_i^{x,y,z}$ are the Pauli matrices acting on the i -th copy of \mathbb{C}^2 , where the system’s periodicity is enforced by the identification $\sigma_{L+1}^{x,y,z} = \sigma_1^{x,y,z}$. The Hamiltonian (1.1) is in fact a special case of the Heisenberg chain known as the XXZ spin chain, referring to the fact that the parameter Δ enforces the a

different coupling in the z -direction.

In a landmark 1931 paper, Bethe provided an exact formulation for the eigenvectors of the Hamiltonian (1.1) in the case $\Delta = 1$ [Bet31]. This solution took the form of an ansatz now known as the *coordinate Bethe ansatz*. This method is extremely versatile and has been applied to many integrable models over the last century [Bax82]. Bethe's ansatz for the eigenvectors takes the following form

$$|\Psi\rangle = \sum_x \psi(x) |x\rangle, \quad \psi(x_1, \dots, x_n) = \sum_{\sigma \in S_n} A_\sigma z_{\sigma(1)}^{x_1} \cdots z_{\sigma(n)}^{x_n}, \quad (1.2)$$

where the sum x is over the state-space of the system, while the coefficient $\psi(x)$ is presented explicitly for the positions of n spins. The functions in the Bethe ansatz $\psi(x)$ from (1.2) are given by a symmetrisation over a product of plane waves with complex momenta z_1, \dots, z_n which are scattered and permuted with some amplitudes A_σ , which may be determined explicitly in terms of the momenta. In a finite system the momenta z_i must obey a quantisation condition which amounts to them being the solution to coupled polynomial equations called the *Bethe ansatz equations*. These equations were initially proposed by Bethe and were subsequently studied for arbitrary Δ in [Orb58], and were used to rigorously prove the ground state of the Heisenberg is given by a Bethe ansatz eigenvector [YY66]. For more details and history of Bethe's ansatz see the book of Baxter [Bax82].

1.1.2 Six-vertex model

Originally considered as an equilibrium statistical mechanical model for two-dimensional ice crystallisation, the *six-vertex model* has since become an archetypal model for quantum integrability in two-dimensions. Early exact solutions were concerned with the partition function of two-dimensional ice subject to periodic boundary conditions. Its solutions via Bethe ansatz are attributed to Lieb [Lie67b, Lie67a, Lie67c] who noticed connections with the diagonalisation of the Hamiltonian (1.1) for a specific value of Δ . This was later generalised by Sutherland [Sut67] for generic values of the anisotropy parameter.

Commuting transfer matrices and integrability

A major conceptual breakthrough occurred when Baxter, while studying the more general *eight-vertex model* in [Bax72], observed that transfer matrices with independent free spectral parameters commute:

$$[T(u), T(v)] = 0, \quad (1.3)$$

for all appropriate $u, v \in \mathbb{C}$ when subject to periodic boundary conditions. This remarkable observation is central to our understanding of quantum integrability, in part because the commutation relation (1.3) implies that transfer matrices may be diagonalised simultaneously. The commutation relation (1.3) may be understood to be a consequence of the *Yang–Baxter equation*¹, whose solution may be understood as a necessary condition of quantum integrability. In fact the XXZ Hamiltonian (1.1) may be obtained as a logarithmic derivative of the transfer matrix of the six-vertex model. This in turn implies that the transfer matrix commutes with the Hamiltonian $[T(u), \mathcal{H}_{\text{XXZ}}] = 0$.

The commutation of transfer matrices has also been crucial tool for the study of alternative boundary conditions. In particular the *domain-wall boundary conditions* (see Section 2.2), whose partition function emerges as a building block of certain scalar products within the quantum inverse scattering method for the six-vertex model in the work by Korepin [Kor82]. Izergin later proved a remarkable formula for this partition function in the form of a determinant [Ize87] (see Theorem 2.13). What is now typically known as the Izergin–Korepin determinant has proven to be fruitful for the analysis of quantum integrable systems, for example in computing the thermodynamic limit of the six-vertex model with domain-wall boundaries [KZJ00].

Open-boundary conditions

A boundary analogue of the Yang–Baxter equation was studied in the works [Che84, Skl88]. This was followed by a formulation involving more general non-diagonal boundaries in [dVGR93] of what is known as the *reflection equation*. Underlying solutions to the Yang–Baxter and reflection equations are crucial in the study of quantum integrable systems with open or reflecting boundary conditions. As such, their solutions are central to this thesis as they allow for the study of integrable systems and commuting transfer matrices in half-space.

There is a natural half-space analogue for the domain-wall partition function which lives on a triangular geometry which we refer to as the *triangular partition function* (see Section 2.4). This partition function was considered [Kup02] for specific boundary conditions, and more recently in [BFK23] for completely generic boundaries where it is shown to have a Pfaffian formula analogous to the Izergin–Korepin determinant (see Theorem 2.28). This triangular geometry has also been used to study other aspects of the six-vertex model with boundaries, see for example [BBCW18, He24].

¹To avoid any ambiguity, throughout this thesis whenever we refer to the Yang–Baxter equation we mean what is often called the quantum Yang–Baxter equation, whose solution is associated with a quantum group. In our case this is usually $U_{\mathfrak{t}}(\widehat{\mathfrak{sl}}_2)$.

Alternating sign matrices

The six-vertex model has been a powerful tool in the study of *alternating sign matrices* (ASM), which are a combinatorial class of square matrices which generalise permutation matrices. In [MRR83] it was conjectured that the number A_n of $n \times n$ ASMs had the following simple enumeration $A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$. Subsequently known as the alternating sign matrix conjecture, the enumeration went unproven until the work of Zeilberger in [Zei96] using connections with other classes of combinatorial objects. A short while later Kuperberg found a bijective proof with configurations of the six-vertex model with domain-wall boundary conditions [Kup96]. This proof is far shorter, and relies heavily on the integrability properties of the six-vertex model.

Kuperberg's bijective method for six-vertex model has proven to be particularly powerful for the enumeration of ASM symmetry classes. These are often formulated using determinantal and Pfaffian formulas for relatives of the domain-wall and triangular partition functions [Kup02] with many classes having simple factorial product enumerations. Recently, the class of diagonally symmetric ASM's (DSASM) were enumerated by a connection with the triangular partition function [BFK23] (see Figure 2.2).

1.1.3 Integrable stochastic processes

The *asymmetric simple exclusion process* (ASEP) is a stochastic interacting particle system consisting of many particles hopping on a one-dimensional lattice with fixed rates. These particles may hop to a neighbouring site, only if that site is free (see Section 6.1). The model was originally considered as a model for DNA replication [MGP68] and its mathematical properties were more thoroughly explored in [Spi70].

The ASEP has become a "default model" for the study of stochastic transport phenomena [Yau04]. Much of this interest has stemmed from the model's integrable properties which have been used in many different physical applications including random growth processes [TW09, Cor12] and large deviation properties [DG09, dGE11]. In infinite volume, coordinate Bethe ansatz solutions have been obtained for the TASEP in [Sch97b] (see Corollary 6.19), for the ASEP with general hopping rates in [TW08] and in half-space with closed boundary conditions in [TW13]. There has also been a significant amount of analysis by the study of duality observables in [Sch97a, BCS14, BC24].

One may also define a Markov process from the six-vertex model, known as the *stochastic six-vertex model*, considered originally in [GS92a, GS92b]. This model has been fruitful in the construction of discrete-time Markov processes like in [BCG16]. There is also an explicit limit

by which the stochastic six-vertex model recovers the dynamics of the ASEP. This connection is well-known, see for example [Agg16]. More connections between the stochastic vertex models and the ASEP have been analysed fruitfully in [BBCW18, He24] in half-space.

1.1.4 Higher-spin generalisations

The quantum integrable models discussed so far may be understood as being fundamental cases of larger families of *higher-spin* models. The solutions of the Yang–Baxter equation which are inherit to these models may be understood as particular representations of the universal R -matrix of the quantum group $U_t(\widehat{\mathfrak{sl}}_2)$. Alternative higher-dimensional representations, i.e. higher-spin representations, may be derived using the algebraic procedure of fusion [KRS81].

A model of similar character to the ASEP, but with multiple occupancies permitted, was studied by Povolotsky in [Pov13] where it was solved using coordinate Bethe ansatz. The model is related to Bethe ansatz solvable models studied earlier [SW98] and further generalised in [Cor15, BC16]. From the model in question, often referred to as the q -Hahn process, Povolotsky demonstrated two kinds of dual orthogonality relations, from which were defined the *Plancherel isomorphisms*. These allow for appropriate functions to be expanded in terms of the eigenfunctions of the model’s Markov generator which are expressible in terms of coordinate Bethe ansatz. These transforms have been shown to lift to a wider class of both higher-spin and fundamental models in [BCPS15] all of which are Bethe ansatz solvable. This has been expressed explicitly in terms of the higher-spin six-vertex model in [Bor17, BP18].

The study of higher-spin vertex models, and their Plancherel theory, has been extended to higher-rank or coloured models, i.e. associated with the quantum group $U_t(\widehat{\mathfrak{sl}}_{n+1})$, in [BW22a], with more progress on the analysis of observables in [BW20, BK21]. There has also been additional work on fermionic vertex models, associated with $U_t(\widehat{\mathfrak{sl}}_{m|n})$, in [ABW23], whose fusion is of a different character.

1.2 Symmetric Functions

A central focus of this thesis is the construction of families of symmetric rational functions and a demonstration of their properties. As such, we will provide a brief overview of relevant concepts and literature concerning symmetric functions.

1.2.1 Classical symmetric polynomials

The study of symmetric polynomials, being multivariate polynomials which are invariant under the permutation of their alphabets, is a classical one. One of the most important families of symmetric polynomial are the *Schur polynomials*, originally considered by Cauchy and later Jacobi in the first half of the 19th century. These polynomials, indexed by integer partitions λ , are traditionally denoted by $s_\lambda(x)$, where they arise as characters of irreducible polynomial representations of the general linear group $\mathrm{GL}(n)$. These functions have many fascinating properties which are beyond the scope of this thesis; for an introduction see the books of Macdonald [Mac98] and Stanley [Sta99]. These functions may be expressed in several ways, one of which is as the ratio of two determinants:

$$s_\lambda(x_1, \dots, x_n) = \frac{\det \left[x_i^{\lambda_j - j + n} \right]_{1 \leq i, j \leq n}}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}.$$

One of the most relevant properties of Schur polynomials is the following *Cauchy-type* summation identity

$$\sum_{\lambda} s_\lambda(x) s_\lambda(y) = \prod_{i,j} \frac{1}{1 - x_i y_j}, \quad (1.4)$$

where the sum on the left-hand side of (1.4) is over all integer partitions λ . Throughout this thesis, we will see several examples of Cauchy summation identities for different families of symmetric functions analogous to (1.4). Schur polynomials have many generalisations including the important two-parameter (\mathbf{q}, \mathbf{t}) -generalisation: the Macdonald polynomials $P_\lambda(x; \mathbf{q}, \mathbf{t})$. Many of the interesting properties of these polynomials may be found in Macdonald's book [Mac98] including their Cauchy identity:

$$\sum_{\lambda} b_\lambda P_\lambda(x; \mathbf{q}, \mathbf{t}) P_\lambda(y; \mathbf{q}, \mathbf{t}) = \prod_{i,j} \frac{(\mathbf{t} x_i y_j; \mathbf{q})_\infty}{(x_i y_j; \mathbf{q})_\infty}, \quad (1.5)$$

where b_λ is a simple \mathbf{t} -dependent coefficient and $(z; \mathbf{q})_\infty = \prod_{i \geq 0} (1 - z \mathbf{q}^i)$ is the \mathbf{q} -Pochhammer symbol. Many other orthogonal bases are obtained via specialisations of the parameters (\mathbf{q}, \mathbf{t}) as in Figure 1.1.

It is also possible to consider polynomials which have different symmetries. For instance, Laurent polynomials which are symmetric under the inversion of their parameters i.e. invariant under $x_i \mapsto x_i^{-1}$. These polynomials are referred to as BC_n -symmetric polynomials because of their association with root systems of the semisimple Lie algebras of type BC . Many families of type A_n symmetric polynomials have BC_n analogues, like the symplectic Schur polynomials which arise as characters of representations of $\mathrm{Sp}(2n)$ [Kin76]. The BC_n analogue of Macdonald polynomials are the Koornwinder polynomials defined in [Koo92] and studied

further, for example, in [Rai04, Ven15].

1.2.2 Integrable constructions of symmetric functions

The observation that transfer matrices commute has led to the construction of symmetric functions from Yang–Baxter integrable systems. Many known families of symmetric function have been constructed using partition functions of vertex models including Hall–Littlewood [Tsi06] and q -Whittaker [Kor13] polynomials, both 1-parameter generalisations of Schur. Notably, Macdonald polynomials have also been constructed using a matrix product ansatz equivalent to a vertex-model partition function living on a cylinder [CdGW15].

There has been significant interest in integrable constructions of symmetric functions following the work of Borodin [Bor17], and subsequent works [BP18]. These works used a higher-spin version of the six-vertex model to construct a family of rational symmetric functions which may be regarded as generalisations of Hall–Littlewood polynomials with an additional spin parameter s . In these works, the orthogonality properties of the vertex models are shown to have an intimate relationship with their Cauchy identities. This work is made possible because of the versatility of the underlying integrable systems to, not only construct symmetric functions, but also to use exchange relations to prove Cauchy-type identities.

In the years since, there has been a large body of work in this direction on spin q -Whittaker rational functions [BW21], and their inhomogeneous versions [MP22, BK24]. Non-symmetric analogues of Hall–Littlewood [BW22a] and Macdonald [BW22b] polynomials have also been studied using higher-rank vertex models. Cauchy and Littlewood-type summation identities of both A_n and BC_n -symmetric polynomials have also been analysed in [BWZJ15, BW16, WZJ16] many of which are written in terms of determinants and Pfaffians of partition functions associated with ASM symmetry classes.

1.2.3 Probability measures and stochastic processes

When interested in probability, it is often convenient to think about symmetric functions via their *specialisations*², ρ , rather than as functions over a specific alphabet. The non-negative specialisations of Schur functions are well-known and are outlined in [Bor11]. Motivated by the Cauchy identity for Schur polynomials (1.4), the following is a probability measure on integer

²When thinking in terms of specialisations, which may be thought of as alphabets of infinite length, we refer to Schur functions rather than Schur polynomials.

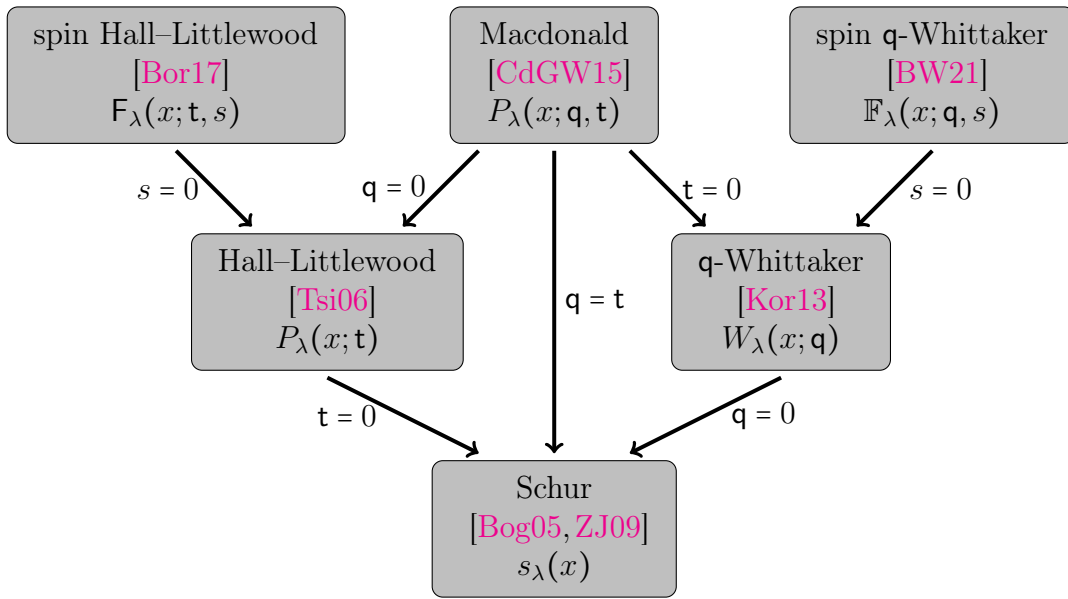


Figure 1.1: Some degenerations of certain families of symmetric rational functions and polynomials with references to their construction using Yang–Baxter integrable models.

partitions, defined by Okounkov [Oko01], which is known as the *Schur measure*:

$$\mathcal{M}(\lambda) = \frac{1}{H(\rho^+; \rho^-)} \cdot s_\lambda(\rho^+) s_\lambda(\rho^-),$$

which depends on two non-negative Schur specialisations ρ^\pm such that

$$H(\rho^+; \rho^-) := \sum_{\lambda} s_\lambda(\rho^+) s_\lambda(\rho^-) \quad (1.6)$$

is finite. Using the skew-Cauchy identity for Schur functions (a generalisation of (1.4) involving functions indexed by skew-partitions), Okounkov and Reshetikhin defined a probability measure on sequences of partitions called the *Schur process* [OR03]. It is defined on sequences of partitions $\vec{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(n)})$ and $\vec{\mu} = (\mu^{(1)}, \dots, \mu^{(n-1)})$ satisfying an interlacing property. The Schur process is the following probability measure

$$\mathcal{M}(\vec{\lambda}, \vec{\mu}) = \frac{1}{Z} s_{\lambda^{(1)}}(\rho_0^+) s_{\lambda^{(1)}/\mu^{(1)}}(\rho_1^-) s_{\lambda^{(2)}/\mu^{(1)}}(\rho_1^+) \cdots s_{\lambda^{(n-1)}/\mu^{(n-1)}}(\rho_{n-1}^-) s_{\lambda^{(n)}/\mu^{(n-1)}}(\rho_{n-1}^+) s_{\lambda^{(n)}}(\rho_n^-), \quad (1.7)$$

which depends on appropriate non-negative specialisations ρ_j^\pm and a normalisation constant defined in terms of (1.6). The Schur measure and process have been extremely fruitful because they both define determinantal point processes [Oko01, OR03], which were used to analyse volume measures of large plane partitions. Since then, they have been used extensively to

analyse various physical phenomena, see for example [Joh03, Joh05] and to define discrete-time Markov processes [Bor11].

In turn, the Cauchy identity for Macdonald polynomials (1.5), and their degenerations, were used to analogously define the *Macdonald measure* and the *Macdonald process* in [BC14]. This may be regarded as a two parameter generalisation of the Schur measure and process. In order to obtain genuine probability measures and stochastic dynamics, we require that the Macdonald measure $\mathcal{M}(\lambda) \geq 0$ for all partitions λ . Fortunately, all non-negative Macdonald specialisations have been obtained in [Mat19].

Pfaffian measures

By defining the symmetric function $\tau_\lambda(x) = \sum_{\kappa' \text{ even}} s_{\lambda/\kappa'}(x)$, one can show that the following Cauchy identity holds

$$\sum_{\lambda} \tau_{\lambda}(x) s_{\lambda}(y) = \prod_{i,j} \frac{1}{1 - x_i y_j} \cdot \prod_{i < j} \frac{1}{1 - y_i y_j}, \quad (1.8)$$

by using properties of skew-Schur functions which may be found in [Mac98]. The identity (1.8) was used, along the same lines as (1.7), by Borodin and Rains to define the *Pfaffian Schur process* [BR05]. As the name suggests, the Pfaffian Schur process is a related process whereby correlation functions take the form of a Pfaffian point process, defined in [Rai00] (see Section 7.1). Since then related processes have been defined including free-boundary Schur processes [BBNV18] and half-space Macdonald Processes [BBC20] which may all be used to define Pfaffian point processes.

1.3 Kardar–Parisi–Zhang universality

In 1986 a breakthrough observation in statistical physics literature led Kardar, Parisi and Zhang (KPZ) [KPZ86] to propose the following non-linear stochastic partial differential equation to characterise the random dynamics of the height of rough surface growth:

$$\frac{\partial h}{\partial t} = \frac{1}{2} \frac{\partial^2 h}{\partial x^2} - \frac{1}{2} \left(\frac{\partial h}{\partial x} \right)^2 + \xi. \quad (1.9)$$

Here, the height $h = h(t, x)$ is a random variable while ξ is uncorrelated space-time white noise satisfying $\langle \xi(t, x) \rangle = 0$ and $\langle \xi(t_1, x_1) \xi(t_2, x_2) \rangle = \delta(x_1 - x_2) \delta(t_1 - t_2)$. The equation (1.9), the one-dimensional version of what is now known as the *KPZ equation*, was derived heuristically to model the dynamics of random surface growth. This crucial observation has spawned a vast array of work on the KPZ equation. Much of the interest has stemmed from the understanding

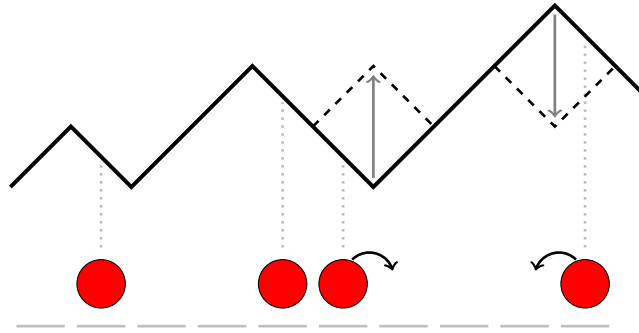


Figure 1.2: ASEP particle dynamics correspondence with random surface growth.

that the KPZ equation (1.9) is universal in a similar way to Gaussian universality. That is, a wide class of interacting particle systems have their large-scale behaviour described by exponents suggested by the KPZ equation. For a great mathematical review on the topic see [Cor12].

1.3.1 Exact statistics

Much of the mathematical interest in KPZ behaviour stems from that fact that certain integrable stochastic processes lie within the KPZ universality class. This includes the ASEP [TW09] and the stochastic six-vertex model [BCG16]. The correspondence of these models with the KPZ equation may be understood by associating a *height function* to the dynamics of the model, as shown in Figure 1.2 for the ASEP. These profiles are expected to display the following features: a global tendency to smooth, a slope-dependent growth rate and roughening due to space-time white noise.

One of the earliest exact results on the statistics of the KPZ class was due to Johansson [Joh00], building on earlier work with Baik and Deift [BDJ99]. Through a matching with another model of the KPZ universality class, Johansson showed that the height function of TASEP subject to an adjacent (step) initial profile experiences linear growth with fluctuations of order $\sim t^{1/3}$, governed by the following:

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{h(t, x) - vt}{ct^{1/3}} < s \right) = F_{\text{GUE}}(s), \quad (1.10)$$

where $v, c > 0$ are constants. The distribution function is the Tracy–Widom distribution of the Gaussian unitary ensemble (GUE). This function has its origin in random matrix theory, where it was considered in [TW94] as the $N \rightarrow \infty$ limit of largest eigenvalue of identically and independently distributed entries of an $N \times N$ Hermitian matrix. Crucially, this function is non-Gaussian and is characteristic of the KPZ universality class. Johansson’s result (1.10) has been extended to other models in the KPZ class like the ASEP [TW09] and the stochastic

six-vertex model [BCG16], when all models are started from the analogous step initial profile.

For different classes of initial data, alternative distribution functions are expected. In the work [BFPS07], the authors considered the TASEP started from a flat initial height profile and were able to obtain a similar limit law to (1.10), where the distribution function becomes F_{GOE} : the Tracy–Widom distribution of the Gaussian orthogonal ensemble (GOE). This follows from an observation of Sasamoto [Sas05], whereby from the determinantal result of Schütz [Sch97b] one may define a determinantal point process whose kernel is written in terms of a family of orthogonal polynomials. This program work has been extended to several models with similar determinantal structures to TASEP, including discrete-time analogues of TASEP, see for example [RS05, DW08, BF14, WW09].

For generic initial height profile, the time-evolved height function of a KPZ model is expected to recover to a universal Markov process known as the *KPZ fixed point* under the so-called 1:2:3 scaling (see [CQR15, Rem22]). Now, the orthogonal polynomial problem which arises in [Sas05] differs depending on the initial height profile of the TASEP. This problem was solved for TASEP with generic initial condition, where the polynomials depend on the hitting probabilities of random walks in [MQR21], ultimately providing a mathematical formulation of the KPZ fixed point. This work has been extended to analyse similar models to TASEP in [MR23, Ara24].

1.3.2 Half-space statistics

The study of the KPZ universality class in the half-space with open-boundary conditions has been a highly active area of research in recent years. Some of the earliest exact statistics involve half-space generalisations of the result of Johansson (1.10) obtained in [BR01a, BR01b, SI04]. Using the Pfaffian Schur process, through a matching with last passage percolation in the half-quadrant, the following distribution function was obtained for the TASEP in half-space with empty initial conditions (with $\alpha > 0$ as a boundary rate) in [BBCS18]:

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{h(t, 0) - \tilde{v}t}{\tilde{c}t^{1/3}} < s \right) = \begin{cases} F_{\text{GSE}}(s) & \text{if } \alpha > 1/2 \\ F_{\text{GOE}}(s) & \text{if } \alpha = 1/2 \end{cases},$$

where $\tilde{v}, \tilde{c} > 0$ are both constants. Here, the Tracy–Widom distribution associated with the Gaussian symplectic ensemble (GSE) emerges whenever $\alpha > 1/2$. Within KPZ universality this function is understood to be associated to systems with open boundary conditions. Whenever $\alpha < 1/2$, the boundary height fluctuations $h(t, 0)$ are of order $\sim t^{1/2}$ and governed by the Gaussian distribution. For fluctuations measured away from the boundary in the half-space TASEP, a limit law involving the F_{GUE} is obtained similar to (1.10).

While there is a good understanding of results with empty initial conditions, there has been less progress on obtaining the exact statistics of models in the KPZ class for more general initial conditions in half-space. Using completely different methods to that developed for the full-space TASEP in [Sas05, BFPS07], recent work by Zhang [Zha24] obtains a Fredholm Pfaffian description of the joint current distribution in the half-space with generic initial conditions. Zhang’s work also ultimately provides a description of the KPZ fixed point in half-space. The mathematical methods used to obtain the joint distribution are in contrast to those used in Chapter 7 of this thesis, which may be viewed as a generalisation of those in [Sas05, BFPS07] (see Corollary 7.21). Nonetheless, it is of significant interest to understand how these approaches are related.

1.4 Overview

Chapter 2 introduces the stochastic six-vertex model and its open-boundary analogue where we outline the algebraic properties of its integrability including the Yang–Baxter and reflection equations. We also construct and connect the domain-wall and triangular partition functions using both integrability and shuffle products. In Chapter 3, two classes of symmetric functions are constructed from commuting transfer matrices indexed by half-space configurations which satisfy a Cauchy-type summation identity. One of these classes admits an explicit contour integral expression with generic boundary parameters which is provided in Chapter 4. Under the specialisation of one boundary parameter, the Cauchy summation identity is interpreted as an eigenvector relation in Chapter 5, wherein the spectral theory and orthogonality relations of the classes are explored. In Chapter 6 the transition probability of the half-space asymmetric simple exclusion process is recovered from a reduction from our symmetric functions. This transition probability admits a single Pfaffian expression in the totally asymmetric specialisation of its jump rates. From this expression, in Chapter 7 we construct a Pfaffian point process supported on Gelfand–Tsetlin patterns. This formalism is used to derive a Fredholm Pfaffian formula for a joint current distribution conditioned on the number of particles in the system which depends implicitly on a class of skew-biorthogonal polynomials.

Chapter 2

The Six-vertex Model

This chapter provides an overview of the algebraic properties of one of the most quintessential quantum integrable models: the six-vertex model. This outline will be extended to include integrability with open-boundary conditions.

Let us initially motivate the six-vertex model as a statistical mechanical model for two-dimensional ice crystals, largely following [Bax82] and references therein. Each molecule in the crystal consists of a central oxygen atom bonded to two hydrogen atoms. Each oxygen atom may be regarded as lying on the vertices of a square lattice $\mathcal{L} \subset \mathbb{Z}^2$. We further consider that the molecules exhibit hydrogen bonding: whereby each oxygen atom will be surrounded by four hydrogen atoms, two of which are connected closely by covalent bonds and another two connected distantly by hydrogen bonds from neighbouring molecules.

This is equivalent to the *ice-rule* whereby each oxygen atom draws two hydrogen atoms closer and two are further away. This guarantees that we only have $\binom{4}{2} = 6$ choices for each vertex configuration, giving the six-vertex model its name. We may then represent a lattice configuration by having an arrow at each edge which points to the vertex which is closer to the hydrogen. We may then extend this to regard right and up arrows as edge occupations while left and down arrows are edge vacancies. When combined with the ice-rule, this allows us to consider the six-vertex model as a model of intersecting connected paths which move up-right along the edges of a square lattice. These correspondences are demonstrated in Figure 2.1.

As a model of classical equilibrium statistical mechanics, many of the essential thermodynamic

Within this chapter, Sections 2.1, 2.3, 2.4.1, 2.4.2 and 2.6 are based on the publication [P1]. Minor modifications have been made for overall coherence and so that the thesis references its other chapters where relevant.

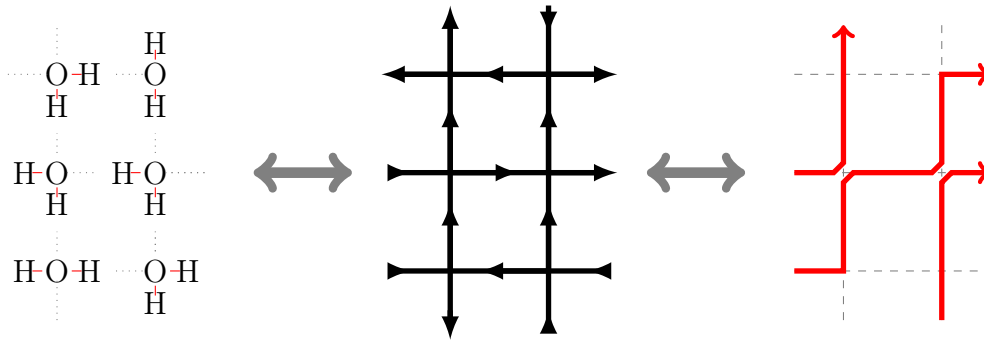


Figure 2.1: The equivalence between a two-dimensional ice crystal and the typical six-vertex model path picture.

properties exhibited by the ice-like crystal may be calculated from of the partition function:

$$\mathcal{Z} = \sum_{\Omega} \exp(-\beta E(\Omega)).$$

The sum is over lattice configurations Ω of \mathcal{L} and $\beta = 1/T$ is the inverse temperature parameter of the lattice. The energy of each configuration, $E(\Omega)$, is assumed to be a sum of energies associated with each vertex configuration of Ω . This assumption allows us to associate individual energies (and Boltzmann weights) to individual configurations at each vertex. The six-vertex model is understood to be integrable because we typically assign Boltzmann weights which lead to solutions of the Yang–Baxter equation.

As we outline in this Chapter, the study of the six-vertex model has developed significantly from its origins as a thermodynamic model for two-dimensional ice crystals. Throughout this thesis, the vertex Boltzmann weights will usually take the form of rational functions, where the computation and properties of the partition functions will be central.

2.1 Stochastic six-vertex model

Here we define the weights and relations of the vertex models that will be used throughout this work.

2.1.1 Stochastic weights


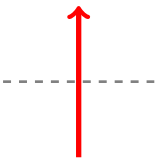
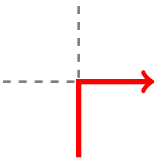
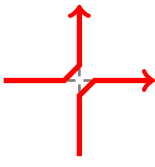
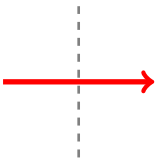
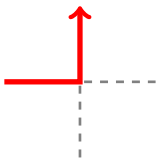
First, we provide a definition of the stochastic six-vertex model [GS92b] and its $U_q(\widehat{\mathfrak{sl}}_2)$ R -matrix. We will follow the conventions of [BW22a].

Definition 2.1. The *stochastic six-vertex model* is an assignment of a rational function to a

vertex³

$$R_{y/x,q}(i, j; k, l) \equiv R_{y/x}(i, j; k, l) = \begin{array}{c} k \\ \uparrow \\ x \rightarrow j \text{ --- } l \\ \downarrow \\ i \\ \uparrow \\ y \end{array} ; \quad i, j, k, l \in \{0, 1\}. \quad (2.1)$$

At any given edge a 1 indicates the presence of a path while a 0 indicates an absence thereof. The values of the weights (2.1) are given pictorially in the following table where $z = y/x$. Any weight which does not appear in the table is defined to be equal to zero.

		
1	$\frac{q(1-z)}{1-qz}$	$\frac{1-q}{1-qz}$
(2.2)		
		
1	$\frac{1-z}{1-qz}$	$\frac{z(1-q)}{1-qz}$

Proposition 2.2 (Stochasticity). The weights of the stochastic six-vertex model satisfy a sum to unity property

$$\sum_{k,l \in \{0,1\}} R_z(i, j; k, l) = 1,$$

for any fixed $i, j \in \{0, 1\}$.

Definition 2.3. The *R-matrix* of the stochastic six-vertex model is $R_{12}(z) \in \text{End}(V_1 \otimes V_2)$ for $V_1, V_2 \cong \mathbb{C}^2$ given by

$$R_{12}(z) = \sum_{i,j,k,l \in \{0,1\}} R_z(i, j; k, l) E_1^{(j,l)} \otimes E_2^{(i,k)}, \quad (2.3)$$

where $E_m^{(i,j)} \in \text{End}(V_m)$ is the 2×2 elementary matrix with a 1 in row i and column j .

We will be interested in *R*-matrices acting on (possibly infinite-dimensional) spaces $V_1 \otimes V_2 \otimes \dots$

³Our conventions for the parameters \mathbf{q}, \mathbf{t} from Chapter 1 has changed. The parameter q here is equivalent to \mathbf{t} while the parameter \mathbf{q} plays no role in the remainder of the text.

where each $V_i \cong \mathbb{C}^2$. We will denote by R_{jk} the R -matrix which acts non-trivially in the space V_j aligned horizontally and V_k aligned vertically as in the picture (2.1).

Proposition 2.4 (Factorisation). At the special value $z = 1$, the R -matrix (2.3) satisfies the relation $R_{z=1}(i, j; k, l) = \delta_{i,l} \delta_{j,k}$. This has pictorial representation

$$R_{x/x}(i, j; k, l) = \begin{array}{c} \begin{array}{ccc} & k & \\ & \uparrow & \\ x \rightarrow j & \text{---} & l \\ & \downarrow & \\ & i & \\ & \uparrow & \\ & x & \end{array} \\ = \\ \begin{array}{ccc} & k & \\ & \uparrow & \\ x \rightarrow j & \text{---} & l \\ & \uparrow & \\ & i & \\ & \uparrow & \\ & x & \end{array} \end{array} .$$

Proposition 2.5 (Yang–Baxter equation). The R -matrix (2.3) satisfies the *Yang–Baxter equation*

$$R_{12}(y/x)R_{13}(z/x)R_{23}(z/y) = R_{23}(z/y)R_{13}(z/x)R_{12}(y/x). \tag{2.4}$$

For fixed $i_1, i_2, i_3, j_1, j_2, j_3 \in \{0, 1\}$, this is represented pictorially as

$$\sum_{k_1, k_2, k_3 \in \{0,1\}} \begin{array}{c} \begin{array}{ccc} & j_3 & \\ & \uparrow & \\ x \rightarrow i_1 & \text{---} & j_2 \\ & \text{---} & \text{---} & \\ y \rightarrow i_2 & \text{---} & j_1 \\ & \downarrow & \\ & i_3 & \\ & \uparrow & \\ & z & \end{array} \\ = \\ \sum_{k_1, k_2, k_3 \in \{0,1\}} \begin{array}{ccc} & j_3 & \\ & \uparrow & \\ x \rightarrow i_1 & \text{---} & j_2 \\ & \text{---} & \text{---} & \\ y \rightarrow i_2 & \text{---} & j_1 \\ & \downarrow & \\ & i_3 & \\ & \uparrow & \\ & z & \end{array} \end{array} .$$

Proposition 2.6 (R -matrix unitarity). The R -matrix (2.3) satisfies the unitarity condition

$$R_{21}(x/y)R_{12}(y/x) = \text{id},$$

where id is the identity within $\text{End}(V_1 \otimes V_2)$. For fixed $i_1, i_2, j_1, j_2 \in \{0, 1\}$, this is represented pictorially as

$$\sum_{k_1, k_2 \in \{0,1\}} \begin{array}{c} \begin{array}{ccc} & j_2 & \\ & \uparrow & \\ y \rightarrow i_2 & \text{---} & j_1 \\ & \text{---} & \text{---} & \\ & \downarrow & \\ & i_1 & \\ & \uparrow & \\ & x & \end{array} \\ = \\ \begin{array}{ccc} & j_2 & \\ & \uparrow & \\ y \rightarrow i_2 & \text{---} & j_1 \\ & \uparrow & \\ & i_1 & \\ & \uparrow & \\ & x & \end{array} \\ = \delta_{i_1, j_1} \delta_{i_2, j_2} . \end{array}$$

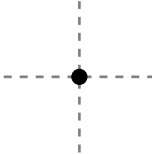
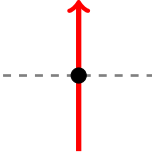
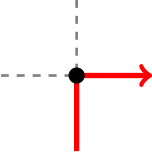
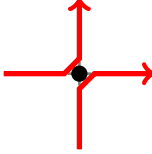
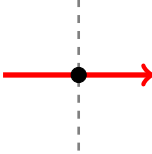
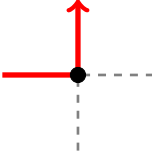
2.1.2 Re-normalised weights

We will also define two versions of re-normalised version of the weights from Definition 2.1. These new weights inherit many of the algebraic properties of the stochastic ones.

Definition 2.7. The *re-normalised vertices* are represented pictorially

$$\dot{R}_{y/x}(i, j; k, l) = \begin{array}{c} k \\ \uparrow \\ \bullet \\ \leftarrow x \rightarrow j \quad \rightarrow l \\ \downarrow \\ i \\ \uparrow \\ y \end{array},$$

for $i, j, k, l \in \{0, 1\}$. We will refer to these vertices as *dotted*. The values of these weights are given in the following table, where $z = y/x$. Any weight which does not appear in the table is defined to be equal to zero.

			
$\frac{1 - qz}{1 - z}$	q	$\frac{1 - q}{1 - z}$	
			(2.5)
$\frac{1 - qz}{1 - z}$	1	$\frac{z(1 - q)}{1 - z}$	

These weights are related to their stochastic counterparts (2.1) through the relation

$$\dot{R}_{y/x}(i, j; k, l) := \frac{1 - qy/x}{1 - y/x} \cdot R_{y/x}(i, j; k, l) \tag{2.6}$$

Since the re-normalised vertices differ from the stochastic ones only by an overall multiplicative factor, we can write versions of the Yang–Baxter equation (2.4) what incorporate mixtures of both R and re-normalised \dot{R} -matrices.

That is:

$$Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) = \begin{array}{c} \begin{array}{cccccc} & & 1 & 1 & 1 & 1 & 1 \\ & & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 1 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \rightarrow 0 \\ & & & & & & \\ x_i^{-1} \rightarrow 1 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \rightarrow 0 \\ & & & & & & \\ x_{i+1}^{-1} \rightarrow 1 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \rightarrow 0 \\ & & & & & & \\ 1 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \rightarrow 0 \\ & & & & & & \\ 1 & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \rightarrow 0 \\ & & & & & & \\ & & 0 & 0 & 0 & 0 & 0 \\ & & \uparrow & \dots & \dots & \dots & \uparrow \\ & & y_1 & & & & y_n \end{array} \end{array}, \quad (2.8)$$

where the spectral variable labels are interchanged on the left-side of the intertwiner so that they match (2.7) on the right-side. Then, through repeated applications of the Yang–Baxter equation (2.4), the intertwining vertex can be moved to the external right-hand side of the diagram. This process interchanges the positions of the i -th and $(i + 1)$ -th lines of the diagram. From here, since it is frozen with weight 1, the intertwiner may be removed at no overall cost to the value of the partition function in a similar way to which it was added. This leaves us with the domain-wall partition function with the roles of x_i and x_{i+1} interchanged. This shows the desired symmetry.

The symmetry in the y -alphabet follows by similarly appending an intertwining vertex to the bottom of the diagram as in (2.8) and removing it at the top of the diagram. \square

Proposition 2.10. The function defined by

$$\tilde{Z}_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) := \prod_{i,j=1}^n (1 - qx_i y_j) \cdot Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) \quad (2.9)$$

is a polynomial of degree at most n in each variable x_1, \dots, x_n and y_1, \dots, y_n .

Proof. Firstly, observe that the the multiplicative factor on the right-hand side of (2.9) may be absorbed into the weight of each vertex in the lattice (2.7). Then, by virtue of weights table (2.2), this absorption precisely cancels the denominators of each weight so that each vertex contributes a polynomial of maximal degree 1 in the appropriate x_i, y_j -variable for each $1 \leq i, j \leq n$. Since there are exactly n vertices whose weights depend on each x and y -variable, it follows that the maximal degree of (2.9) is n for each variable. \square

The following collection of defining properties were considered by Korepin in [Kor82].

Proposition 2.11. The domain-wall partition function (2.7) both satisfies and is defined by the following recursion relations

$$Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) \Big|_{x_i=0} = 0, \quad (2.10)$$

$$Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) \Big|_{y_j=0} = 0, \quad (2.11)$$

$$Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) \Big|_{x_i=1/y_j} = Z_{n-1}^{\text{DW}}(x_1, \dots, \hat{x}_i, \dots, x_n; y_1, \dots, \hat{y}_j, \dots, y_n). \quad (2.12)$$

Proof. By virtue of Proposition 2.10, the domain-wall partition function (2.7) is defined by a symmetric polynomial of degree at most n in each variable $x_1, \dots, x_n, y_1, \dots, y_n$. And so, it is completely defined by $n+1$ independent specialisations. The relations (2.10), (2.11) and (2.12) constitute $n+1$ the required specialisations and so they may be regarded as the defining relations.

We will now proceed by showing each of the relation individually.

- (i) **Proof of (2.10).** Recall that, using Proposition 2.9, the domain-wall partition function is symmetric in the x -alphabet. And so, it suffices to consider its specialisation in the parameter x_1 :

$$Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) \Big|_{x_1=0},$$

where x_1^{-1} is the horizontal spectral parameter associated with the bottom row of the square lattice (2.7). It is straightforward to see that, due to the boundary conditions, the following vertex appears exactly once in the bottom row of each allowed configuration:



$$, \quad (2.13)$$

which has weight $\frac{x_1 y_j (1-q)}{1-q x_1 y_j}$, depending on which column the vertex configuration appears. Since this weight appears exactly once for each configuration and each other weight in (2.2) is analytic at $x_1 = 0$, the partition function must vanish whenever $x_1 = 0$.

- (ii) **Proof of (2.11).** This recursion relation follows by a similar argument as above, were it suffices to consider the specialisation $y_n = 0$. It follows similar argument that the vertex configuration (2.13) must occur exactly once in the right-most column with vanishing weight at $y_n = 0$.

- (iii) **Proof of (2.12).** Using the partition function's symmetry, it suffices to consider the specialisation $x_1 = y_1^{-1}$. By applying the factorisation property of Proposition 2.4, the

specialisation becomes:

$$Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) \Big|_{x_1=1/y_1} = \begin{array}{c} \begin{array}{c} \uparrow 1 \quad \uparrow 1 \quad \uparrow 1 \quad \uparrow 1 \quad \uparrow 1 \\ \hline \begin{array}{c} x_n^{-1} \rightarrow 1 \quad \rightarrow 0 \\ \vdots \quad 1 \quad \rightarrow 0 \\ \vdots \quad 1 \quad \rightarrow 0 \\ x_2^{-1} \rightarrow 1 \quad \rightarrow 0 \\ y_1 \rightarrow 1 \quad \rightarrow 0 \end{array} \\ \hline \begin{array}{c} 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ \uparrow \quad \uparrow \quad \cdots \quad \cdots \quad \uparrow \\ y_1 \quad y_2 \quad \cdots \quad \cdots \quad y_n \end{array} \end{array} \end{array}. \quad (2.14)$$

Observe that both the split left-most column and bottom row of (2.14) are entirely frozen, both with weight 1. It follows that the following specialisation takes place:

$$Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) \Big|_{x_1=1/y_1} = Z_{n-1}^{\text{DW}}(x_2, \dots, x_n; y_2, \dots, y_n),$$

which completes the argument. □

Remark 2.12. Although we do not present a complete argument in this direction, an alternative defining collection of properties of the domain-wall partition function may also be formulated as the following:

- (i) The function $Z_n^{\text{DW}}(x; y)$ is a symmetric function separately symmetric in both alphabets x and y .
- (ii) The function $Z_n^{\text{DW}}(x; y)$ is a meromorphic function of each x_i whose singularities are all simple poles at the points $x_i = q^{-1}y_j^{-1}$ for all $1 \leq i, j \leq n$.
- (iii) The following residue evaluation holds:

$$\begin{aligned} \text{Res}_{x_n=q^{-1}y_n^{-1}} Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) &= \frac{q-1}{q^2 y_n} \prod_{i=1}^{n-1} \left[\frac{y_i - q y_n}{y_i - y_n} \frac{1 - x_i y_n}{1 - q x_i y_n} \right] \\ &\quad \times Z_{n-1}^{\text{DW}}(x_1, \dots, x_{n-1}; y_1, \dots, y_{n-1}). \end{aligned}$$

- (iv) When $n = 1$, the domain-wall partition function is given by $Z_1^{\text{DW}}(x_1; y_1) = \frac{(1-q)x_1 y_1}{1 - q x_1 y_1}$.

(v) The following limit holds:

$$\lim_{x_n \rightarrow \infty} Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) = 0.$$

Each of these properties may be identified using properties of the six-vertex model construction (2.7). Given a candidate expression for the domain-wall partition function $\mathfrak{Z}_n(x; y)$, these properties may be used to prove that \mathfrak{Z}_n is equal to Z_n^{DW} by induction on n . We may proceed by showing that the difference $P_n(x; y) := Z_n^{\text{DW}}(x; y) - \mathfrak{Z}_n(x; y)$ has only removable singularities at $x_i = q^{-1}y_j^{-1}$ and is therefore an entire function of each x_i . The final property (v) may be used to show that the difference P_n is bounded and exactly equal to zero by Liouville's Theorem. This technique is quite versatile and we will use it in Sections 4 and 5 to prove certain formulas for other partition functions of the six-vertex model.

2.2.2 A determinantal formula

The domain-wall partition function has played a central role in the study of quantum integrable systems. Indeed one of the most celebrated results in the field is the explicit evaluation of the domain-wall partition function in terms of the so-called *Izergin–Korepin determinant* (also often referred to as *Izergin's determinant*) due to Izergin in [Ize87].

Theorem 2.13 (Izergin–Korepin determinant). Given a fixed integer n and alphabets $(x_1, \dots, x_n), (y_1, \dots, y_n)$, the domain wall partition function has the following determinantal expression:

$$Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) = \frac{\prod_{i,j=1}^n (1 - x_i y_j)}{\prod_{1 \leq i < j \leq n} [(x_i - x_j)(y_i - y_j)]} \det \left[\frac{(1 - q)x_i y_j}{(1 - x_i y_j)(1 - q x_i y_j)} \right]_{1 \leq i, j \leq n}. \quad (2.15)$$

Although we do not present it explicitly here, Theorem 2.13 may be proven by either demonstrating that the right-hand side of (2.15) satisfies the properties of Proposition 2.11 or, alternatively, as outlined in Remark 2.12. Instead, we will find it convenient to construct an alternative proof of this important result using an algebraic technique known as *shuffle products* in Section 2.5.

The following expression provides an equivalent expression for the domain-wall partition function by expanding the determinant (2.15). This expansion can also be found in [BW16].

Corollary 2.14. The domain-wall partition function may be written as the following:

$$Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) = \sum_{r=0}^n \sum_{\substack{S, T \subseteq [1, n] \\ |S|, |T|=r}} (-1)^{n-r} q^{\binom{n-r}{2}} \prod_{\substack{i \in S \\ j \in T}} \frac{1 - x_i y_j}{1 - q x_i y_j}$$

$$\times \prod_{\substack{i \in S \\ j \in T^c}} (1 - x_i y_j) \prod_{\substack{i \in S^c \\ j \in T}} (1 - x_i y_j) \prod_{\substack{i \in S \\ j \in S^c}} \frac{1}{x_i - x_j} \prod_{\substack{i \in T \\ j \in T^c}} \frac{1}{y_i - y_j} \quad (2.16)$$

Proof. Let first note the partial fraction expansion of the kernel of the Izergin–Korepin determinant (2.15):

$$\frac{(1 - q)x_i y_j}{(1 - x_i y_j)(1 - qx_i y_j)} = \frac{1}{1 - x_i y_j} - \frac{1}{1 - qx_i y_j}. \quad (2.17)$$

Then note the following determinantal expansion, equivalent to Lemma 3.6 of [Oka06]:

$$\det(A - B) = \sum_{r=0}^n \sum_{\substack{S, T \subseteq [1, n] \\ |S|, |T|=r}} (-1)^{n-r} (-1)^{\sum_i S_i + \sum_i T_i} \det(A_{S, T}) \det(B_{S^c, T^c}),$$

which has a natural application as the expansion of the determinant (2.17). The fact that the two expressions (2.15) and (2.16) are equal follows, after some manipulation, by application of the Cauchy determinant identity:

$$\det \left[\frac{1}{1 - x_i y_j} \right]_{\substack{i \in S \\ j \in T}} = \frac{\prod_{\substack{i, j \in S \\ i < j}} (x_i - x_j) \prod_{\substack{i, j \in T \\ i < j}} (y_i - y_j)}{\prod_{\substack{i \in S \\ j \in T}} (1 - x_i y_j)}.$$

□

2.3 Boundary weights

We also define weights of a boundary vertex. Such boundary vertices and their matrices were introduced in [Che84, Skl88] and in a more general non-diagonal case in [dVGR93]. We consider the non-diagonal case in which the boundary vertex weights depend on two free parameters.

Definition 2.15. A *stochastic boundary weight* is an assignment of a rational function to a half-vertex

$$K_x(i; j) = \begin{array}{c} \nearrow j \rightarrow x^{-1} \\ \bullet \\ \searrow i \leftarrow x \end{array} \quad ; \quad i, j \in \{0, 1\}. \quad (2.18)$$

As previously, a 0/1 at a given edge indicates the absence/presence of a path. With the use of the rational function

$$h(x) = \frac{ac(1 - x^2)}{(a - x)(c - x)}, \quad (2.19)$$

these weights are tabulated below.

$$\begin{array}{cccc}
 \begin{array}{c} \diagup \text{---} \\ \bullet \\ \diagdown \text{---} \end{array} & \begin{array}{c} \text{---} \nearrow \\ \bullet \\ \text{---} \searrow \end{array} & \begin{array}{c} \diagup \text{---} \\ \bullet \\ \text{---} \searrow \end{array} & \begin{array}{c} \text{---} \nearrow \\ \bullet \\ \text{---} \searrow \end{array} \\
 1 - h(x) & h(x) & \frac{-h(x)}{ac} & 1 + \frac{h(x)}{ac}
 \end{array} \tag{2.20}$$

Proposition 2.16 (Stochasticity). The stochastic boundary weights also satisfy a sum to unity property

$$\sum_{j \in \{0,1\}} K_x(i; j) = 1,$$

for any fixed $i \in \{0, 1\}$.

Definition 2.17. The K -matrix of the stochastic boundary weights is $K_1(x) \in \text{End}(V_1)$ for $V_1 \cong \mathbb{C}^2$ given by

$$K_1(x) = \sum_{i,j \in \{0,1\}} K_x(i; j) E_1^{(i,j)}.$$

Just as with the R -matrix, while acting on spaces $V_1 \otimes V_2 \otimes \dots$ we will denote by K_j the K -matrix which acts non-trivially on the space V_j . The boundary vertices (2.18), together with the bulk vertices (2.1), combine to give the reflection equation.

Proposition 2.18 (Sklyanin reflection equation). The boundary vertex weights from Definition 2.15 and the bulk weights from Definition 2.1 satisfy the *Sklyanin reflection equation*

$$R_{21} \left(\frac{x}{y} \right) K_1(x) R_{12}(xy) K_2(y) = K_2(y) R_{21}(xy) K_1(x) R_{12} \left(\frac{x}{y} \right). \tag{2.21}$$

For fixed $i_1, i_2, j_1, j_2 \in \{0, 1\}$ this can be represented pictorially as

$$\sum_{k_1, k_2, \ell_1, \ell_2 \in \{0,1\}} \begin{array}{c} \text{---} \rightarrow j_2 \rightarrow y^{-1} \\ \bullet \\ \text{---} \rightarrow j_1 \rightarrow x^{-1} \\ \bullet \\ \text{---} \leftarrow i_1 \leftarrow x \\ \text{---} \leftarrow i_2 \leftarrow y \end{array} = \sum_{k_1, k_2, \ell_1, \ell_2 \in \{0,1\}} \begin{array}{c} \text{---} \rightarrow j_2 \rightarrow y^{-1} \\ \bullet \\ \text{---} \rightarrow j_1 \rightarrow x^{-1} \\ \bullet \\ \text{---} \leftarrow i_1 \leftarrow x \\ \text{---} \leftarrow i_2 \leftarrow y \end{array}$$

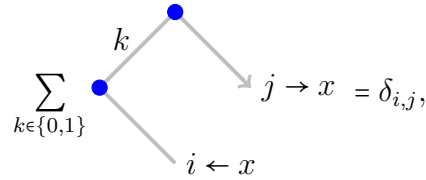
This relation is sometimes referred to as the *boundary Yang–Baxter equation*.

Proposition 2.19 (K -matrix unitarity). The K -matrix from Definition 2.15 satisfies its own

unitarity relation

$$K_1(x)K_1(x^{-1}) = \text{id},$$

where id is the identity within $\text{End}(V_1)$. This can be represented pictorially as



for all $i, j \in \{0, 1\}$.

2.4 Triangular partition function

Definition 2.20. We refer to the following partition function, with generic boundary parameters, as the *triangular partition function*:

$$Z_m(x_1, \dots, x_m) = \begin{array}{c} \begin{array}{cccccc} & & & & \bullet & \longrightarrow 0 \rightarrow x_m^{-1} \\ & & & & \bullet & \longrightarrow 0 \quad \vdots \\ & & & & \bullet & \longrightarrow 0 \quad \vdots \\ & & & & \bullet & \longrightarrow 0 \rightarrow x_2^{-1} \\ \bullet & \longrightarrow 0 \rightarrow x_1^{-1} \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 0 & 0 & 0 & 0 & 0 \\ \uparrow & \uparrow & \cdots & \cdots & \uparrow \\ x_1 & x_2 & \cdots & \cdots & x_m \end{array} \end{array} \quad (2.22)$$

This function plays an analogous role to the domain wall partition function [Ize87] of the six-vertex model on a square geometry. In fact, this function directly generalises a partition function related to diagonally symmetric alternating-sign matrices introduced in [Kup02]. The correspondence between DSASMs and lattice path configurations of the triangular partition function is demonstrated in Figure 2.2.

We first note that for any $m \in \mathbb{N}$, the partition function $Z_m(x_1, \dots, x_m)$ is symmetric and satisfies certain defining recursion relations. It is instructive to derive these properties directly from (2.22) and we will do so below. Moreover, the triangular partition function is a rational function in its alphabet where the degree of numerator and denominator can be easily established. All

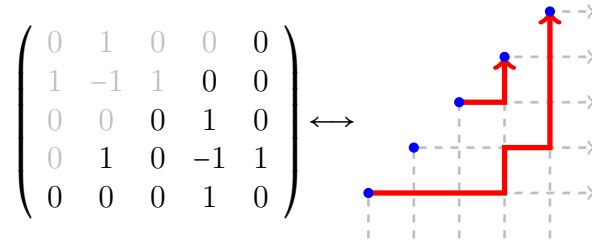


Figure 2.2: Correspondence between a 5×5 DSASM and a lattice path configuration of Z_5 .

these properties together completely determine $Z_m(x_1, \dots, x_m)$ and we shall find several closed formulas for it.

2.4.1 Properties and recursion relations

Proposition 2.21. The triangular partition function $Z_m(x_1, \dots, x_m)$ from (2.22) is a symmetric rational function in the alphabet (x_1, \dots, x_m) .

Proof. Considering $Z_m(x_1, \dots, x_m)$, the symmetry can be seen by inserting an intertwining vertex at the bottom of the diagram (2.22) to cross the adjacent lines with spectral parameters x_i and x_{i+1} . This can be done at no overall cost to the partition function since the boundary conditions enforce that the only possible vertex configuration is the all-empty one, which carries weight 1.

Using repeated applications of the Yang–Baxter equation (2.4) and reflection equation (2.21), the intertwining R -matrix can be pulled through to the right hand side of the lattice between the i -th and $(i+1)$ -th lines of the lattice. In the process of shifting this intertwiner, the x_i and x_{i+1} spectral parameters swap positions. The intertwiner can then be removed from the right hand side of the lattice at no overall cost, again enforced by the boundary conditions, leaving us with

$$Z_m(x_1, \dots, x_i, x_{i+1}, \dots, x_m) = Z_m(x_1, \dots, x_{i+1}, x_i, \dots, x_m),$$

which generates symmetry over the whole alphabet. \square

Proposition 2.22. The function defined by

$$\tilde{Z}_m(x_1, \dots, x_m) := \prod_{i=1}^m [(a - x_i)(c - x_i)] \prod_{1 \leq i < j \leq m} (1 - qx_i x_j) \cdot Z_m(x_1, \dots, x_m) \quad (2.23)$$

is a symmetric polynomial of degree at most $m+1$ in each of the variables x_1, \dots, x_m .

Proof. The pre-factors remove all possible denominators of bulk and boundary vertex weights

of any lattice configuration, so that we can conclude that \tilde{Z}_m is a polynomial in all x_1, \dots, x_m . Once the denominators are removed, the vertex on the boundary with argument x_i contributes a factor of power 2. Each of the $m - 1$ bulk vertices also contribute a factor of power of at most 1, giving the leading order power of $m + 1$. \square

Proposition 2.23. The triangular partition function (2.22) both satisfies, and is completely determined by, the following recursion relations

$$Z_m(x_1, \dots, x_m) \Big|_{x_i=0} = 0, \quad (2.24)$$

$$Z_m(x_1, \dots, x_m) \Big|_{x_i=\pm 1} = Z_{m-1}(x_1, \dots, \hat{x}_i, \dots, x_m), \quad (2.25)$$

$$Z_m(x_1, \dots, x_m) \Big|_{x_i=1/x_j} = Z_{m-2}(x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_m). \quad (2.26)$$

Proof. Because of Proposition 2.22, the triangular partition function Z_m is defined by a polynomial of degree $m + 1$ in each variable. The $m + 2$ recursions for each variable in the statement of Proposition 2.23 therefore completely determine the rational function Z_m .

We now prove the recursion relations individually.

(i) **Proof of (2.24).** We make use of the symmetry of the partition function and consider

$$Z_m(x_i, x_1, \dots, \hat{x}_i, \dots, x_m) \Big|_{x_i=0},$$

so that the boundary vertex corresponding to the parameter x_i is at the bottom-left-most position. Setting $x_i = 0$ forces the bottom-left boundary vertex to generate a path with weight $h(0) = 1$, where we recall the definition of $h(x)$ in (2.19). Since $x_i = 0$, the bottom-right weight in (2.2) vanishes and hence this path cannot turn to any vertical edge on any vertex on the bottom line in the diagram (2.22). And so this path must proceed to the right hand side of the lattice. However, due to the imposed boundary conditions there cannot be any occupations on the external boundary edges on the right hand side and so we conclude that there are no allowed configurations when $x_i = 0$ and hence that the partition function is equal to zero.

(ii) **Proof of (2.25).** Again making use of symmetry we consider

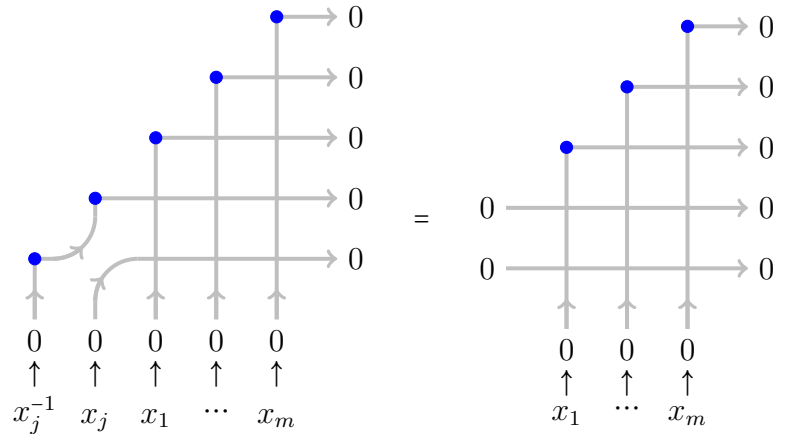
$$Z_m(x_i, x_1, \dots, \hat{x}_i, \dots, x_m) \Big|_{x_i=\pm 1}.$$

This forces the boundary vertex at the bottom-left of the diagram in (2.22) to generate no paths and be of weight $1 - h(\pm 1) = 1$. This causes there to be no occupations along any edges of the bottom line, and hence this line may be removed at no cost leaving us with $Z_{m-1}(x_1, \dots, \hat{x}_i, \dots, x_m)$.

(iii) **Proof of (2.26).** Again using symmetry we consider

$$Z_m(x_i, x_j, x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_m) \Big|_{x_i=1/x_j}.$$

Using the R -matrix factorisation, Lemma 2.4, the partition function becomes



Here, we have used the unitarity of the K -matrix (Proposition 2.19) to obtain the second diagram. We note here that the two bottom rows in the second diagram are completely frozen with no occupations and total weight 1, so that they can be removed at no cost to the partition function. This yields the result.

□

Remark 2.24. Another set of recursions can be derived for the numerator (2.23) of Z_m ,

$$\begin{aligned} \tilde{Z}_m(x_1, \dots, x_m) \Big|_{x_i=1/qx_j} &= -ac(1-q)q^{m-3}(1-x_j^2)(1-1/q^2x_j^2) \prod_{\substack{k=1 \\ k \neq i,j}}^m (1-x_k/qx_j)(1-x_kx_j) \\ &\times \tilde{Z}_{m-2}(x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_m). \end{aligned} \tag{2.27}$$

These relations follow from the observation that the bottom-right vertex in diagram (2.22) completely freezes when $qx_1x_m = 1$, and as a consequence so does the bottom row and right-most column, leaving a partition function of size $m - 2$ multiplied by the pre-factors in (2.27) that arise from the weights of the frozen vertices. By symmetry a similar result follows for $qx_ix_j = 1$ for any i and j .

2.4.2 Solution to recursion relations

This section provides solutions to the recursion relations of Proposition 2.23 which in turn provide closed form solutions to the triangular partition function.

Let Z_m^K be the partition function corresponding to the off-diagonal boundary conditions, which can be realised by setting $a = -c = 1$, set

$$Z_m^K(x_1, \dots, x_m) := Z_m(x_1, \dots, x_m)|_{a=-c=1}. \quad (2.28)$$

This partition function admits a Pfaffian formula due to Kuperberg [Kup02]:

$$Z_m^K(x_1, \dots, x_m) = \prod_{i=1}^m x_i \prod_{1 \leq i < j \leq m} \frac{1 - x_i x_j}{x_i - x_j} \text{Pf}(M(x_i, x_j))_{1 \leq i, j \leq m},$$

$$M(x_i, x_j) = \frac{(1 - q)(x_i - x_j)}{(1 - x_i x_j)(1 - q x_i x_j)}.$$

We note that when m is odd the partition function Z_m^K vanishes.

Theorem 2.25. The triangular partition function (2.22) with the general boundary weights can be expressed as,

$$Z_m(x_1, \dots, x_m) = H_m(x) \sum_{r=0}^{\lfloor m/2 \rfloor} \left(\frac{1}{-ac} \right)^r \sum_{\substack{S \subseteq [1, m] \\ |S|=2r}} \prod_{i \in S} \frac{h(x_i)}{1 - h(x_i)} \prod_{\substack{i \in S \\ j \in S^c}} \frac{1 - x_i x_j}{x_i - x_j} Z_{2r}^K(x_S), \quad (2.29)$$

where

$$H_m(x) := \prod_{i=1}^m (1 - h(x_i)), \quad h(x) = \frac{ac(1 - x^2)}{(a - x)(c - x)}.$$

The proof of Theorem 2.25 is presented in the next section using shuffle algebra techniques.

Corollary 2.26. Expression (2.29) for Z_m as a sum over subsets can routinely be converted to a contour integration over a family of contours \mathcal{L}_i which all enclose each pole at x_1, \dots, x_m but omit all other singularities of the integrand

$$Z_m(x_1, \dots, x_m) = H_m(x) \sum_{r=0}^m \frac{1}{r!} \left(\frac{1}{-ac} \right)^{r/2} \oint_{\mathcal{L}_1} \frac{dv_1}{2\pi i} \dots \oint_{\mathcal{L}_r} \frac{dv_r}{2\pi i} \prod_{1 \leq i < j \leq r} \frac{v_j - v_i}{1 - v_i v_j}$$

$$\times \prod_{i=1}^r \left(\frac{v_i h(v_i)}{(1 - v_i^2)(1 - h(v_i))} \prod_{j=1}^m \frac{1 - v_i x_j}{v_i - x_j} \right) \text{Pf}[M(v_k, v_\ell)]_{1 \leq k, \ell \leq r}.$$

Corollary 2.27. For all $m \geq 1$, one has that

$$\lim_{c \rightarrow \infty} Z_m(x_1, \dots, x_m) = \lim_{c \rightarrow \infty} \prod_{i=1}^m (1 - h(x_i)) = \prod_{i=1}^m \frac{x_i(1 - ax_i)}{x_i - a}. \quad (2.30)$$

Proof. Examining the sum-over-subsets formula (2.29), it is easily verified that the limit $c \rightarrow \infty$ eliminates all terms in the sum over r except that corresponding to $r = 0$. The claim (2.30) is then immediate.

Alternatively, one may prove (2.30) directly from the definition of the partition function (2.22), by noting that the $c \rightarrow \infty$ limit causes the third vertex in the table (2.20) to vanish. Since the boundary vertices then only have the option to inject (but never eject) paths, and no paths exit the partition function (2.22) via its right-outgoing edges, it follows that the whole partition function is frozen as a product of empty vertices. The factorisation (2.30) follows trivially. \square

The sum over subsets (2.29) can be compactly written in terms of Pfaffians in various ways. A particularly elegant expression is the following single Pfaffian expression for Z_m from [BFK23].

Theorem 2.28. When m is even, the triangular partition function (2.22) with the generic open boundary weights can be expressed in terms of a Pfaffian,

$$Z_m(x_1, \dots, x_m) = \prod_{1 \leq i < j \leq m} \frac{1 - x_i x_j}{x_i - x_j} \cdot \text{Pf} \left[\frac{x_i - x_j}{1 - x_i x_j} Q(x_i, x_j) \right]_{1 \leq i, j \leq m}, \quad (2.31)$$

where Q is a symmetric function in two variables given by

$$Q(x_i, x_j) = (1 - h(x_i))(1 - h(x_j)) - \frac{h(x_i)h(x_j)}{ac} \frac{(1 - q)x_i x_j}{1 - qx_i x_j}. \quad (2.32)$$

Proof. Theorem 2.28 follows from Theorem 2.25 and the Pfaffian summation identity (A.3) in Lemma A.3. \square

Remark 2.29. In order to obtain an odd-sized solution from (2.31) we would write, for $m = 2\ell$,

$$Z_{2\ell-1}(x_1, \dots, x_{2\ell-1}) := Z_{2\ell}(x_1, \dots, x_{2\ell-1}, 1),$$

which makes use of the recursion relation (2.25).

We note that the Pfaffian kernel (2.32) bears some resemblance to the one appearing in a refined Littlewood identity for spin Hall–Littlewood symmetric rational functions [Gav23], though is quite different due to the boundary factors.

2.4.3 Reduction to domain-wall partition function

We will now present a connection between the triangular and domain-wall partition functions.

Proposition 2.30. For fixed integer n , the triangular partition function (2.22) reduces to the domain-wall partition function (2.7) as:

$$\lim_{\epsilon \rightarrow 0} Z_{2n} \left(\epsilon x_1, \dots, \epsilon x_n, \frac{y_1}{\epsilon}, \dots, \frac{y_n}{\epsilon} \right) = Z_n^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n). \quad (2.33)$$

Second proof of Proposition 2.30. Let us consider, for some $\epsilon \neq 0$, the left-hand side of (2.33) using the Pfaffian expression of Theorem 2.28. Let us denote this as

$$Z_{2n} \left(\epsilon x_1, \dots, \epsilon x_n, \frac{y_1}{\epsilon}, \dots, \frac{y_n}{\epsilon} \right) = \prod_{1 \leq i, j \leq n} \frac{(1 - \epsilon^2 x_i x_j)(1 - \epsilon^{-2} y_i y_j)}{(x_i - x_j)(y_i - y_j)} \prod_{i, j=1}^n \frac{1 - x_i y_j}{\epsilon x_i - \epsilon^{-1} y_j} \\ \times \text{Pf} \begin{bmatrix} \tilde{Q}(\epsilon x_i, \epsilon x_j)_{1 \leq i, j \leq n} & \tilde{Q}(\epsilon x_i, \epsilon^{-1} y_j)_{1 \leq i, j \leq n} \\ \tilde{Q}(\epsilon^{-1} y_i, \epsilon x_j)_{1 \leq i, j \leq n} & \tilde{Q}(\epsilon^{-1} y_i, \epsilon^{-1} y_j)_{1 \leq i, j \leq n} \end{bmatrix}, \quad (2.35)$$

where $\tilde{Q}(z_i, z_j) := \frac{z_i - z_j}{1 - z_i z_j} Q(z_i, z_j)$ is the kernel of the Pfaffian in (2.31). The pre-factor outside of the Pfaffian in (2.35) has the following asymptotic behaviour as $\epsilon \rightarrow 0$:

$$\prod_{1 \leq i, j \leq n} \frac{(1 - \epsilon^2 x_i x_j)(1 - \epsilon^{-2} y_i y_j)}{(x_i - x_j)(y_i - y_j)} \prod_{i, j=1}^n \frac{1 - x_i y_j}{\epsilon x_i - \epsilon^{-1} y_j} \\ \sim (-1)^{\binom{n}{2}} \prod_{i=1}^n \frac{1}{\epsilon x_i - \epsilon^{-1} y_i} \cdot \frac{\prod_{i, j=1}^n (1 - x_i y_j)}{\prod_{1 \leq i < j \leq n} [(x_i - x_j)(y_i - y_j)]}.$$

And so, by simultaneously absorbing the factors of $(\epsilon x_i - \epsilon^{-1} y_j)^{-1}$ in to the last n rows and columns of (2.35), we obtain the expression

$$Z_{2n} \left(\epsilon x_1, \dots, \epsilon x_n, \frac{y_1}{\epsilon}, \dots, \frac{y_n}{\epsilon} \right) = \frac{\prod_{i, j=1}^n (1 - x_i y_j)}{\prod_{1 \leq i < j \leq n} [(x_i - x_j)(y_i - y_j)]} \prod_{1 \leq i < j \leq n} \frac{(1 - \epsilon^2 x_i x_j)(1 - \epsilon^{-2} y_i y_j)}{(\epsilon x_i - \epsilon^{-1} y_j)(\epsilon x_j - \epsilon^{-1} y_i)} \\ \times (-1)^{\binom{n}{2}} \cdot \text{Pf} \begin{bmatrix} \tilde{Q}(\epsilon x_i, \epsilon x_j)_{1 \leq i, j \leq n} & \frac{\tilde{Q}(\epsilon x_i, \epsilon^{-1} y_j)}{\epsilon x_j - \epsilon^{-1} y_j}_{1 \leq i, j \leq n} \\ \frac{\tilde{Q}(\epsilon^{-1} y_i, \epsilon x_j)}{\epsilon x_i - \epsilon^{-1} y_i}_{1 \leq i, j \leq n} & \tilde{Q}(\epsilon^{-1} y_i, \epsilon^{-1} y_j)_{1 \leq i, j \leq n} \end{bmatrix}, \quad (2.36)$$

where the second pre-factor on the right-hand side of (2.36) converges to 1 as $\epsilon \rightarrow 0$. And so, assuming that each individual limit exists, we may evaluate the $\epsilon \rightarrow 0$ limit on (2.36) by taking the limit on the individual matrix entries. Using the form of (2.32), these limits are readily checked as:

$$\lim_{\epsilon \rightarrow 0} \tilde{Q}(\epsilon x_i, \epsilon x_j) = \lim_{\epsilon \rightarrow 0} \frac{\tilde{Q}(\epsilon^{-1} y_i, \epsilon^{-1} y_j)}{(\epsilon x_i - \epsilon^{-1} y_i)(\epsilon x_j - \epsilon^{-1} y_j)} = 0,$$

whereas

$$\lim_{\epsilon \rightarrow 0} \frac{\tilde{Q}(\epsilon x_i, \epsilon^{-1} y_j)}{\epsilon x_j - \epsilon^{-1} y_j} = \frac{(1 - q)x_i y_j}{(1 - x_i y_j)(1 - q x_i y_j)}.$$

It follows that this is equal to the Izergin–Korepin determinant of (2.15) by noting the Pfaffian property (A.2). \square

2.5 Shuffle Products

2.5.1 Single alphabet shuffle

The partition function Z_m and its generating function

$$Z(v) := \sum_{m=0}^{\infty} v^m Z_m,$$

can both be conveniently written in terms of a *shuffle product*.

Shuffle product

Definition 2.31. Let $f(x_1 \dots x_k)$ and $g(x_1 \dots x_\ell)$ be two symmetric rational functions. We define the *shuffle product* $f * g$ to be the symmetric rational function given by

$$f * g := \sum_{\substack{S \subseteq [1, k+\ell] \\ |S|=k}} f(x_S) g(x_{S^c}) \prod_{\substack{i \in S \\ j \in S^c}} \frac{1 - x_i x_j}{x_i - x_j}. \quad (2.37)$$

The identity with respect to the shuffle product is the rational symmetric function 1 in zero number of arguments and $f * 1 = 1 * f = f$. Further, for any rational function $f(x_1 \dots x_k)$ we define the *shuffle power* and the *shuffle exponential*, \exp_* , by

$$f^{*j} := \underbrace{f * f * \dots * f}_{j \text{ times}}, \quad \exp_*(f) := 1 + f + \frac{1}{2!} f^{*2} + \frac{1}{3!} f^{*3} + \dots \quad (2.38)$$

From Definition 2.31 it follows that the shuffle product of $f(x_1 \dots x_k)$ and $g(x_1 \dots x_l)$ is commutative unless both k and l are odd

$$f * g = (-1)^{kl} g * f. \quad (2.39)$$

It can also be easily shown that this shuffle product is associative $(f * g) * h = f * (g * h)$. This shuffle product can be used to construct an algebra of functions and constitutes a convenient notation.

Shuffle proof for triangular partition function

Proposition 2.32. Consider the first three triangular partition functions Z_0, Z_1 and Z_2 . These can be explicitly calculated from the the diagram (2.22) as

$$Z_0 = 1, \quad Z_1 = 1 - h(x_1), \quad Z_2 = (1 - h(x_1))(1 - h(x_2)) - \frac{h(x_1)h(x_2)(1-q)x_1x_2}{ac(1-qx_1x_2)} \quad (2.40)$$

which are rational symmetric functions in 0, 1 and 2 arguments x respectively. The generating function $Z(v)$ takes form

$$Z(v) = \exp_*(v^2 Z_2 + v Z_1). \quad (2.41)$$

This exponential formula is equivalent to

$$Z_{2m} = \frac{1}{m!} Z_2^{*m}, \quad Z_{2m+1} = \frac{1}{m!} Z_1 * Z_2^{*m}. \quad (2.42)$$

Proof. First we note that the two terms in the exponent in (2.41) commute with each other and $Z_1^{*k} = 0$ for $k > 1$ due to (2.39). Applying definitions (2.38) to the generating function (2.41) leads to (2.42). Let us examine the expression for Z_{2m} given by (2.42). Computing Z_2^{*m} produces a rational function with the (minimal) denominator

$$\prod_{i=1}^{2m} (a - x_i)(c - x_i) \prod_{1 \leq i < j \leq 2m} (1 - qx_i x_j). \quad (2.43)$$

This denominator is a polynomial of degree $2m + 1$ in the individual x_l , $l = 1, \dots, 2m$. Let us fix l and show that the limit $x_l \rightarrow \infty$ of Z_2^{*m} exists. By writing Z_2^{*m} using (2.38) and (2.37) we can see that Z_2^{*m} is of the form

$$Z_2^{*m} = \sum \cdots Z_2(x_{i_1}, x_{i_2}) \cdots Z_2(x_{j_1}, x_{j_2}) \cdots \times \prod_{a,b=1,2} \frac{1 - x_{i_a} x_{j_b}}{x_{i_a} - x_{j_b}} \times \cdots, \quad (2.44)$$

therefore in each summand the dependence on x_ℓ is of the form

$$Z_2(x_\ell, x_a) \prod_b \frac{1 - x_\ell x_b}{x_\ell - x_b},$$

where a and b are some indices not equal to l . Computing $x_\ell \rightarrow \infty$ in both of these factors shows that this limit exists. Therefore Z_2^{*m} is given by a ratio of a polynomial of degree at most $2m + 1$ in x_ℓ and the polynomial in (2.43). This implies that in order to prove (2.42) for Z_{2m} we need to show that $Z_2^{*m}/m!$ satisfies the recursion relations of Proposition 2.23.

The specialisations given in (2.24) and (2.25) follow from

$$Z_1(0) = 0, \quad Z_2(x, 0) = Z_2(0, x) = 0, \quad (2.45)$$

$$Z_1(1) = Z_0 = 1, \quad Z_2(x, 1) = Z_2(1, x) = Z_1(x). \quad (2.46)$$

We have (2.42) satisfies (2.24) due to (2.45) and it satisfies (2.25) due to (2.46). Consider next (2.26) and set $x_k = 1/x_\ell$ in Z_2^{*m} as written in (2.44) for any distinct $k, \ell = 1 \dots 2m$. In each term of the sum in (2.44) the arguments $x_1 \dots x_{2m}$ are distributed between various factors Z_2 . Considering a generic summand we encounter two cases: either $k \in \{i_1, i_2\}$ and $\ell \in \{j_1, j_2\}$ or $k, \ell \in \{i_1, i_2\}$. In the first case the contribution is zero because of the factor which is explicitly written in (2.44) and in the second case we compute

$$\begin{aligned} \frac{1}{m!} Z_2^{*m}|_{x_k=1/x_\ell} &= \frac{1}{m!} \sum \dots Z_2\left(x_\ell, \frac{1}{x_\ell}\right) \dots Z_2(x_{j_1}, x_{j_2}) \dots \times \prod_{b=1,2} \frac{1 - x_\ell x_{j_b}}{x_\ell - x_{j_b}} \frac{1 - \frac{1}{x_\ell} x_{j_b}}{\frac{1}{x_\ell} - x_{j_b}} \times \dots \\ &= \frac{1}{(m-1)!} \left(Z_2^{*(m-1)} \right) (\dots, \hat{x}_k, \dots, \hat{x}_\ell, \dots), \end{aligned}$$

where we noted that $Z_2(x_\ell, 1/x_\ell) = 1$ and the explicitly written rational function is also equal to 1. There are in total m different summands for which $k, \ell \in \{i_1, i_2\}$. All these summands are equal to each other and to the symmetric function $Z_2^{*(m-1)}$ which depends on x_1, \dots, x_{2m} with x_k, x_ℓ omitted. These computations show that Z_{2m} given by (2.42) satisfies the conditions of Proposition 2.23. The case of Z_{2m+1} can be proven analogously. \square

Corollary 2.33. The summation formula for Z_m given in Theorem 2.25 holds as a consequence of (2.42).

Proof. We note that Z_2 in (2.40) is given by a sum of two terms and therefore (2.42) can be expanded using the binomial theorem

$$\begin{aligned} Z_{2m} &= \frac{1}{m!} \left((1 - h(x_1))(1 - h(x_2)) + \frac{-1}{ac} h(x_1)h(x_2) Z_2^K(x_1, x_2) \right)^{*m} \\ &= \sum_{r=0}^m \left(\frac{-1}{ac} \right)^r \frac{1}{(m-r)!r!} ((1 - h(x_1))(1 - h(x_2)))^{*(m-r)} * (h(x_1)h(x_2) Z_2^K(x_1, x_2))^{*r}. \end{aligned}$$

The two terms given by the shuffle powers $*(m-r)$ and $*r$ can be computed. For example, the second term is computed by observing that $(Z_2^K)^{*r} = r! Z_{2r}^K$ as a consequence of (2.42) and (2.28). After this we can write the shuffle product of these two terms using (2.31) and match the outcome with (2.29). \square

2.5.2 Double alphabet shuffle

In this section we will provide an overview of a shuffle product defined between functions of two alphabets which is ultimately related to the domain-wall partition function of Definition 2.8. Shuffle products of two alphabets have been considered before, see for example [FO97].

Definition 2.34. Let $f(x_1, \dots, x_k; y_1, \dots, y_k)$ and $g(x_1, \dots, x_\ell; y_1, \dots, y_\ell)$ be two rational functions which are both symmetric in their x and y -alphabets. We define their *shuffle product* as

$$f \odot g := \sum_{\substack{S, T \subseteq [k+\ell] \\ |S|, |T|=k}} f(x_S; y_T) g(x_{S^c}; y_{T^c}) \prod_{\substack{i \in S \\ j \in T^c}} (1 - x_i y_j) \prod_{\substack{i \in S^c \\ j \in T}} (1 - x_i y_j) \prod_{\substack{i \in S \\ j \in S^c}} \frac{1}{x_i - x_j} \prod_{\substack{i \in T \\ j \in T^c}} \frac{1}{y_i - y_j}. \quad (2.47)$$

The *shuffle power* and *shuffle exponential* are defined as

$$f^{\odot j} := \underbrace{f \odot f \odot \dots \odot f}_{j \text{ times}}, \quad \exp_{\odot}(f) := 1 + f + \frac{1}{2!} f^{\odot 2} + \frac{1}{3!} f^{\odot 3} + \dots.$$

It is easily verifiable that the double alphabet shuffle product between $f(x_1, \dots, x_k; y_1, \dots, y_k)$ and $g(x_1, \dots, x_\ell; y_1, \dots, y_\ell)$ is commutative:

$$f \odot g = g \odot f,$$

which holds for all alphabet lengths k, ℓ . Now, for any integer $k \geq 0$, let us denote the constant function by $\mathbf{1}_k(x_1, \dots, x_k; y_1, \dots, y_k) := 1$. In other words, $\mathbf{1}_k$ is the function in two alphabets, each with k variables, which is exactly equal to 1. It is also easily verifiable that

$$\mathbf{1}_k = \frac{1}{k!} (\mathbf{1}_1)^{\odot k}. \quad (2.48)$$

It also follows that the identity with respect to shuffle multiplication is the constant function in zero arguments: i.e. $f \odot \mathbf{1}_0 = \mathbf{1}_0 \odot f = f$ for all appropriate symmetric functions in two alphabets f .

This double alphabet shuffle product is particularly useful because of its description construction of the domain-wall partition function (2.7). Let us denote the generating function of the domain-wall partition function as

$$Z^{\text{DW}}(v) = \sum_{n=0}^{\infty} v^n Z_n^{\text{DW}}.$$

Using the weights (2.2), we explicitly calculate the following expression for the domain-wall

partition function of size 1

$$Z_1^{\text{DW}}(x_1; y_1) = \frac{x_1 y_1 (1 - q)}{1 - q x_1 y_1}. \quad (2.49)$$

Proposition 2.35. The generating function of the domain-wall partition function may be formulated in terms of the double-alphabet shuffle as

$$Z^{\text{DW}}(v) = \exp_{\odot} (v Z_1^{\text{DW}}).$$

This exponential form is equivalent to

$$Z_n^{\text{DW}} = \frac{1}{n!} (Z_1^{\text{DW}})^{\odot n}. \quad (2.50)$$

We will proceed with two separate proofs of Proposition 2.35. The first will be a direct proof of the shuffle power formula (2.50) by demonstrating the defining relations of the domain-wall partition function Z_n^{DW} . The second proof will rely on the single alphabet shuffle product formulation of the triangular partition function (Proposition 2.32), and its reduction to the domain-wall partition function (Proposition 2.30).

First proof of Proposition 2.35. It suffices to prove the shuffle power expression (2.50). The proof will proceed by first demonstrating that the function defined by

$$\prod_{i,j} (1 - q x_i y_j) \cdot \frac{1}{n!} (Z_1^{\text{DW}})^{\odot n} (x_1, \dots, x_n; y_1, \dots, y_n) \quad (2.51)$$

is a polynomial in each x and y variable of maximal degree n . Observe the form of the function Z_1^{DW} in (2.49) and the shuffle product (2.47). It is not hard to see that the shuffle power expression (2.50) will produce a rational function with the denominator $\prod_{i,j} (1 - q x_i y_j)$. It is also easily verified that the following limits exist

$$\lim_{x_k \rightarrow \infty} \frac{1}{n!} (Z_1^{\text{DW}})^{\odot n}, \quad \lim_{y_\ell \rightarrow \infty} \frac{1}{n!} (Z_1^{\text{DW}})^{\odot n},$$

for all $1 \leq k, \ell \leq n$, from which we may conclude that (2.51) is indeed a polynomial of maximal degree n in each x, y variable.

Now, it remains to show that the shuffle power expression satisfied the defining recursion relations of the domain-wall partition function as in Proposition 2.11. The first two recursion relations, (2.10) and (2.11), follow straightforwardly by the observation that

$$Z_1^{\text{DW}}(0; y_1) = Z_1^{\text{DW}}(x_1; 0) = 0. \quad (2.52)$$

In order to prove the remaining relation (2.12) we will proceed by induction n . First note that

$$Z_1^{\text{DW}}(x_1; y_1) \Big|_{y_1=1/x_1} = 1, \quad (2.53)$$

which serves as our base case and assume, for some fixed $n \geq 2$ that $Z_{m-1}^{\text{DW}} = \frac{1}{(m-1)!} (Z_1^{\text{DW}})^{\odot(m-1)}$ for each $1 < m \leq n$. Observe that this is equivalent to the assumption that

$$Z_{n-1}^{\text{DW}} = \frac{1}{n-1} Z_1^{\text{DW}} \odot Z_{n-2}^{\text{DW}}. \quad (2.54)$$

It will suffice to show that

$$\frac{1}{n} Z_1^{\text{DW}} \odot Z_{n-1}^{\text{DW}}(x_1, \dots, x_n; y_1, \dots, y_n) \Big|_{x_k=y_\ell} = Z_{n-1}^{\text{DW}}(x_1, \dots, \hat{x}_k, \dots, x_n; y_1, \dots, \hat{y}_\ell, \dots, y_n) \quad (2.55)$$

for any $1 \leq k, \ell \leq n$ under the assumption of our induction hypothesis (2.54). Let us consider the following

$$\begin{aligned} \frac{1}{n} Z_1^{\text{DW}} \odot Z_{n-1}^{\text{DW}} &= \frac{1}{n} \sum_{\substack{S, T \subseteq [n] \\ |S|, |T|=1}} Z_1^{\text{DW}}(x_S; y_T) Z_{n-1}^{\text{DW}}(x_{S^c}; y_{T^c}) \\ &\quad \times \prod_{\substack{i \in S \\ j \in T^c}} (1 - x_i y_j) \prod_{\substack{i \in S^c \\ j \in T}} (1 - x_i y_j) \prod_{\substack{i \in S \\ j \in S^c}} \frac{1}{x_i - x_j} \prod_{\substack{i \in T \\ j \in T^c}} \frac{1}{y_i - y_j}, \end{aligned} \quad (2.56)$$

where the only terms which survive the specialisation $x_k = 1/y_\ell$ on the left-hand side of (2.55) are terms satisfying either both $k \in S, \ell \in T$ or both $k \in S^c, \ell \in T^c$. Let us firstly consider the specialisation $x_k = 1/y_\ell$ of the sole term in (2.56) corresponding to $k \in S, \ell \in T$ which, using (2.53) and

$$\prod_{\substack{i \in S \\ j \in T^c}} (1 - x_i y_j) \prod_{\substack{i \in S^c \\ j \in T}} (1 - x_i y_j) \prod_{\substack{i \in S \\ j \in S^c}} \frac{1}{x_i - x_j} \prod_{\substack{i \in T \\ j \in T^c}} \frac{1}{y_i - y_j} \Big|_{x_k=1/y_\ell} = 1,$$

can be seen to contribute $\frac{1}{n} z_{n-1}^{\text{DW}}(x_1, \dots, \hat{x}_k, \dots, x_n; y_1, \dots, \hat{y}_\ell, \dots, y_n)$. Next, consider the $x_k = 1/y_\ell$ specialisation of the sum of terms in (2.56) corresponding to $k \in S^c, \ell \in T^c$. These contribute

$$\begin{aligned} \frac{1}{n} \sum_{\substack{S \subseteq [n] \setminus \{k\}, T \subseteq [n] \setminus \{\ell\} \\ |S|, |T|=1}} Z_1^{\text{DW}}(x_S; y_T) Z_{n-2}^{\text{DW}}(x_{S^c \setminus \{k\}}; y_{T^c \setminus \{\ell\}}) \\ \times \prod_{\substack{i \in S \\ j \in T^c \setminus \{\ell\}}} (1 - x_i y_j) \prod_{\substack{i \in S^c \setminus \{k\} \\ j \in T}} (1 - x_i y_j) \prod_{\substack{i \in S \\ j \in S^c \setminus \{k\}}} \frac{1}{x_i - x_j} \prod_{\substack{i \in T \\ j \in T^c \setminus \{\ell\}}} \frac{1}{y_i - y_j} \\ = \frac{1}{n} Z_1^{\text{DW}} \odot Z_{n-2}^{\text{DW}}(x_1, \dots, \hat{x}_k, \dots, x_n; y_1, \dots, \hat{y}_\ell, \dots, y_n). \end{aligned} \quad (2.57)$$

Now, using the induction hypothesis (2.54), the right-hand side of (2.57) may be identified as

the function $\frac{n-1}{n} Z_{n-1}^{\text{DW}}$. Finally, the contribution of both terms will be

$$\frac{1}{n} Z_{n-1}^{\text{DW}} + \frac{n-1}{n} Z_{n-1}^{\text{DW}} = Z_{n-1}^{\text{DW}}$$

as functions of the double alphabet $(x_1, \dots, \hat{x}_k, \dots, x_n; y_1, \dots, \hat{y}_\ell, \dots, y_n)$, which completes the argument. \square

Second proof of Proposition 2.35. The proof follows by the reduction of the triangular partition function in Proposition 2.30. Namely, the result will follow from the limit

$$\lim_{\epsilon \rightarrow 0} \frac{1}{n!} (Z_2)^{*n} \left(\epsilon x_1, \dots, \epsilon x_n, \frac{y_1}{\epsilon}, \dots, \frac{y_n}{\epsilon} \right) = \frac{1}{n!} (Z_1^{\text{DW}})^{\odot n} (x_1, \dots, x_n; y_1, \dots, y_n), \quad (2.58)$$

which we will demonstrate by induction on n . Firstly, the limit (2.58) holds for $n = 1$ directly from Proposition 2.30 at $n = 1$. Let us assume that (2.58) holds for $n = k - 1$, for some fixed $k > 1$. Then let us consider

$$Z_2 * Z_{2k-2}(w_1, \dots, w_{2k}) = \sum_{\substack{P \subseteq [2k] \\ |P|=2}} Z_2(w_P) Z_{2k-2}(w_{P^c}) \prod_{\substack{i \in P \\ j \in P^c}} \frac{1 - w_i w_j}{w_i - w_j}. \quad (2.59)$$

Let us then specialise the alphabet in (2.59) so that, for fixed $\epsilon \neq 0$, we set

$$w_i = \begin{cases} \epsilon x_i, & \text{for } i \leq k \\ \epsilon^{-1} y_{i-k} & \text{for } i > k \end{cases}, \quad (2.60)$$

where we note that the limits $\lim_{\epsilon \rightarrow 0} Z_2(w_P), \lim_{\epsilon \rightarrow 0} Z_{2k-2}(w_{P^c})$ both exist for any subset $P \subseteq [2k]$. And so, consider now the product $\prod_{i \in P, j \in P^c} \frac{1 - w_i w_j}{w_i - w_j}$ which appears in the summand of (2.59). Under the specialisation (2.60), one can see that this product vanishes in the limit $\epsilon \rightarrow 0$ unless the subset P (with cardinality $|P| = 2$) has exactly one element in the set $[1, k]$ and one in $[k + 1, 2k]$. For each such P , we may identify $S, T \subseteq [k]$ (with cardinality $|S| = |T| = 1$) such that the sole elements, $s_1 \in S, t_1 \in T$, are also $\{s_1, t_1 + k\} = P$. In this way, we may calculate

$$\lim_{\epsilon \rightarrow 0} \prod_{\substack{i \in P \\ j \in P^c}} \frac{1 - w_i w_j}{w_i - w_j} = \prod_{\substack{i \in S \\ j \in T^c}} (1 - x_i y_j) \prod_{\substack{i \in S^c \\ j \in T}} (1 - x_i y_j) \prod_{\substack{i \in S \\ j \in S^c}} \frac{1}{x_i - x_j} \prod_{\substack{i \in T \\ j \in T^c}} \frac{1}{y_i - y_j}, \quad (2.61)$$

where the complements $S^c, T^c \subseteq [k]$ are defined in the obvious way. Under the same subsets P , we also may recognise

$$\lim_{\epsilon \rightarrow 0} Z_2(w_P) = Z_1^{\text{DW}}(x_S, y_T), \quad \lim_{\epsilon \rightarrow 0} Z_{2k-2}(w_{P^c}) = Z_{k-1}^{\text{DW}}(x_{S^c}, y_{T^c}), \quad (2.62)$$

from Proposition 2.30. Finally, using (2.61) and (2.62), we conclude that

$$\lim_{\epsilon \rightarrow 0} Z_2 * Z_{2k-2}(w_1, \dots, w_{2k}) = Z_1^{\text{DW}} \odot Z_{k-1}^{\text{DW}}(x_1, \dots, x_k; y_1, \dots, y_k),$$

which completes the argument. \square

The determinantal structure of the domain-wall partition function, through the Izergin–Korepin determinant of Theorem 2.13, is related to the double alphabet shuffle power formula (2.50) through the following result.

Corollary 2.36. The summation formula for the domain-wall partition function of Corollary 2.14 holds as a consequence of (2.50).

Proof. Let us begin by noting the following shuffle power property:

$$q^{\binom{k}{2}} \prod_{i,j=1}^k \frac{1 - x_i y_j}{1 - q x_i y_j} = \frac{1}{k!} (\mathbf{1}_1 - Z_1^{\text{DW}})^{\odot k}, \quad (2.63)$$

which holds for all integers $k \geq 0$. We do not present an explicit proof of the property (2.63) since it follows by an analogous argument to that presented in the first proof of Proposition 2.35. That is, firstly under multiplication by $\prod_{i,j} (1 - q x_i y_j)$, both the left-hand side and right-hand side of (2.63) become polynomials of maximal degree k in each x and y variable. Then, both the left and right-hand sides both can be seen to obey the following recursion relations

$$\begin{aligned} f_k(x_1, \dots, x_k; y_1, \dots, y_k) \Big|_{x_i=0} &= q^{k-1} \prod_{\ell \neq i} \frac{1 - x_\ell y_i}{1 - q x_\ell y_i} \cdot f_{k-1}(x_1, \dots, \widehat{x}_i, \dots, x_k; y_1, \dots, \widehat{y}_i, \dots, y_k), \\ f_k(x_1, \dots, x_k; y_1, \dots, y_k) \Big|_{y_j=0} &= q^{k-1} \prod_{\ell \neq j} \frac{1 - x_j y_\ell}{1 - q x_j y_\ell} \cdot f_{k-1}(x_1, \dots, \widehat{x}_j, \dots, x_k; y_1, \dots, \widehat{y}_j, \dots, y_k), \\ f_k(x_1, \dots, x_k; y_1, \dots, y_k) \Big|_{x_i=1/y_j} &= 0, \end{aligned} \quad (2.64)$$

for all $1 \leq i, j \leq k$. The relations are immediate on the left-hand side of (2.63). Consider the right-hand side of (2.63), and note the properties of the $k = 1$ version:

$$(\mathbf{1}_1 - Z_1^{\text{DW}}) \Big|_{x_1=0} = (\mathbf{1}_1 - Z_1^{\text{DW}}) \Big|_{y_1=0} = 1, \quad (\mathbf{1}_1 - Z_1^{\text{DW}}) \Big|_{x_1=1/y_1} = 0,$$

which follow from the properties (2.52) and (2.53). Now, for $k \geq 1$, the right-hand side of (2.63) can be seen to satisfy the recursions (2.64) by a similar calculation to the first proof of Proposition 2.35 whose details we omit.

Now, consider the result of Corollary 2.14. Using property (2.48) as an evaluation for the number 1, we can free insert a factor of $\frac{1}{r!} (\mathbf{1}_1)^{\odot r}$ into the summand of the expression (2.16).

Then, we may use property (2.63) to obtain the expression

$$Z_n^{\text{DW}} = \sum_{r=0}^n \sum_{\substack{S, T \subseteq [1, n] \\ |S|, |T|=r}} \frac{1}{(n-r)!} (Z_1^{\text{DW}} - \mathbf{1}_1)^{\odot(n-r)}(x_{S^c}; y_{T^c}) \cdot \frac{1}{r!} (\mathbf{1}_1)^{\odot r}(x_S; y_T) \\ \times \prod_{\substack{i \in S \\ j \in T^c}} (1 - x_i y_j) \prod_{\substack{i \in S^c \\ j \in T}} (1 - x_i y_j) \prod_{\substack{i \in S \\ j \in S^c}} \frac{1}{x_i - x_j} \prod_{\substack{i \in T \\ j \in T^c}} \frac{1}{y_i - y_j}, \quad (2.65)$$

whose sums over S, T , for each fixed r , may be identified as a shuffle product (2.47). Finally, the expression (2.65) may be simplified using the shuffle binomial theorem as

$$Z_n^{\text{DW}} = \sum_{r=0}^n \frac{1}{r!(n-r)!} (Z_1^{\text{DW}} - \mathbf{1}_1)^{\odot(n-r)} \odot (\mathbf{1}_1)^{\odot r} = \frac{1}{n!} (Z_1^{\text{DW}})^{\odot n},$$

which completes the argument. \square

2.6 Alternative expression for the triangular partition function

Theorem 2.37 below contains an alternative explicit expression for the triangular partition function $Z_m(x_1, \dots, x_m)$ in terms of subset-sums over factorised expressions and valid for both m even and odd. We first define S by

$$S(x_i, x_j) = \frac{x_i - x_j}{1 - x_i x_j},$$

and let

$$Q^e(x_i, x_j) = S(x_i, x_j) + \frac{u^2 q^{1/2}}{ac} x_i x_j h(x_i) h(x_j) S(q^{1/2} x_i, q^{1/2} x_j), \\ Q^o(x_i, x_j) = x_i x_j S(x_i, x_j) + \frac{u^2}{q^{1/2} ac} h(x_i) h(x_j) S(q^{1/2} x_i, q^{1/2} x_j),$$

where u is a generating parameter. Furthermore, we define the following functions in terms of Pfaffians

$$Z_{2m}^e(u; x_1, \dots, x_{2m}) = \prod_{1 \leq i < j \leq 2m} \frac{1 - x_i x_j}{x_i - x_j} \cdot \text{Pf} \left((Q_{2m}^e(x_i, x_j))_{1 \leq i, j \leq 2m} \right), \\ Z_{2m-1}^o(u; x_1, \dots, x_{2m-1}) = \prod_{1 \leq i < j \leq 2m-1} \frac{1 - x_i x_j}{x_i - x_j} \cdot \text{Pf} \begin{pmatrix} (Q_{2m-1}^o(x_i, x_j))_{1 \leq i < j \leq 2m-1} & (-uh(x_i))_{1 \leq i \leq 2m-1} \\ (uh(x_j))_{1 \leq j \leq 2m-1} & 0 \end{pmatrix}.$$

We also set

$$\begin{aligned} Z_{2m-1}^e(u; x_1, \dots, x_{2m-1}) &= Z_{2m}^e(u; x_1, \dots, x_{2m-1}, 1), \\ Z_{2m}^o(u; x_1, \dots, x_{2m}) &= Z_{2m+1}^o(u; x_1, \dots, x_{2m}, 1). \end{aligned}$$

Using these definitions we can now state the following theorem and corollary.

Theorem 2.37. The triangular partition function (2.22) is recovered by $Z_m(x_1, \dots, x_m) = Z_m(u = 1; x_1, \dots, x_m)$ for $m \in \mathbb{N}$ with

$$Z_m(u; x_1, \dots, x_m) = Z_m^e(u; x_1, \dots, x_m) + Z_m^o(u; x_1, \dots, x_m). \quad (2.66)$$

Furthermore, the partition function with generating parameter, $Z_m(u; x_1, \dots, x_m)$, can be written as

$$Z_m(u; x_1, \dots, x_m) = \sum_{S \subseteq [1, m]} (-u)^{|S|} g_S(x) \prod_{i \in S} h(x_i) \prod_{i \in S} \prod_{j \in S^c} \frac{x_i x_j - 1}{x_i - x_j} \prod_{\substack{1 \leq i < j \leq m \\ i, j \in S}} \frac{1 - x_i x_j}{1 - q x_i x_j}, \quad (2.67)$$

where $r_S = \lfloor |S|/2 \rfloor$ and

$$g_S(x) = \frac{q^{r_S^2}}{(ac)^{r_S}} \begin{cases} \prod_{i \in S} x_i, & |S| \text{ is even} \\ \prod_{i \in S^c} x_i, & |S| \text{ is odd} \end{cases}.$$

Remark 2.38. The recursions (2.27) appear in (2.67) as residues of the simple poles at $x_k = 1/qx_\ell$.

Proof of Theorem 2.37. The equivalence of (2.66) and (2.67) follows in a straightforward manner from the Pfaffian definitions of Z_m^e and Z_m^o , the Pfaffian identity (A.3) and from the fact that the Pfaffian of S factorises [Ste90],

$$\text{Pf}(S(x_i, x_j))_{1 \leq i, j \leq 2m} = \prod_{1 \leq i < j \leq 2m} \frac{x_i - x_j}{1 - x_i x_j}.$$

Next we need to show that $Z_m(x_1, \dots, x_m) = Z_m(u = 1; x_1, \dots, x_m)$. We do that by computing the generating function

$$Z(u; v) := \sum_{m=0}^{\infty} v^m Z_m(u; x_1, \dots, x_m)$$

at $u = 1$ with $Z_m(u = 1; x_1, \dots, x_m)$ given by (2.67). We will show that this generating function is equal to the generating function of $Z_m(x_1, \dots, x_m)$ (2.41) from Proposition 2.32. Using the

definition of the shuffle product (2.31) we rewrite (2.67) with $u = 1$ as

$$\begin{aligned} Z_{2m}(u = 1; x_1, \dots, x_{2m}) &= \sum_{j=0}^{2m} V_{2m-j} * W_j^e, \\ Z_{2m+1}(u = 1; x_1, \dots, x_{2m+1}) &= \sum_{j=0}^{2m+1} V_{2m+1-j} * W_j^o, \end{aligned} \quad (2.68)$$

where we introduced symmetric functions $V_m = V_m(x_1, \dots, x_m)$

$$\begin{aligned} V_{2m} &= \frac{q^{m^2}}{(ac)^m} \prod_{i=1}^{2m} x_i h(x_i) \prod_{1 \leq i < j \leq 2m} \frac{1 - x_i x_j}{1 - qx_i x_j}, \\ V_{2m+1} &= -\frac{q^{m^2}}{(ac)^m} \prod_{i=1}^{2m+1} h(x_i) \prod_{1 \leq i < j \leq 2m+1} \frac{1 - x_i x_j}{1 - qx_i x_j}, \end{aligned}$$

and $W_m^{e/o} = W_m^{e/o}(x_1, \dots, x_m)$ are defined by

$$W_{2m}^e = 1, \quad W_{2m+1}^e = -x_1 \cdots x_{2m+1}, \quad W_{2m}^o = x_1 \cdots x_{2m}, \quad W_{2m+1}^o = 1.$$

All of these functions also factorise with respect to the shuffle product

$$X_{2m} = \frac{1}{m!} X_2^{*m}, \quad X_{2m+1} = \frac{1}{m!} X_1 * X_2^{*m}, \quad \text{for } X = V, W^e, W^o. \quad (2.69)$$

From these formulas it follows that the generating functions of V, W^e, W^o can be expressed in terms of the shuffle exponential (2.38). We compute the generating function $Z(u = 1; v)$

$$Z(u = 1; v) = \sum_{m=0}^{\infty} v^{2m} Z_{2m}(u = 1; x_1, \dots, x_{2m}) + \sum_{m=0}^{\infty} v^{2m+1} Z_{2m+1}(u = 1; x_1, \dots, x_{2m+1}), \quad (2.70)$$

using (2.68) and by representing each function $V_k, W_j^{e/o}$ in the form (2.69). The first summand in (2.70) is computed as follows

$$\begin{aligned} \sum_{m=0}^{\infty} v^{2m} Z_{2m}(u = 1; x_1, \dots, x_{2m}) &= (1 + v^2 V_1 * W_1^e) * \exp_*(v^2 (V_2 + W_2^e)) \\ &= \exp_*(v^2 (V_2 + W_2^e + V_1 * W_1^e)) = \exp_*(v^2 Z_2), \end{aligned} \quad (2.71)$$

where $Z_2 = Z_2(x_1, x_2)$ in the last expression is the triangular partition function for two sites. In (2.71) the second equality is due to the nilpotency of the shuffle product and the third equality is a consequence of the identity

$$V_2 + W_2^e + V_1 * W_1^e = Z_2. \quad (2.72)$$

Let us remark that the numerator on the right hand side of (2.72) is a polynomial of degree 3

in each x_i while on the left hand side some terms have numerators which are polynomials of degree 4 in individual x_i . In the above equation it is easy to check that the degree 4 terms cancel on the left hand side. This phenomenon manifests itself if one tries to evaluate the degrees produced by the formula (2.67). The apparent degree is higher than expected and, in order to show the connection with Z_m , it is required to argue that (2.67) actually produces the correct degree.

In the next step we calculate the generating function of the second term in (2.70)

$$\begin{aligned} \sum_{m=0}^{\infty} v^{2m+1} Z_{2m+1}(u=1; x_1, \dots, x_{2m+1}) &= v (V_1 + W_1^{\circ}) * \exp_* (v^2 (V_2 + W_2^{\circ})) \\ &= v Z_1 * \exp_* (v^2 (Z_2 + Z_1 * x_1)) = v Z_1 * \exp_* (v^2 Z_2) \end{aligned} \quad (2.73)$$

where $Z_1 = Z_1(x_1)$ and $Z_2 = Z_2(x_1, x_2)$ in the second line are the triangular partition functions for one and two sites. In the second equality in (2.73) we used

$$V_1 + W_1^{\circ} = Z_1 \quad V_2 + W_2^{\circ} = Z_2 + Z_1 * x_1$$

and the last equality of (2.73) is due to the nilpotency $Z_1^{*n} = 0$, $n > 1$. By combining (2.73) with (2.71) in (2.70) we obtain the full generating function

$$Z(u=1; v) = \exp_* (v^2 Z_2 + v Z_1).$$

which coincides with (2.41) and therefore proves the statement of the Theorem. \square

Chapter 3

Half-space Symmetric Functions

3.1 Double row-operators

In this section we construct double-row operators which serve as our transfer matrices. Our operators will act on the vector space with basis elements indexed by configurations in the set

$$\mathbb{W} = \left\{ S \subset \mathbb{N} : \sum_i S_i \text{ is finite} \right\}.$$

We denote a configuration $\mu \in \mathbb{W}$ with $m \geq 0$ parts by $\mu = (\mu_1, \dots, \mu_m)$ where $\mu_m < \dots < \mu_2 < \mu_1$. By agreement $\mu = \emptyset$ is defined when $m = 0$. We will only consider finite configurations $\mu, \nu \in \mathbb{W}$. That means that there are only finitely-many occupations in these states and that these occur at finite positions. We also define an orthogonal inner product on \mathbb{W} by $\langle \mu | \nu \rangle = \delta_{\mu, \nu}$.

Definition 3.1 (Occupation notation). For $\mu \in \mathbb{W}$ we define the occupation at site $i \in \mathbb{N}$ as

$$\eta_i^\mu = \begin{cases} 1 & \text{if } i \in \mu \\ 0 & \text{otherwise} \end{cases}.$$

This chapter is based on the publication [P1]. Minor modifications have been made for overall coherence and so that the thesis references its other chapters where relevant.

Definition 3.2. A *double-row operator* on \mathbb{W} is defined by its action on co-vectors:

$$\langle \mu | A(x | \mathbf{Y}) := \sum_{\nu \in \mathbb{W}} \text{weight} \left(\begin{array}{c} \eta_1^\nu \quad \eta_2^\nu \quad \eta_3^\nu \quad \dots \quad \dots \quad \dots \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ 0 \rightarrow x^{-1} \\ \leftarrow 0 \leftarrow x \\ \eta_1^\mu \quad \eta_2^\mu \quad \eta_3^\mu \quad \dots \quad \dots \quad \dots \\ \uparrow \quad \uparrow \quad \uparrow \\ y_1 \quad y_2 \quad y_3 \end{array} \right) \langle \nu |, \quad (3.1)$$

$$\langle \mu | \dot{B}(z | \mathbf{Y}) := \sum_{\nu \in \mathbb{W}} \text{weight} \left(\begin{array}{c} \eta_1^\nu \quad \eta_2^\nu \quad \eta_3^\nu \quad \dots \quad \dots \quad \dots \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ 1 \rightarrow z^{-1} \\ \leftarrow 0 \leftarrow z \\ \eta_1^\mu \quad \eta_2^\mu \quad \eta_3^\mu \quad \dots \quad \dots \quad \dots \\ \uparrow \quad \uparrow \quad \uparrow \\ y_1 \quad y_2 \quad y_3 \end{array} \right) \langle \nu |. \quad (3.2)$$

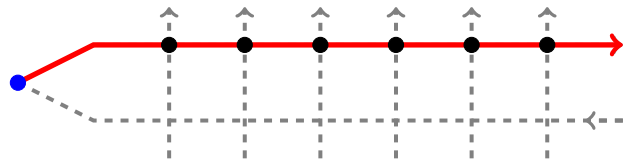
where $x, z \in \mathbb{C}$ are horizontal spectral parameters and $\mathbf{Y} = (y_1, y_2, \dots)$ is an infinite collection of vertical spectral parameters. For conciseness, we will often omit the family of vertical parameters from our notation by writing $A(x | \mathbf{Y}) = A(x)$ and $\dot{B}(z | \mathbf{Y}) = \dot{B}(z)$.

Proposition 3.3. The empty state $\emptyset \in \mathbb{W}$ corresponds to a left-eigenvector of the double-row operator (3.2)

$$\langle \emptyset | \dot{B}(z) = h(z) \langle \emptyset |,$$

where $h(z)$ is given by (2.19).

Proof. When the bottom state is empty in (3.2) there is only one possible configuration on the double-row. This single state is depicted as



which propagates the empty state from below to above the double-row. The weight of all bulk vertices in this picture are 1, while the boundary vertex has weight $h(z)$. \square

A crucial property of the double-row operators of Definition 3.2 is their algebra of commutation and exchange relations. In order to prove these relations, we must first define a version of the double-row operators with finitely many columns.

Definition 3.4. For some fixed $N \in \mathbb{N}$, we define the monodromy matrices $\mathcal{T}^{(N)}(x | \mathbf{Y})$ and $\dot{\mathcal{T}}^{(N)}(z | \mathbf{Y})$. The elements of these matrices are double-row row transfer matrices with N

columns indexed by $i, j \in \{0, 1\}$. For fixed states $\mu, \nu \in \mathbb{N}$ with $\mu_1, \nu_1 \leq N$ these matrices are represented as

$$\langle \mu | \mathcal{T}_{i,j}^{(N)}(x | \mathbf{Y}) | \nu \rangle :=$$

$$\langle \mu | \dot{\mathcal{T}}_{i,j}^{(N)}(z | \mathbf{Y}) | \nu \rangle :=$$

where $x, z \in \mathbb{C}$ are horizontal spectral parameters and $\mathbf{Y} = (y_1, \dots, y_N)$ is a collection of vertical spectral parameters. As with the definition of the double-row operators, we will omit the family of vertical spectral parameters from our notation by writing $\mathcal{T}^{(N)}(x | \mathbf{Y}) = \mathcal{T}^{(N)}(x)$ and $\dot{\mathcal{T}}^{(N)}(z | \mathbf{Y}) = \dot{\mathcal{T}}^{(N)}(z)$. The elements of these matrices are represented as

$$\mathcal{T}^{(N)}(x) = \begin{pmatrix} A^{(N)}(x) & B^{(N)}(x) \\ C^{(N)}(x) & D^{(N)}(x) \end{pmatrix}, \quad \dot{\mathcal{T}}^{(N)}(z) = \begin{pmatrix} \dot{A}^{(N)}(z) & \dot{B}^{(N)}(z) \\ \dot{C}^{(N)}(z) & \dot{D}^{(N)}(z) \end{pmatrix}$$

For states $\mu, \nu \in \mathbb{W}$, we can recover the infinite column double row operators by

$$\begin{aligned} \lim_{N \rightarrow \infty} \langle \mu | A^{(N)}(x | \mathbf{Y}) | \nu \rangle &= \langle \mu | A(x | \mathbf{Y}) | \nu \rangle, \\ \lim_{N \rightarrow \infty} \langle \mu | \dot{B}^{(N)}(z | \mathbf{Y}) | \nu \rangle &= \langle \mu | \dot{B}(z | \mathbf{Y}) | \nu \rangle, \end{aligned}$$

where the infinite column row-double row operators depend on the infinite collection of vertical parameters $\mathbf{Y} = (y_1, y_2, \dots)$ and we regard the finite column operators as having dependence on the first N elements of the collection, i.e. (y_1, \dots, y_N) .

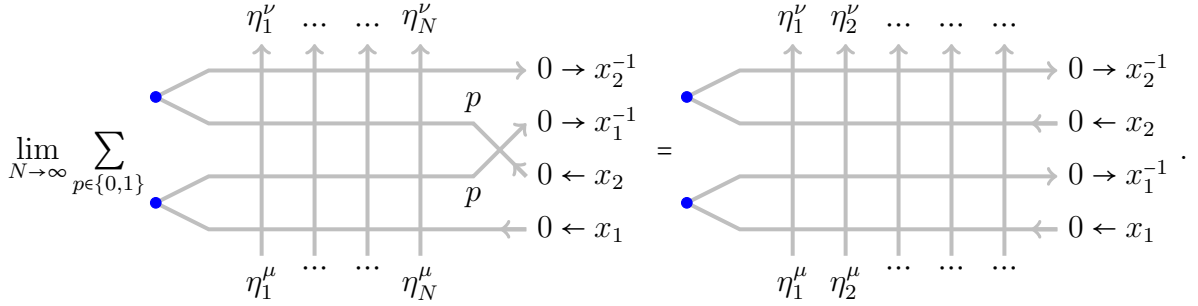
Proposition 3.5. Fix $x_1, x_2 \in \mathbb{C}$ and assume that there exists $\rho > 0$ such that

$$\left| \frac{1 - x_1 y_k}{1 - q x_1 y_k} \frac{q(1 - x_2 / y_k)}{1 - q x_2 / y_k} \right| \leq \rho < 1, \quad (3.3)$$

for all $k \in \mathbb{N}$. Then the following limit holds for states $\mu, \nu \in \mathbb{W}$

$$\lim_{N \rightarrow \infty} \sum_{p \in \{0,1\}} \langle \mu | \mathcal{T}_{0,p}^{(N)}(x_1) \mathcal{T}_{p,0}^{(N)}(x_2) | \nu \rangle R_{x_1 x_2}(0, p; p, 0) = \langle \mu | A(x_1) A(x_2) | \nu \rangle. \quad (3.4)$$

Which has a graphical interpretation



Proof. Consider $\mu, \nu \in \mathbb{W}$ to be finite states. We denote their maximum occupation by $\tau = \max\{\mu_1, \nu_1\}$. Let N be an integer satisfying $N > \tau$ which is also independent of τ . By expanding the sum on the left side of (3.4) as

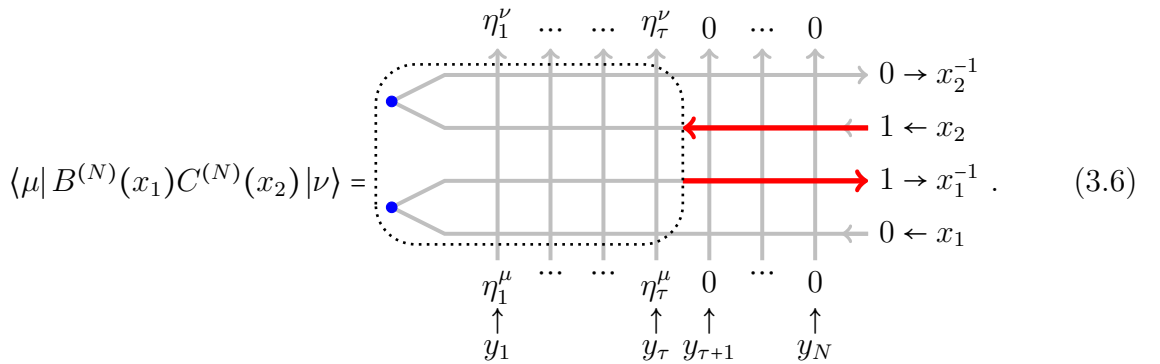
$$\lim_{N \rightarrow \infty} \langle \mu | A^{(N)}(x_1) A^{(N)}(x_2) | \nu \rangle + \frac{x_1 x_2 (1 - q)}{1 - q x_1 x_2} \lim_{N \rightarrow \infty} \langle \mu | B^{(N)}(x_1) C^{(N)}(x_2) | \nu \rangle. \quad (3.5)$$

The first term, which corresponds to $p = 0$, has the limit

$$\lim_{N \rightarrow \infty} \langle \mu | A^{(N)}(x_1) A^{(N)}(x_2) | \nu \rangle = \langle \mu | A(x_1) A(x_2) | \nu \rangle,$$

which is our final result. So all that remains to prove is that the second term ($p = 1$) in (3.5) vanishes under the limit.

Since we are interested in the large N limit, we consider for $N > \tau$ the partition function



Since $N > \tau$, it follows that $\eta_i^\mu, \eta_i^\nu = 0$ for all $\tau < i \leq N$. This freezes the columns to the right of the rectangle as shown in (3.6). The rectangle itself can be identified as a double row partition

with τ columns. By evaluating the frozen section, (3.6) is reduced to

$$\langle \mu | B^{(N)}(x_1)C^{(N)}(x_2) | \nu \rangle = \prod_{k=\tau+1}^N \left[\frac{1-x_1y_k}{1-qx_1y_k} \frac{q(1-x_2/y_k)}{1-qx_2/y_k} \right] \langle \mu | B^{(\tau)}(x_1)C^{(\tau)}(x_2) | \nu \rangle. \quad (3.7)$$

Then with condition (3.3), we can bound (3.7) as

$$|\langle \mu | B^{(N)}(x_1)C^{(N)}(x_2) | \nu \rangle| \leq \rho^{N-\tau} |\langle \mu | B^{(\tau)}(x_1)C^{(\tau)}(x_2) | \nu \rangle|,$$

so that the $p = 1$ term of (3.5) vanishes as $N \rightarrow \infty$. □

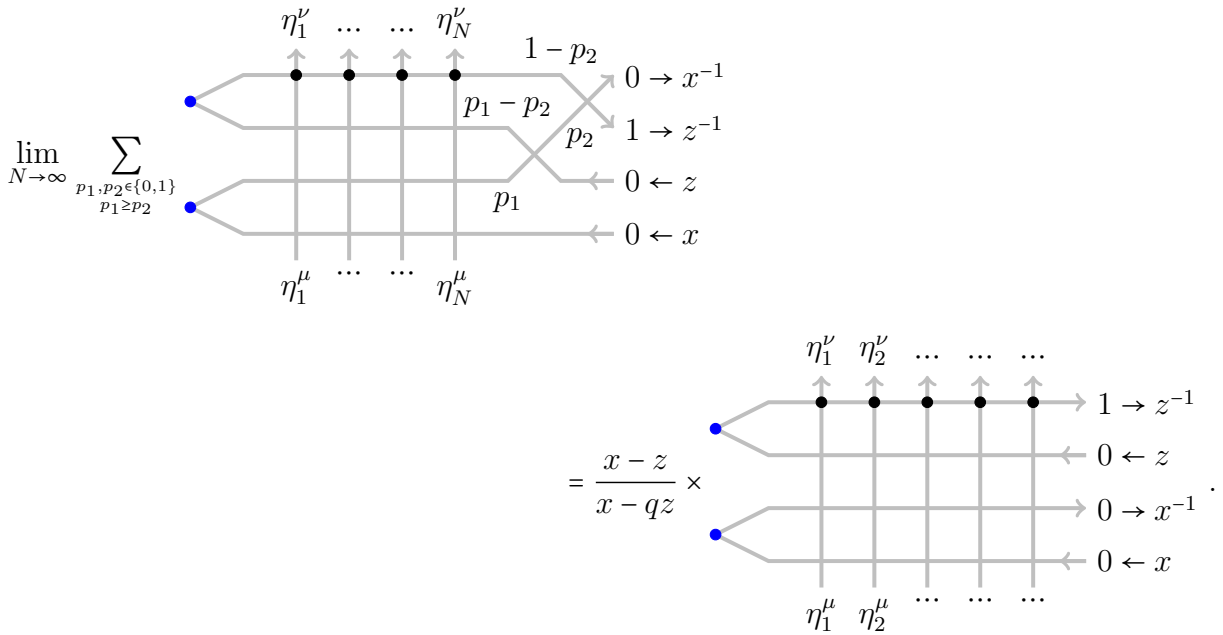
Proposition 3.6. Fix $x, z \in \mathbb{C}$ and assume that there exists $\rho > 0$ such that

$$\left| \frac{1-xy_k}{1-qxy_k} \frac{q(1-z/y_k)}{1-qz/y_k} \right| \leq \rho < 1, \quad \left| \frac{1-xy_k}{1-qxy_k} \frac{1-qzy_k}{1-z/y_k} \right| \leq \rho < 1, \quad (3.8)$$

for all $k \in \mathbb{N}$. Then the following limit holds

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{\substack{p_1, p_2 \in \{0,1\} \\ p_1 \geq p_2}} \langle \mu | \mathcal{T}_{0,p_1}^{(N)}(x) \dot{\mathcal{T}}_{p_1-p_2, 1-p_2}^{(N)}(z) | \nu \rangle R_{xz}(0, p_1; p_1-p_2, p_2) R_{z/x}(p_2, 1-p_2; 0, 1) \\ = \frac{x-z}{x-qz} \langle \mu | A(x) \dot{B}(z) | \nu \rangle. \end{aligned} \quad (3.9)$$

Which has graphical interpretation



Proof. In a similar manner to the proof of Proposition 3.5, we fix $\mu, \nu \in \mathbb{W}$ with maximum occupation $\tau = \max\{\mu_1, \nu_1\}$. We may write the terms in the sum over p_1, p_2 explicitly as

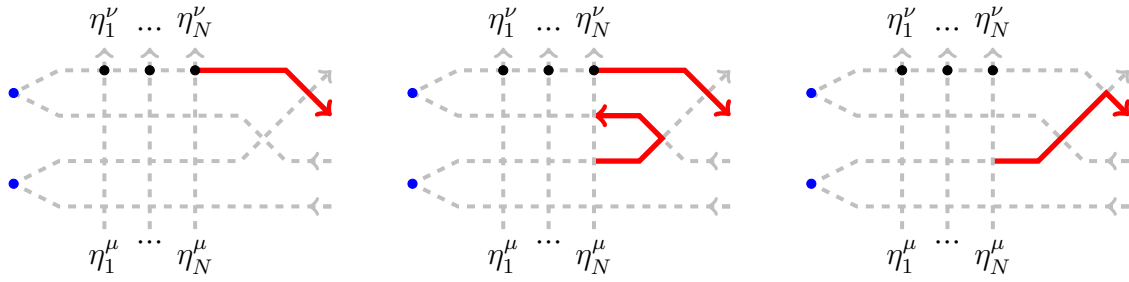


Figure 3.1: The terms of the sum (3.9) corresponding to $(p_1, p_2) = (0, 0), (1, 0), (1, 1)$ respectively.

$$\lim_{N \rightarrow \infty} \langle \mu | \left[\frac{x-z}{x-qz} A^{(N)}(x) \dot{B}^{(N)}(z) + \frac{xz(1-q)}{1-qxz} \frac{x-z}{x-qz} B^{(N)}(x) \dot{D}^{(N)}(z) + \frac{1-xz}{1-qxz} \frac{x(1-q)}{x-qz} B^{(N)}(x) \dot{A}^{(N)}(z) \right] | \nu \rangle, \quad (3.10)$$

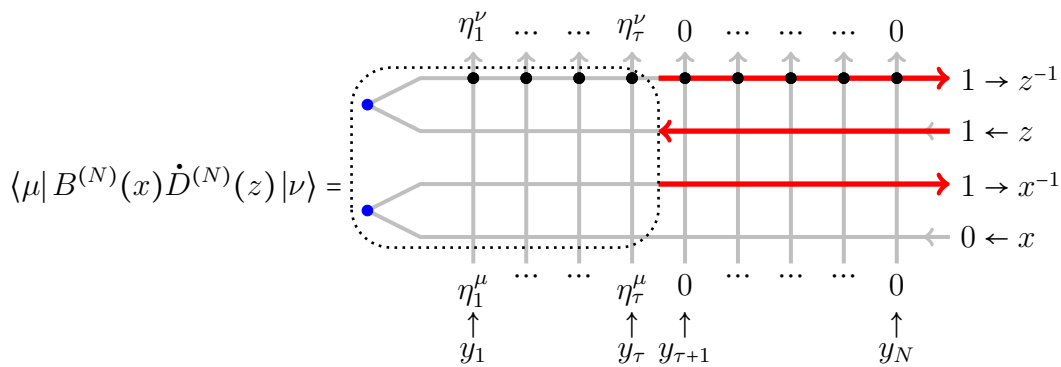
where each of the terms corresponds to $(p_1, p_2) = (0, 0), (1, 0), (1, 1)$ respectively. Each of these three terms are depicted in Figure 3.1. We will analyse each term individually.

Firstly, the $p_1 = p_2 = 0$ term has the limit

$$\lim_{N \rightarrow \infty} \langle \mu | \frac{x-z}{x-qz} A^{(N)}(x) \dot{B}^{(N)}(z) | \nu \rangle = \frac{x-z}{x-qz} \langle \mu | A(x) \dot{B}(z) | \nu \rangle,$$

which is the desired result. So it remains to show that the other terms vanish in the limit.

Consider now the term associated to $p_1 = 1, p_2 = 0$. By virtue of the configurations being finite, this term can be decomposed into two possible configuration types. For some integer $N > \tau$ which is independent of τ , these configurations are summed over



$$+ \sum_{\ell=\tau+1}^N \left(\text{Diagram} \right), \quad (3.11)$$

where the dotted rectangles can be identified as stacked double row partition functions with τ columns. The columns attached to the right of these rectangles in (3.11) can be explicitly evaluated as

$$\begin{aligned} \langle \mu | B^{(N)}(x) \dot{D}^{(N)}(z) | \nu \rangle &= \prod_{k=\tau+1}^N \left[\frac{1 - xy_k}{1 - qxy_k} \frac{q(1 - z/y_k)}{1 - qz/y_k} \right] \langle \mu | B^{(\tau)}(x) \dot{D}^{(\tau)}(z) | \nu \rangle \\ &+ \prod_{k=\tau+1}^N \frac{1 - xy_k}{1 - qxy_k} \sum_{\ell=\tau+1}^N \frac{1 - q}{1 - qz/y_\ell} \frac{1 - q}{1 - zy_\ell} \prod_{k=\tau+1}^{\ell-1} \frac{1 - qzy_k}{1 - zy_k} \prod_{k=\ell+1}^N \frac{q(1 - z/y_k)}{1 - qz/y_k} \langle \mu | B^{(\tau)}(x) \dot{A}^{(\tau)}(z) | \nu \rangle. \end{aligned}$$

Using condition (3.9), we can then bound this term effectively as

$$\begin{aligned} |\langle \mu | B^{(N)}(x) \dot{D}^{(N)}(z) | \nu \rangle| &\leq \rho^{N-\tau} |\langle \mu | B^{(\tau)}(x) \dot{D}^{(\tau)}(z) | \nu \rangle| \\ &+ \rho^{N-\tau-1} (N-\tau) \max_{\ell \in \{\tau+1, \dots, N\}} \left\{ \left| \frac{1 - xy_\ell}{1 - qxy_\ell} \right| \right\} \max_{\ell \in \{\tau+1, \dots, N\}} \left\{ \left| \frac{1 - q}{1 - qz/y_\ell} \frac{1 - q}{1 - zy_\ell} \right| \right\} |\langle \mu | B^{(\tau)}(x) \dot{A}^{(\tau)}(z) | \nu \rangle|. \end{aligned} \quad (3.12)$$

We note here that the conditions (3.8) imply that, for all $N > \tau$, the points $qxy_\ell, qz/y_\ell, zy_\ell$ are all bounded uniformly away 1 for all $\ell \in \{\tau + 1, \dots, N\}$, and hence the maxima in the second term in (3.12) remain finite as $N \rightarrow \infty$, and therefore both terms in (3.12) vanish as $N \rightarrow \infty$.

It remains to show that the third term in (3.10) vanishes. This follows similarly as

$$\langle \mu | B^{(N)}(x) \dot{A}^{(N)}(z) | \nu \rangle = \prod_{k=\tau+1}^N \left[\frac{1 - xy_k}{1 - qxy_k} \frac{1 - qzy_k}{1 - zy_k} \right] \langle \mu | B^{(\tau)}(x) \dot{A}^{(\tau)}(z) | \nu \rangle,$$

which can be bounded using condition (3.9) as

$$|\langle \mu | B^{(N)}(x) \dot{A}^{(N)}(z) | \nu \rangle| \leq \rho^{N-\tau} |\langle \mu | B^{(\tau)}(x) \dot{A}^{(\tau)}(z) | \nu \rangle|.$$

The limit of this term vanishes also, so we may conclude that only the $p_1 = p_2 = 0$ term remains in the limit $N \rightarrow \infty$ which implies the result. \square

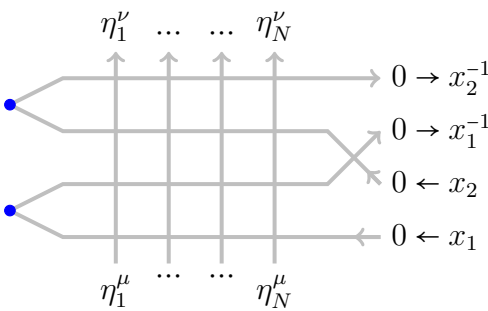
Proposition 3.7. Fix $x_1, x_2 \in \mathbb{C}$ and assume there exists $\rho > 0$ such that

$$\left| \frac{1 - x_i y_k}{1 - q x_i y_k} \frac{q(1 - x_j/y_k)}{1 - q x_j/y_k} \right| \leq \rho < 1, \tag{3.13}$$

for all $i \neq j$ and $k \in \mathbb{N}$. Then the double-row operators from (3.1) commute:

$$A(x_1)A(x_2) = A(x_2)A(x_1). \tag{3.14}$$

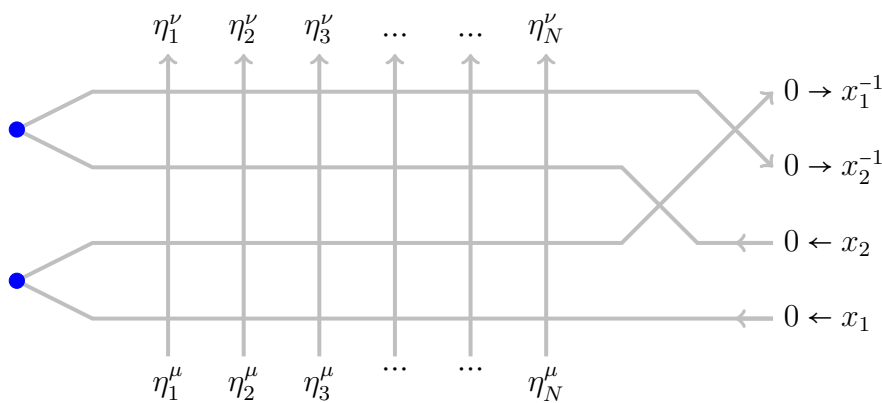
Proof. Let $\mu, \nu \in \mathbb{W}$ with maximum occupation $\tau = \max\{\mu_1, \nu_1\}$. Also let N be an integer satisfying $N \geq \tau$ and consider the double row partition functions with N columns

$$\sum_{p \in \{0,1\}} \langle \mu | \mathcal{T}_{0,p}^{(N)}(x_1) \mathcal{T}_{p,0}^{(N)}(x_2) | \nu \rangle R_{x_1 x_2}(0, p; p, 0) =$$


$$, \tag{3.15}$$

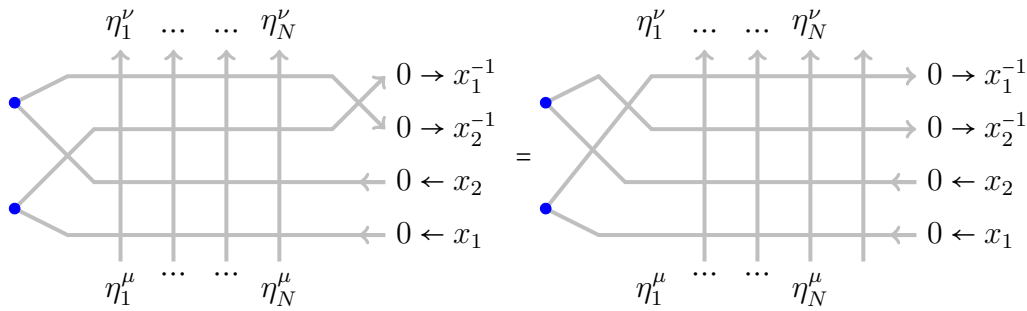
which, due to Proposition 3.5, has limit $\langle \mu | A(x_1)A(x_2) | \nu \rangle$ as $N \rightarrow \infty$.

We may then append an additional intertwining vertex to the lattice after the last column. The boundary conditions of this vertex mean that there is only one allowed vertex configuration on the intertwiner so that it can be added at no overall cost to the partition function. We have the (3.15) is equal to

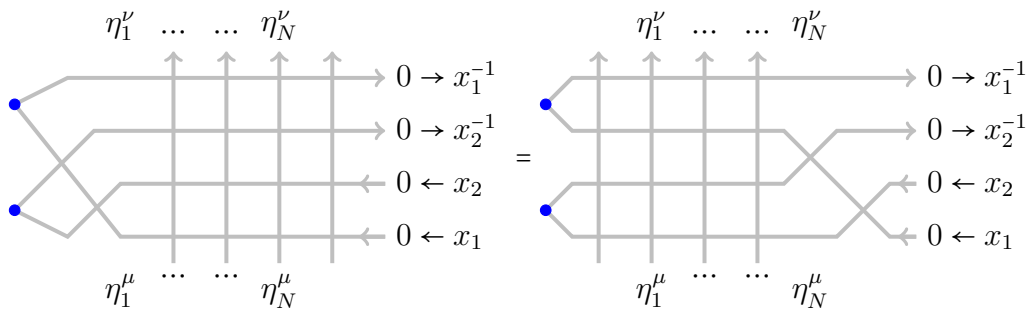


We may then repeatedly apply the Yang–Baxter equation (2.4) to manipulate the diagram. This equation can be applied successively to each column of partition function leading to the

relation



At this point the reflection equation (2.21) can be applied, followed by the Yang–Baxter equation to push the intertwining vertices back to the right edge of the partition function. This yields



At this point, the intertwiner at the bottom-right of the diagram can be removed at no cost the partition function due to the boundary conditions. This yields

(3.16)

which we recognise as (3.15) with x_1 and x_2 interchanged. The limit of (3.16) can then be evaluated as $\langle \mu | A(x_2)A(x_1) | \nu \rangle$ as $N \rightarrow \infty$ due to condition (3.13). Since (3.15) and (3.16) are equal for all $N \geq \tau$, we can conclude that their limits must be equal. This is the result (3.14). \square

Proposition 3.8. Given configurations $\mu, \nu \in \mathbb{W}$ let N be an integer $N \geq \max\{\mu_1, \nu_1\}$. For

z_1, z_2 the double-row operators with N columns commute

$$\langle \mu | \dot{B}^{(N)}(z_1) \dot{B}^{(N)}(z_2) | \nu \rangle = \langle \mu | \dot{B}^{(N)}(z_2) \dot{B}^{(N)}(z_1) | \nu \rangle. \quad (3.17)$$

This can be extended to the case of infinite columns to obtain the commutation relation of the double-row operators (3.2)

$$\dot{B}(z_1) \dot{B}(z_2) = \dot{B}(z_2) \dot{B}(z_1). \quad (3.18)$$

Proof. Let $\mu, \nu \in \mathbb{W}$ with maximum occupation $\tau = \max\{\mu_1, \nu_1\}$. Also let $N \geq \tau$ and consider the following double row partition functions with N columns

$$f^{(N)}(z_1, z_2) := \begin{array}{c} \begin{array}{cccc} & \eta_1^\nu & \dots & \dots & \eta_N^\nu \\ & \uparrow & \uparrow & \uparrow & \uparrow \\ \bullet & \text{---} & \text{---} & \text{---} & \text{---} & \rightarrow 1 \rightarrow z_1^{-1} \\ & \uparrow & \uparrow & \uparrow & \uparrow & \rightarrow 1 \rightarrow z_2^{-1} \\ \bullet & \text{---} & \text{---} & \text{---} & \text{---} & \leftarrow 0 \leftarrow z_2 \\ & \uparrow & \uparrow & \uparrow & \uparrow & \leftarrow 0 \leftarrow z_1 \\ & \eta_1^\mu & \dots & \dots & \eta_N^\mu \end{array} \end{array}, \quad (3.19)$$

with two intertwining vertices appended to the right of the diagram. We note that these intertwiners are frozen in their own right and can simply be evaluated. For any $N \geq \tau$ this yields

$$f^{(N)}(z_1, z_2) = \frac{1 - z_1 z_2}{1 - q z_1 z_2} \langle \mu | \dot{B}^{(N)}(z_1) \dot{B}^{(N)}(z_2) | \nu \rangle. \quad (3.20)$$

It is important to note here that this holds for all $N \geq \tau$ here rather than in just under the large N limit as with the proof of Proposition 3.7.

Then following the same procedure as the proof of Proposition 3.7, we may apply the Yang–Baxter equation (2.4) and reflection equation (2.21) to manipulate the diagram (3.19) to obtain

$$f^{(N)}(z_1, z_2) = \begin{array}{c} \begin{array}{cccc} & \eta_1^\nu & \dots & \dots & \eta_N^\nu \\ & \uparrow & \uparrow & \uparrow & \uparrow \\ \bullet & \text{---} & \text{---} & \text{---} & \text{---} & \rightarrow 1 \rightarrow z_1^{-1} \\ & \uparrow & \uparrow & \uparrow & \uparrow & \rightarrow 1 \rightarrow z_2^{-1} \\ \bullet & \text{---} & \text{---} & \text{---} & \text{---} & \leftarrow 0 \leftarrow z_2 \\ & \uparrow & \uparrow & \uparrow & \uparrow & \leftarrow 0 \leftarrow z_1 \\ & \eta_1^\mu & \dots & \dots & \eta_N^\mu \end{array} \end{array}. \quad (3.21)$$

The intertwiners on the right side of (3.21) are also frozen in their own right and can be

evaluated as

$$f^{(N)}(z_1, z_2) = \frac{1 - z_1 z_2}{1 - q z_1 z_2} \langle \mu | \dot{B}^{(N)}(z_2) \dot{B}^{(N)}(z_1) | \nu \rangle. \quad (3.22)$$

The result (3.17) is obtained by comparing (3.20) and (3.22). Taking the large N limit yields the result (3.18). \square

Proposition 3.9. Fix $x, z \in \mathbb{C}$ and assume there exists $\rho > 0$ such that

$$\left| \frac{1 - xy_k}{1 - qxy_k} \frac{q(1 - z/y_k)}{1 - qz/y_k} \right| \leq \rho < 1, \quad \left| \frac{1 - xy_k}{1 - qxy_k} \frac{1 - qzy_k}{1 - zy_k} \right| \leq \rho < 1, \quad (3.23)$$

for all $k \in \mathbb{N}$. Then the double-row operators from Definition 3.2 obey the exchange relation

$$A(x) \dot{B}(z) = \frac{x - qz}{x - z} \frac{1 - xz}{1 - qxz} \dot{B}(z) A(x). \quad (3.24)$$

Proof. Let $\mu, \nu \in \mathbb{W}$ with maximum occupation $\tau = \max\{\mu_1, \nu_1\}$. Also let N be an integer satisfying $N \geq \tau$ and consider the double row partition functions with N columns

$$\sum_{\substack{p_1, p_2 \in \{0, 1\} \\ p_1 \geq p_2}} \langle \mu | \mathcal{T}_{0, p_1}^{(N)}(x) \dot{\mathcal{T}}_{p_1 - p_2, 1 - p_2}^{(N)}(z) | \nu \rangle R_{xz}(0, p_1; p_1 - p_2, p_2) R_{z/x}(p_2, 1 - p_2; 0, 1)$$

$$= f^{(N)}(x, z). \quad (3.25)$$

Due to condition (3.23) and Proposition 3.6, the limit of (3.25) is

$$\lim_{N \rightarrow \infty} f^{(N)}(x, z) = \frac{x - z}{x - qz} \langle \mu | A(x) \dot{B}(z) | \nu \rangle \quad (3.26)$$

We may manipulate the diagram of (3.25) to obtain an exchange relation in a similar way to the proof of Proposition 3.7.

By following the same steps as in the proof of Proposition 3.7, we may arrive at the following

diagram which is equal to (3.25) as a partition function

$$f^{(N)}(x, z) = \begin{array}{c} \eta_1^\nu \quad \dots \quad \dots \quad \eta_N^\nu \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ \begin{array}{c} \text{Diagram 1: A grid of paths with blue dots at the top-left and bottom-left corners. The top row has arrows pointing right labeled } 0 \rightarrow x^{-1}, 1 \rightarrow z^{-1}, 0 \leftarrow z, 0 \leftarrow x. \text{ The bottom row has arrows pointing left labeled } 0 \leftarrow z, 1 \rightarrow z^{-1}, 0 \leftarrow x. \text{ Vertical lines connect the top and bottom rows, with some crossings.} \end{array} \\ \eta_1^\mu \quad \dots \quad \dots \quad \eta_N^\mu \end{array} = \frac{1-xz}{1-qxz} \begin{array}{c} \eta_1^\nu \quad \dots \quad \dots \quad \eta_N^\nu \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ \begin{array}{c} \text{Diagram 2: Similar to Diagram 1, but the top and bottom rows are swapped. The top row has arrows pointing left labeled } 0 \leftarrow x, 0 \leftarrow z, 1 \rightarrow z^{-1}, 0 \rightarrow x^{-1}. \text{ The bottom row has arrows pointing right labeled } 0 \rightarrow x^{-1}, 1 \rightarrow z^{-1}, 0 \leftarrow z, 0 \leftarrow x. \end{array} \\ \eta_1^\mu \quad \dots \quad \dots \quad \eta_N^\mu \end{array} .$$

The last equality follows from noticing that the intertwining vertices are both frozen by the boundary conditions, so they may be evaluated as a factor and removed from the diagram. The limit of the right diagram yields

$$\lim_{N \rightarrow \infty} f^{(N)}(x, z) = \frac{1-xz}{1-qxz} \langle \mu | \dot{B}(z) A(x) | \nu \rangle,$$

which can be combined with (3.26) to give the result. □

3.2 Symmetric functions

We will now define a partition function which will be central to much the remainder of this work.

Definition 3.10. Fix two alphabets (x_1, \dots, x_L) and (z_1, \dots, z_M) , and configurations $\mu, \nu \in \mathbb{W}$. We define

$$G_{\nu/\mu}(x_1, \dots, x_L | \mathbf{Y}) = \langle \mu | A(x_1 | \mathbf{Y}) \cdots A(x_L | \mathbf{Y}) | \nu \rangle, \tag{3.27}$$

$$F_{\mu/\nu}(z_1, \dots, z_M | \mathbf{Y}) = \langle \mu | \dot{B}(z_1 | \mathbf{Y}) \cdots \dot{B}(z_M | \mathbf{Y}) | \nu \rangle. \tag{3.28}$$

The functions (3.27) and (3.28) can be represented diagrammatically by stacking double-row

operators (3.1) and (3.2) appropriately. We find

$$G_{\nu/\mu}(x_1, \dots, x_L | \mathbf{Y}) = \begin{array}{c} \begin{array}{cccccc} & \eta'_1 & \eta'_2 & \eta'_3 & \dots & \dots \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & \eta''_1 & \eta''_2 & \eta''_3 & \dots & \dots \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & y_1 & y_2 & y_3 & \dots & \dots \end{array} \\ \begin{array}{l} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{l} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} \begin{array}{l} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} \begin{array}{l} 0 \rightarrow x_L^{-1} \\ 0 \leftarrow x_L \\ \vdots \\ \vdots \\ 0 \rightarrow x_1^{-1} \\ 0 \leftarrow x_1 \end{array} \end{array}, \quad (3.29)$$

and

$$F_{\mu/\nu}(z_1, \dots, z_M | \mathbf{Y}) = \begin{array}{c} \begin{array}{cccccc} & \eta'_1 & \eta'_2 & \eta'_3 & \dots & \dots \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & \eta''_1 & \eta''_2 & \eta''_3 & \dots & \dots \\ & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ & y_1 & y_2 & y_3 & \dots & \dots \end{array} \\ \begin{array}{l} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{l} \nearrow \\ \searrow \\ \nearrow \\ \searrow \end{array} \begin{array}{l} \longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \end{array} \begin{array}{l} 1 \rightarrow z_M^{-1} \\ 0 \leftarrow z_M \\ \vdots \\ \vdots \\ 1 \rightarrow z_1^{-1} \\ 0 \leftarrow z_1 \end{array} \end{array}. \quad (3.30)$$

We will also, where convenient, omit the family of parameters \mathbf{Y} from our notation.

The primary focus for the remainder of this work will be the partition function depicted in (3.29). This will ultimately be shown to reduce to describing the behaviour of the ASEP on the half-line with generic open boundary conditions.

In many cases we will be interested in the partition function (3.29) whose bottom state is empty, so that $\mu = \emptyset$. While for the partition function (3.30) we will often be interested in cases when the top state is empty, so that $\nu = \emptyset$. In such cases we will write

$$\begin{aligned} G_{\nu/\emptyset}(x_1, \dots, x_L | \mathbf{Y}) &=: G_\nu(x_1, \dots, x_L) \\ F_{\mu/\emptyset}(z_1, \dots, z_M | \mathbf{Y}) &=: F_\mu(z_1, \dots, z_M) \end{aligned}$$

Corollary 3.11 (of Propositions 3.7 and 3.8). Fix $\mu, \nu \in \mathbb{W}$. Given parameters $x_1, \dots, x_L \in \mathbb{C}$ and $\mathbf{Y} \in \mathbb{C}^{\mathbb{N}}$ all satisfying the conditions (3.13), the partition function $G_{\nu/\mu}(x_1, \dots, x_L | \mathbf{Y})$ is symmetric under permuting its x -alphabet. Given $z_1, \dots, z_M \in \mathbb{C}$, the partition function $F_{\mu/\nu}(z_1, \dots, z_M | \mathbf{Y})$ is symmetric in permuting its z -alphabet.

Proposition 3.12 (Branching relations). The partition functions from Definition 3.10 obey the branching relations

$$\begin{aligned} G_{\nu/\mu}(x_1, \dots, x_{L+M}) &= \sum_{\kappa} G_{\nu/\kappa}(x_{M+1}, \dots, x_{L+M}) G_{\kappa/\mu}(x_1, \dots, x_M), \\ F_{\mu/\nu}(z_1, \dots, z_{L+M}) &= \sum_{\lambda} F_{\lambda/\nu}(z_{M+1}, \dots, z_{L+M}) F_{\mu/\lambda}(z_1, \dots, z_M). \end{aligned} \quad (3.31)$$

Proof. This can be seen by inserting a sum over a complete set of states between the double-row operators in (3.27) and (3.28). \square

Proposition 3.13. Fix a configuration $\mu \in \mathbb{W}$. The partition function (3.27) obeys the sum-to-unity property

$$\sum_{\nu} G_{\nu/\mu}(x_1, \dots, x_L) = 1.$$

Proof. This follows from the stochasticity of the bulk and boundary vertices, Propositions 2.2 and 2.16. \square

3.2.1 Cauchy summation identity

In this section, we use the exchange relation (3.24) to prove an infinite summation identity of Cauchy type between the functions (3.27) and (3.28). This identity is the hint of a deeper orthogonality theory behind these functions that we plan to explore in a future text.

Theorem 3.14. Fix the alphabets $(x_1, \dots, x_L), (z_1, \dots, z_M)$ and assume there exists $\rho > 0$ such that

$$\left| \frac{1 - x_i y_k}{1 - q x_i y_k} \frac{q(1 - z_j / y_k)}{1 - q z_j / y_k} \right| \leq \rho < 1, \quad \left| \frac{1 - x_i y_k}{1 - q x_i y_k} \frac{1 - q z_j y_k}{1 - z_j y_k} \right| \leq \rho < 1, \quad (3.32)$$

for all $1 \leq i \leq L, 1 \leq j \leq M$ and $k \in \mathbb{N}$. Then the partition functions from Definition 3.10 satisfy the skew Cauchy identity

$$\begin{aligned} \sum_{\kappa} G_{\kappa/\mu}(x_1, \dots, x_L) F_{\kappa/\nu}(z_1, \dots, z_M) \\ = \prod_{i=1}^M \prod_{j=1}^L \left[\frac{x_j - qz_i}{x_j - z_i} \frac{1 - z_i x_j}{1 - qz_i x_j} \right] \sum_{\lambda} F_{\mu/\lambda}(z_1, \dots, z_M) G_{\nu/\lambda}(x_1, \dots, x_L). \end{aligned} \quad (3.33)$$

where the sum on the left is an infinite sum over $\kappa \in \mathbb{W}$ while the sum on the right is a finite sum is over $\lambda \in \mathbb{W}$ contained within μ . That is, $\lambda_i < \mu_i$ for all i less than the lengths of both λ and μ .

Proof. We begin by writing the left-hand side of (3.33) in double-row operator notation as

$$\sum_{\kappa} G_{\kappa/\mu}(x_1, \dots, x_L) F_{\kappa/\nu}(z_1, \dots, z_M) = \langle \mu | A(x_1) \cdots A(x_L) \dot{B}(z_1) \cdots \dot{B}(z_M) | \nu \rangle.$$

From here we may commute the A operators through the \dot{B} operators using Proposition 3.9. Each commutation will generate a multiplicative rational factor; collecting all of these, we have

$$\begin{aligned} \langle \mu | A(x_1) \cdots A(x_L) \dot{B}(z_1) \cdots \dot{B}(z_M) | \nu \rangle \\ = \prod_{i=1}^M \prod_{j=1}^L \left[\frac{x_j - qz_i}{x_j - z_i} \frac{1 - z_i x_j}{1 - qz_i x_j} \right] \langle \mu | \dot{B}(z_1) \cdots \dot{B}(z_M) A(x_1) \cdots A(x_L) | \nu \rangle. \end{aligned} \quad (3.34)$$

The right side of this may be recognised as the right-hand side of (3.33) □

Corollary 3.15. With the same set of assumptions as in Theorem 3.14, one has the following summation identity

$$\sum_{\kappa} G_{\kappa}(x_1, \dots, x_L) F_{\kappa/\nu}(z_1, \dots, z_M) = \prod_{i=1}^M h(z_i) \prod_{i=1}^M \prod_{j=1}^L \left[\frac{x_j - qz_i}{x_j - z_i} \frac{1 - z_i x_j}{1 - qz_i x_j} \right] G_{\nu}(x_1, \dots, x_L). \quad (3.35)$$

Proof. This is the $\mu = \emptyset$ case of Theorem 3.14. Indeed, following the same steps as in the previous proof we use the fact, due to Proposition 3.3, that

$$\langle \emptyset | \dot{B}(z_1) \cdots \dot{B}(z_M) = \prod_{i=1}^M h(z_i) \langle \emptyset |$$

in (3.34). This recovers precisely (3.35). □

Remark 3.16. Subject to the positivity, Propositions 3.12 and 3.13 motivate the understanding of $G_{\nu/\mu}$ as the propagator of a discrete-time Markov process from initial state μ to state ν . In order to define an appropriate probability measure, the spectral parameters (x_1, \dots, x_L) and

$Y = (y_1, y_2, \dots)$ need to be fixed so that the weights from table (2.2) and (2.20) are all real, non-negative and less than or equal to 1.

We may also regard the symmetric function F as an observable of this Markov process; indeed, the left-side of (3.35) can be interpreted as the formal definition of the expectation value of the observable $F_{\kappa/\nu}$ with respect to the discrete measure G_κ . Provided that the right-hand side of (3.35) can be evaluated explicitly, Corollary 3.15 then provides a systematic method for evaluating the expectation value of the observable $F_{\kappa/\nu}$.

3.2.2 Recursion relations

In this section we demonstrate a series of recursion relations for the symmetric function (3.27) which will prove important in deriving a formula for the function. These relations follow from the unitary of the R and K -matrices (Propositions 2.6 and 2.19) and the R -matrix factorisation (Proposition 2.4).

Proposition 3.17. We have the following relations for the row-operator (3.1):

$$A(0) = 0, \tag{3.36}$$

$$A(\pm 1) = \text{id}, \tag{3.37}$$

where id is the identity within $\text{End}(\text{Span } \mathbb{W})$. Further, for fixed $x \in \mathbb{C}$, if there exists $\rho > 0$ such that

$$\left| \frac{1 - xy_k}{1 - qxy_k} \frac{q(xy_k - 1)}{xy_k - q} \right| \leq \rho < 1$$

for all $j \in \mathbb{N}$, then it holds that

$$A(x)A(x^{-1}) = \text{id}. \tag{3.38}$$

Proof. We will proceed with the proof of each identity separately. In each proof, we will consider arbitrary finite configurations $\mu, \nu \in \mathbb{W}$.

(i) **Proof of (3.36).** It is sufficient here to consider the partition function

$$f(x) = \langle \mu | A(x) | \nu \rangle = \bullet \begin{array}{c} \begin{array}{cccccc} \eta_1^\nu & \eta_2^\nu & \eta_3^\nu & \dots & \dots & \dots \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ \text{---} & \text{---} & \text{---} & \text{---} & \text{---} & \text{---} \rightarrow 0 \rightarrow x^{-1} \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \eta_1^\mu & \eta_2^\mu & \eta_3^\mu & \dots & \dots & \dots \\ \leftarrow 0 \leftarrow x \end{array} \end{array}, \tag{3.39}$$

where we will be interested in the case $x = 0$. We note here that all contributions from the weights which make up (3.39) are non-singular at $x = 0$. In this case, the vertex

configuration with weight $z(1-q)/(1-qz) = 0$ from (2.2) cannot appear. This means that the lower row of (3.39) cannot carry any occupations on any horizontal edges.

This will mean that the only possible boundary vertex configuration will be that of weight $h(x) = 1$. Since the weight $z(1-q)/(1-qz) = 0$ will not appear on the upper row, each horizontal edge on the upper row must be occupied.

However, this is not permitted since the boundary conditions on the right edge must be vacant. This means that there is no possible path configuration in (3.39). It follows that $f(0) = 0$, which gives the result.

- (ii) **Proof of (3.37).** We will again consider (3.39), this time with $x = \pm 1$. In this case the function $h(\pm 1) = 0$ so that the boundary vertex must have both entry and exits either both occupied or both vacant. Both of these configurations carry weight 1.

This is equivalent to the fact that that $K(\pm 1)$ is the identity matrix. We are then left with

$$f(\pm 1) = \begin{array}{c} \eta'_1 \quad \eta'_2 \quad \eta'_3 \quad \dots \quad \dots \quad \dots \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \rightarrow 0 \\ \leftarrow 0 \\ \eta''_1 \quad \eta''_2 \quad \eta''_3 \quad \dots \quad \dots \quad \dots \end{array} = \begin{array}{c} \eta'_1 \quad \eta'_2 \quad \eta'_3 \quad \dots \quad \dots \quad \dots \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ \eta''_1 \quad \eta''_2 \quad \eta''_3 \quad \dots \quad \dots \quad \dots \end{array} \begin{array}{c} \rightarrow 0 \\ \leftarrow 0 \end{array},$$

where we have made repeated use of the unitarity condition of R -matrices (Proposition 2.6), to produce the final equality. From here we may conclude that $f(\pm 1) = 1$ which gives the result.

- (iii) **Proof of (3.38).** Let $\tau = \max\{\mu_1, \nu_1\}$ and let N be an integer satisfying $N \geq \tau$. Then for $x \neq 0$, consider

$$\sum_{p \in \{0,1\}} \langle \mu | \mathcal{T}_{0,p}^{(N)}(x) \mathcal{T}_{p,0}^{(N)}(x^{-1}) | \nu \rangle R_{x/x}(0, p, p, 0) = \begin{array}{c} \eta'_1 \quad \dots \quad \dots \quad \eta'_N \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \rightarrow 0 \rightarrow x \\ \rightarrow 0 \rightarrow x^{-1} \\ \leftarrow 0 \leftarrow x^{-1} \\ \leftarrow 0 \leftarrow x \\ \eta''_1 \quad \dots \quad \dots \quad \eta''_N \end{array}, \tag{3.40}$$

which is equal to $\langle \mu | A(x) A(x^{-1}) | \nu \rangle$ in the limit $N \rightarrow \infty$ from Proposition 3.5. Due to the factorisation property of the R -matrix (Proposition 2.4), we may recognise that the

intertwining vertex in (3.40) is the identity. We may simplify (3.40) as

$$= \quad , \quad (3.41)$$

where we have use the unitarity property of the R -matrix (Proposition 2.6) over the N columns to obtain the second diagram. Within this diagram, we may use the unitarity property of the K -matrix (Proposition 2.19) to simplify (3.41) as

$$,$$

from which we may again use R -matrix unitarity to remove all vertices. This shows that (3.44) is equal to $\langle \mu | \nu \rangle$, which yields the result under the limit $N \rightarrow \infty$.

□

The relations in Proposition 3.17 for the row-operators lead to recursion relations for the partition function (3.27).

Corollary 3.18. The partition function $G_{\nu/\mu}(x_1, \dots, x_L | \mathbf{Y})$ from (3.27) satisfies the following recursion relations

$$\begin{aligned}
 G_{\nu/\mu}(x_1, \dots, x_L) \Big|_{x_i=0} &= 0, \\
 G_{\nu/\mu}(x_1, \dots, x_L) \Big|_{x_i=\pm 1} &= G_{\nu/\mu}(x_1, \dots, \hat{x}_i, \dots, x_L), \\
 G_{\nu/\mu}(x_1, \dots, x_L) \Big|_{x_j=1/x_k} &= G_{\nu/\mu}(x_1, \dots, \hat{x}_j, \dots, \hat{x}_k, \dots, x_L),
 \end{aligned} \quad (3.42)$$

for all $1 \leq i \leq L$ and $1 \leq j < k \leq L$. Here, \hat{x}_i means that x_i is omitted from the alphabet (x_1, \dots, x_L) .

Proof. These follow from the row-operator identities, (3.36), (3.37), (3.38), in Proposition 3.17 as well as the fact that $G_{\nu/\mu}$ is symmetric in its alphabet. □

the lattice to obtain the following object

$$g_\nu^{(N)}(x_1, \dots, x_L) := \sum_{p_1, p_2, \dots \in \{0,1\}} \text{Diagram} \quad , \quad (3.44)$$

where the sum is over all occupations of the edges of the appended triangle. Absorbing the sum into the notation of the partition function, we can apply the Yang–Baxter equation to move the intertwiners to be adjacent to the boundary. This is depicted as

$$g_\nu^{(N)}(x_1, \dots, x_L) \text{ Diagram}$$

We note that the left-moving sector of the vertical columns is frozen with no occupations. As a result, these vertices may be evaluated to 1 and removed from the diagram. In the limit $N \rightarrow \infty$ this results in the desired diagram (3.43).

Now directly consider the large N limit of (3.44). Due to the conditions (3.13) the only term which survives this limit is the one with $p_1 = \dots = p_{2L-2} = 0$. This forces all $p_i = 0$ so that the sum of (3.44) collapses into a single term where all intertwining vertices are equal to 1. This

yields

$$\lim_{N \rightarrow \infty} g_\nu^{(N)}(x_1, \dots, x_L) = \lim_{N \rightarrow \infty} \langle \emptyset | A^{(N)}(x_1) \cdots A^{(N)}(x_L) | \nu \rangle = G_\nu(x_1, \dots, x_L).$$

□

The diagram in (3.43) leads to the following result.

Theorem 3.21. When both the initial and final configurations are empty, i.e. $\mu = \nu = \emptyset$, the symmetric function (3.27) reduces to the triangular partition function (2.22)

$$G_\emptyset(x_1, \dots, x_L) = Z_L(x_1, \dots, x_L).$$

In particular, G_\emptyset does not depend on the collection of vertical spectral parameters Y .

Proof. This result is immediate from diagram (3.43) in Lemma 3.20 when we set ν to be empty. This causes there to be no occupations on any of columns or rows past the point of y_i -dependence. This means that the entire bulk on the right evaluates to 1 and may be removed without effect on the partition function evaluation. □

3.2.4 Cauchy summation identity revisited

Corollary 3.22 (of Theorem 3.14). Fix alphabets $(x_1, \dots, x_L), (z_1, \dots, z_M)$ and assume that there exists $\rho > 0$ such that conditions (3.32) hold. Then the following Cauchy summation identity holds

$$\begin{aligned} \sum_{\kappa} G_{\kappa}(x_1, \dots, x_L) F_{\kappa}(z_1, \dots, z_M) &= \prod_{i=1}^M h(z_i) \prod_{i=1}^M \prod_{j=1}^L \left[\frac{x_j - qz_i}{x_j - z_i} \frac{1 - z_i x_j}{1 - qz_i x_j} \right] \\ &\quad \times \prod_{1 \leq i < j \leq L} \frac{1 - x_i x_j}{x_i - x_j} \cdot \text{Pf} \left(\frac{x_i - x_j}{1 - x_i x_j} Q(x_i, x_j) \right)_{1 \leq i, j \leq L}, \end{aligned} \quad (3.45)$$

where Q is given by (2.32).

Proof. The proof follows by using Theorem 2.28 in the Cauchy summation identity of Corollary 3.15. □

We note here the parallel of (3.45) to the refined Cauchy identity of Macdonald polynomials from [KN99], which is expressed as the product of the Macdonald Cauchy kernel and the Izergin–Korepin determinant in [War08].

Chapter 4

Half-space Markov Processes with Generic Boundary Parameters

The central objects of this work are the two symmetric functions of Definition 3.10. Theorem 3.21 shows that the function $G_{\nu/\mu}$ reduces to the triangular partition function when both bottom and top configurations are empty. The previous section demonstrates how even when both conditions are empty this symmetric function is highly non-trivial. In this section we provide more insight into this behaviour by providing two equivalent evaluations of $G_{\nu/\mu}$ for arbitrary ν from an empty $\mu = \emptyset$. The form of this function leads to a striking conjecture on the orthogonality of the dual family F_κ .

4.1 Subset formula

Theorem 4.1. Let the state on the bottom be empty while the arbitrary state on top $\nu = (\nu_1, \dots, \nu_n)$ consist of n occupations at positions finitely far from the origin. Assume that $L \geq n$ and there exists $\rho > 0$ such that

$$\left| \frac{1 - x_i y_k}{1 - q x_i y_k} \frac{q(1 - x_j / y_k)}{1 - q x_j / y_k} \right| \leq \rho < 1,$$

for all $1 \leq i \neq j \leq L, k \in \mathbb{N}$. Then the partition function (3.27) is calculated explicitly as

$$G_\nu(x_1, \dots, x_L | \mathbf{Y}) = \sum_{\substack{K \subseteq [L] \\ |K|=n}} Z_{L-n}(x_{\bar{k}_1}, \dots, x_{\bar{k}_{L-n}}) \prod_{i \in K} h(x_i)$$

This chapter is based on the publication [P1]. Minor modifications have been made for overall coherence and so that the thesis references its other chapters where relevant.

The coordinate $\nu_1 \in \mathbb{N}$ denotes the right-most occupation in ν .

- (i) $G_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1})$ is a meromorphic function in y_{ν_1} . Its poles are all simple and occur at the points $y_{\nu_1} = qx_i$ for $1 \leq i \leq L$.
- (ii) $G_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1})$ is symmetric in its alphabet (x_1, \dots, x_L) .
- (iii) The residue of $G_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1})$ at its simple pole $y_{\nu_1} = qx_1$ is given by

$$\begin{aligned} \text{Res}_{y_{\nu_1}=qx_1} \left[G_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1}) \right] &= (1-q)x_1 h(x_1) \prod_{j=2}^L \frac{x_j - qx_1}{x_j - x_1} \frac{1 - x_1 x_j}{1 - qx_1 x_j} \\ &\quad \times \prod_{j=1}^{\nu_1-1} \frac{y_j - x_1}{y_j - qx_1} G_{(\nu_2, \dots, \nu_n)}(x_2, \dots, x_L | \mathbf{Y}^{-1}). \end{aligned} \quad (4.4)$$

- (iv) The limit in y_{ν_1} is

$$\lim_{y_{\nu_1} \rightarrow \infty} G_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1}) = 0.$$

- (v) When the coordinate is empty

$$G_\emptyset(x_1, \dots, x_L | \mathbf{Y}^{-1}) = Z_L(x_1, \dots, x_L).$$

Proof. We will demonstrate the properties diagrammatically on the partition function (4.2).

- (i) From the diagram, the only dependence on y_{ν_1} is from the ν_1 'th column. The weights which contribute to the partition function from this column will be from (2.2) with $z = x_i/y_{\nu_1}$, where i corresponds to the rows $1 \leq i \leq L$. All of the vertex configurations carry weights which are either entire functions of y_{ν_1} or are analytic except at the isolated point $y_{\nu_1} = qx_i$. Since these weights contribute the only dependence on y_{ν_1} we can conclude that $G_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1})$ is a meromorphic function for all y_{ν_1} with possible singularities at the isolated points $y_{\nu_1} = qx_i$ for $1 \leq i \leq L$.

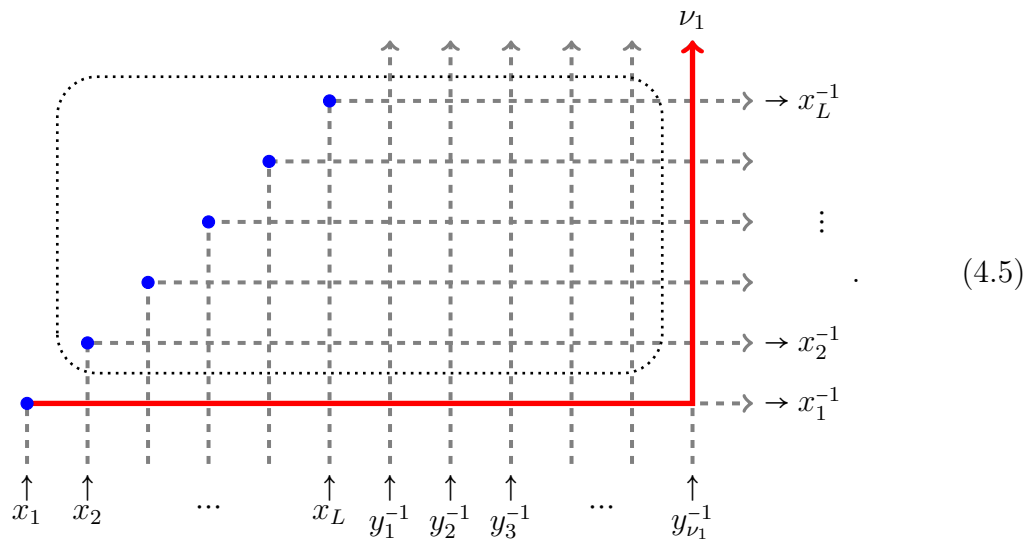
Each global path configuration on (4.2) is will feature a weight from each vertex in the ν_1 'th column at most once. Since each weight generates at most a simple pole at $y_{\nu_1} = qx_i$, we can conclude that the partition function will be a sum of rational functions with simple poles at $y_{\nu_1} = qx_i$. Therefore the poles at these points will be simple.

- (ii) This property follows from Corollary 3.11.
- (iii) Observing the boundary conditions, the vertex in the ν_1 'th column and first row has two

possible vertex configurations. these are shown in the table below.



Since the all-empty configuration on the left has weight 1, lattice configurations where this vertex is empty will have a partition function contribution which are analytic at $y_{\nu_1} = qx_1$. When the other weight is involved the contribution will have a simple pole at $y_{\nu_1} = qx_1$. By taking the residue of the whole partition function at the point $y_{\nu_1} = qx_1$ we isolate contributions where this vertex is non-empty. Such configurations are depicted in the following diagram



By fixing the configuration at this vertex, we really freeze the contribution along the whole line associated with spectral parameter x_1 and the line associated with vertical spectral parameter y_{ν_1} . This freezing passes on the empty boundary conditions from below the first line to the second line. Likewise it enforces the empty conditions on the right of the ν_1 'th column to the $(\nu_1 - 1)$ 'th column.

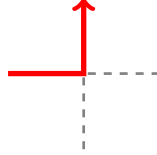
After removing the frozen contribution when taking the residue, what is left in the rectangle in (4.5) is that of the same partition function (4.2) with a $n - 1$ coordinates (ν_2, \dots, ν_n) and $L - 1$ rows with spectral parameters (x_2, \dots, x_L) .

Taking this residue can be written as

$$\begin{aligned} \text{Res}_{y_{\nu_1}=qx_1} \left[G_{\nu} (x_1, \dots, x_L | \mathbf{Y}^{-1}) \right] &= h(x_1) \prod_{j=2}^n \frac{1-x_1x_j}{1-qx_1x_j} \prod_{j=1}^{\nu_1-1} \frac{y_j-x_1}{y_j-qx_1} \\ &\times G_{(\nu_2, \dots, \nu_n)} (x_2, \dots, x_L | \mathbf{Y}^{-1}) \lim_{y_{\nu_1} \rightarrow qx_1} \left[(y_{\nu_1}-qx_1) \frac{x_1(1-q)}{y_{\nu_1}-qx_1} \prod_{j=2}^n \frac{q(y_{\nu_1}-x_j)}{y_{\nu_1}-qx_j} \right], \end{aligned}$$

which can be easily manipulated to take the form of (4.4). We note here we can include the entire inverted alphabet \mathbf{Y}^{-1} and remove it from the evaluation of the limit since the smaller partition function will only have explicit dependence on y_j for $1 \leq j \leq \nu_2$.

(iv) From the table of weights (2.2), the configuration



has weight $(1-q)x_i/(y_{\nu_1}-qx_i)$ when the horizontal and vertical spectral parameters are x_i^{-1} and $y_{\nu_1}^{-1}$ respectively. In the limit $y_{\nu_1} \rightarrow \infty$ this weight is equal to zero and will not occur with the ν_1 'th column of the partition function.

However when observing the boundary conditions of (4.2), we note that there is a path exit through the top of the ν_1 'th column while there are no occupations on the bottom entry or to the right of this column. Consequently, this weight must appear in the ν_1 'th column exactly once for any configuration to provide a non-zero contribution.

Meanwhile, all other vertex configurations within (2.2) will not diverge under the same limit. This is sufficient to conclude that the limit of the whole partition function will evaluate to zero.

(v) This property follows from Theorem 3.21.

□

We will now proceed with the proof of the theorem.

Proof of Theorem 4.1. Let us write (4.3) in the more compact form

$$\mathfrak{G}_{\nu} (x_1, \dots, x_L | \mathbf{Y}^{-1}) = \sum_{\substack{K \subseteq [L] \\ |K|=n}} Z_{L-n} (x_{K^c}) h(x_K) \Delta(x_K | x_{K^c}) \Phi_{\nu} (x_K | \mathbf{Y}^{-1}), \quad (4.6)$$

where we have defined

$$\begin{aligned}
h(x_K) &= \prod_{i \in K} h(x_i), \\
\Delta(x_K | x_{K^c}) &= \prod_{i \in K} \prod_{j \in K^c} \left[\frac{x_j - qx_i}{x_j - x_i} \frac{1 - x_i x_j}{1 - qx_i x_j} \right] \prod_{\substack{1 \leq i < j \leq L \\ i, j \in K}} \frac{1 - x_i x_j}{1 - qx_i x_j}, \\
\Phi_\nu(x_K | \mathbf{Y}^{-1}) &= \sum_{\sigma \in S_n} \prod_{1 \leq i < j \leq n} \frac{x_{k_{\sigma(j)}} - qx_{k_{\sigma(i)}}}{x_{k_{\sigma(j)}} - x_{k_{\sigma(i)}}} \prod_{i=1}^n \left[\frac{(1-q)x_{k_{\sigma(i)}}/y_{\nu_i}}{1 - qx_{k_{\sigma(i)}}/y_{\nu_i}} \prod_{j=1}^{\nu_i-1} \frac{1 - x_{k_{\sigma(i)}}/y_j}{1 - qx_{k_{\sigma(i)}}/y_j} \right].
\end{aligned}$$

We shall begin by proving that (4.6) obeys the same set of properties as $G_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1})$ from (4.2). These are the properties in Lemma 4.2 which provide a recursive construction for the formula (4.3) with an initial condition, and so completely define the formula for (4.2) through an inductive argument.

- (i) $\mathfrak{G}_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1})$ is a meromorphic function in y_{ν_1} . Its poles are all simple and occur at the points $y_{\nu_1} = qx_i$, $1 \leq i \leq L$. This property is immediate from the formula for $\Phi_\nu(x_K | \mathbf{Y}^{-1})$, which is the only place where $\mathfrak{G}_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1})$ has dependence on the family \mathbf{Y} .
- (ii) $\mathfrak{G}_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1})$ is symmetric in its alphabet (x_1, \dots, x_L) . This is manifest from the form (4.6) of $\mathfrak{G}_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1})$.
- (iii) The residue of $\mathfrak{G}_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1})$ at its simple pole $y_{\nu_1} = qx_1$ is given by

$$\begin{aligned}
\text{Res}_{y_{\nu_1}=qx_1} \left[\mathfrak{G}_\nu(x_1, \dots, x_L | \mathbf{Y}^{-1}) \right] &= (1-q)x_1 h(x_1) \prod_{j=2}^L \frac{x_j - qx_1}{x_j - x_1} \frac{1 - x_1 x_j}{1 - qx_1 x_j} \\
&\quad \times \prod_{j=1}^{\nu_1-1} \frac{y_j - x_1}{y_j - qx_1} \mathfrak{G}_{(\nu_2, \dots, \nu_n)}(x_2, \dots, x_L | \mathbf{Y}^{-1}).
\end{aligned}$$

This is easily seen by computing

$$\begin{aligned}
&\text{Res}_{y_{\nu_1}=qx_1} \left[\Phi_\nu(x_K | \mathbf{Y}^{-1}) \right] \\
&= \begin{cases} (1-q)x_1 \prod_{j \in K \setminus \{1\}} \frac{x_j - qx_1}{x_j - x_1} \prod_{j=1}^{\nu_1-1} \frac{y_j - x_1}{y_j - qx_1} \Phi_{(\nu_2, \dots, \nu_n)}(x_{K \setminus \{1\}} | \mathbf{Y}^{-1}) & \text{if } 1 \in K \\ 0 & \text{if } 1 \notin K \end{cases},
\end{aligned}$$

and noting that for $1 \in K$, we have

$$\Delta(x_K|x_{K^c}) = \prod_{j \in K^c} \frac{x_j - qx_1}{x_j - x_1} \prod_{j=2}^L \frac{1 - x_1 x_j}{1 - qx_1 x_j} \Delta(x_{K \setminus \{1\}}|x_{K^c}), \quad h(x_K) = h(x_1)h(x_{K \setminus \{1\}}).$$

(iv) $\mathfrak{G}_\nu(x_1, \dots, x_L|\mathbf{Y}^{-1}) \rightarrow 0$ as $y_{\nu_1} \rightarrow \infty$. This follows by computing this limit directly on $\Phi_\nu(x_K|\mathbf{Y}^{-1})$.

(v) $\mathfrak{G}_\emptyset(x_1, \dots, x_L|\mathbf{Y}^{-1}) = Z_L(x_1, \dots, x_L)$. This is simply the $n = 0$ case of the formula (4.6).

We have shown that $\mathfrak{G}_\nu(x_1, \dots, x_L|\mathbf{Y}^{-1})$ obeys the same set of properties as $G_\nu(x_1, \dots, x_L|\mathbf{Y}^{-1})$ does according to Lemma 4.2. It remains to show that these properties imply the equality of the two objects; we do this by induction on the length of ν . To that end, define the function

$$\mathfrak{Z}_\nu(x_1, \dots, x_L|\mathbf{Y}) = \mathfrak{G}_\nu(x_1, \dots, x_L|\mathbf{Y}^{-1}) - G_\nu(x_1, \dots, x_L|\mathbf{Y}^{-1}).$$

By construction, $\mathfrak{Z}_\emptyset(x_1, \dots, x_L|\mathbf{Y}) = 0$. It follows that there exists an integer $m \geq 0$ such that $\mathfrak{Z}_\mu(x_1, \dots, x_L|\mathbf{Y}) = 0$ for all strict partitions $\mu = (\mu_1 > \dots > \mu_m)$ of length m (with L being arbitrary); this is our inductive hypothesis.

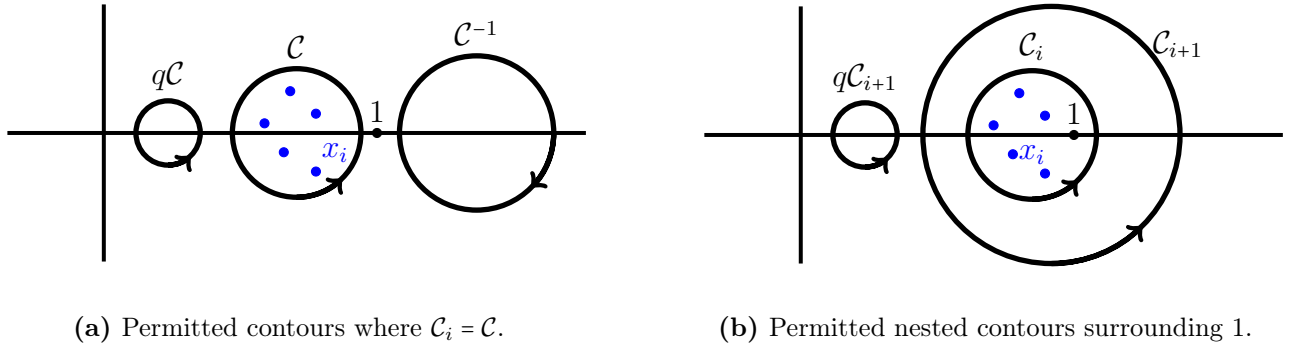
Now let $\lambda = (\lambda_1 > \dots > \lambda_{m+1})$ be a strict partition of length $m+1$. We know that $\mathfrak{Z}_\lambda(x_1, \dots, x_L|\mathbf{Y})$ is a meromorphic function in y_{λ_1} , its poles are all simple, and it vanishes as $y_{\lambda_1} \rightarrow \infty$. However, from the recursion relation obeyed by $\mathfrak{G}_\lambda(x_1, \dots, x_L|\mathbf{Y}^{-1})$ and $G_\lambda(x_1, \dots, x_L|\mathbf{Y}^{-1})$, as well as the inductive hypothesis, all poles have vanishing residue. This means that $\mathfrak{Z}_\lambda(x_1, \dots, x_L|\mathbf{Y})$ is entire and bounded in y_{λ_1} and therefore constant. This constant must be zero in view of the known $y_{\lambda_1} \rightarrow \infty$ behaviour. It follows that $\mathfrak{Z}_\lambda(x_1, \dots, x_L|\mathbf{Y}) = 0$ for all strict partitions $\lambda = (\lambda_1 > \dots > \lambda_{m+1})$ of length $m+1$, and the inductive step of the proof is complete. \square

4.2 Integral formula

Here we will present the sum over subset expression (4.1) as an equivalent nested integral formula.

Definition 4.3. Fix an alphabet $(x_1, \dots, x_L) \in \mathbb{C}^L$. We denote by $\mathcal{C}_1, \dots, \mathcal{C}_n$ a collection of positively oriented closed complex contours satisfying

- For all $1 \leq i < j \leq n$, we have that $q\mathcal{C}_j$ lies completely outside the interior of \mathcal{C}_i , where $q\mathcal{C}_j$ denotes the image of \mathcal{C}_j under multiplication by q . In addition, if 1 is within the interior of \mathcal{C}_i we also require that \mathcal{C}_i is completely contained in the interior of \mathcal{C}_j .

(a) Permitted contours where $\mathcal{C}_i = \mathcal{C}$.

(b) Permitted nested contours surrounding 1.

Figure 4.1: Diagrams depicting arrangements of contours allowed by Definition 4.3.

- For all $1 \leq i \leq L$, the contour \mathcal{C}_i surrounds all points x_j and does not surround the points $qx_j, q^{-1}x_j^{-1}, q^{-1}y_k^{-1}, a, c$ for all $1 \leq j \leq L$ and $k \in \mathbb{N}$.

Examples of contours satisfying the conditions of Definition 4.3 are shown in Figure 4.1. These conditions allow for some freedom with contour choice. In particular we may choose all contours to be equal, $\mathcal{C}_i = \mathcal{C}$ for all i , provided that the contours neither lie upon nor enclose $1 \in \mathcal{C}$.

Provided that the integrand considered has no singularity at 1, the contours may surround 1 if we choose that they are nested. That is, for all $1 \leq i \leq n - 1$ the contour \mathcal{C}_i is completely contained within the interior of \mathcal{C}_{i+1} . This choice is convenient as it allows us to choose elements of our alphabet, x_i , to be arbitrarily close to 1.

Theorem 4.4. The partition function expression (4.1) can be expressed as the following n -fold integral:

$$\begin{aligned}
 G_\nu(x_1, \dots, x_L | \mathbf{Y}) &= \oint_{\mathcal{C}_1} \frac{dw_1}{2\pi i} \dots \oint_{\mathcal{C}_n} \frac{dw_n}{2\pi i} Z_{L+n}(x_1, \dots, x_L, w_1^{-1}, \dots, w_n^{-1}) \\
 &\times \prod_{i=1}^n \prod_{j=1}^L \left[\frac{qw_i - x_j}{w_i - x_j} \frac{1 - w_i x_j}{1 - qw_i x_j} \right] \prod_{1 \leq i < j \leq n} \left[\frac{w_j - w_i}{qw_j - w_i} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] \\
 &\times \prod_{i=1}^n \left[\frac{ac(qw_i^2 - 1)}{(w_i - a)(w_i - c)} \frac{y_{\nu_i}}{1 - qw_i y_{\nu_i}} \prod_{j=1}^{\nu_i - 1} \frac{1 - w_i y_j}{1 - qw_i y_j} \right], \quad (4.7)
 \end{aligned}$$

where the contours $\mathcal{C}_1, \dots, \mathcal{C}_n$ satisfy the conditions of Definition 4.3.

Proof. The idea of the proof is to replace each x_i for $i \in K$ in the sum in (4.1) by an auxiliary variable w_i that will be integrated over a contour surrounding simple poles at all x_1, \dots, x_L . The sum over K then dictates which n of the L possible residues are evaluated, whilst the sum over $\sigma \in S_n$ dictates the order in which the residues are evaluated for a given K .

For this to work, only the residues at $w_i = x_j$ in (4.7) should be evaluated. All other poles in (4.7) therefore need to be excluded from the contours, and this is guaranteed by Definition 4.3.

Firstly, it is obvious that the explicit poles in (4.7) at a , c , $q^{-1}y_j^{-1}$ and $q^{-1}x_j^{-1}$ need to lie outside each contour. Secondly, the poles at qx_j also need to be avoided because these cause singularities in Z_{L+n} , see below. Moreover, for $1 \leq i < j \leq n$, the factors of the form

$$\frac{w_j - w_i}{qw_j - w_i} \frac{1 - qw_i w_j}{1 - w_i w_j},$$

produce potential residues which will be avoided when $q\mathcal{C}_j$ lies outside the interior of \mathcal{C}_i .

In order to reproduce the triangular partition function Z_{L-n} that appears in the summand of (4.1) we extend this function to Z_{L+n} in the combined alphabet $(x_1, \dots, x_L, w_1^{-1}, \dots, w_n^{-1})$. During the evaluation of the residue of the simple pole at each $w_i = x_j$, the recursion relation (2.26) ensures that we re-obtain Z_{L-n} in the complement alphabet of x -variables of (4.1). Furthermore, according to Proposition 2.22, the rational function $Z_{L+n}(x_1, \dots, x_L, w_1^{-1}, \dots, w_n^{-1})$ has poles at $w_i = a^{-1}$, $w_i = c^{-1}$ and $w_i = qx_j$ and so has no singularities at $w_i = x_j$ that could affect the residue evaluation.

In order to proceed, we rewrite the following factor that occurs in (4.1),

$$\begin{aligned} \prod_{i \in K} \prod_{j \in K^c} \left[\frac{x_j - qx_i}{x_j - x_i} \frac{1 - x_i x_j}{1 - qx_i x_j} \right] &= \prod_{i \in K} \prod_{j \in K^c} \frac{1}{x_i - x_j} \prod_{i \in K} \prod_{j=1}^L \frac{(qx_i - x_j)(1 - x_i x_j)}{1 - qx_i x_j} \\ &\quad \times \prod_{\substack{i \neq j \\ i, j \in K}} \frac{1 - qx_i x_j}{(qx_i - x_j)(1 - x_i x_j)} \prod_{i \in K} \frac{qx_i^2 - 1}{(1 - q)x_i(1 - x_i^2)}. \end{aligned} \quad (4.8)$$

Incorporating this we notice that for each K the second line in (4.1) is manifestly symmetric in the variables x_i for $i \in K$. This allows us to replace each x_i for $i \in K$ with w_i , i.e. the right hand side of (4.8) is replaced by

$$\prod_{\substack{i, j=1 \\ i \neq j}}^n (w_i - w_j) \prod_{i=1}^n \prod_{j=1}^L \frac{(qw_i - x_j)(1 - w_i x_j)}{(w_i - x_j)(1 - qw_i x_j)} \prod_{\substack{i, j=1 \\ i \neq j}}^n \frac{1 - qw_i w_j}{(qw_i - w_j)(1 - w_i w_j)} \prod_{i=1}^n \frac{qw_i^2 - 1}{(1 - q)w_i(1 - w_i^2)},$$

and each $x_{\sigma(i)}$ and $x_{\sigma(j)}$ in (4.1) is similarly replaced by w_i and w_j respectively. Simplifying and cancelling common factors we thus obtain the integrand of (4.7). Finally we note that the factors $w_j - w_i$ in (4.7) ensure that after evaluating the residue of w_i at x_k the singularity at $w_j = x_k$ is removable for all $j \neq i$, and hence that the residue for each w_i is evaluated at a different simple pole. \square

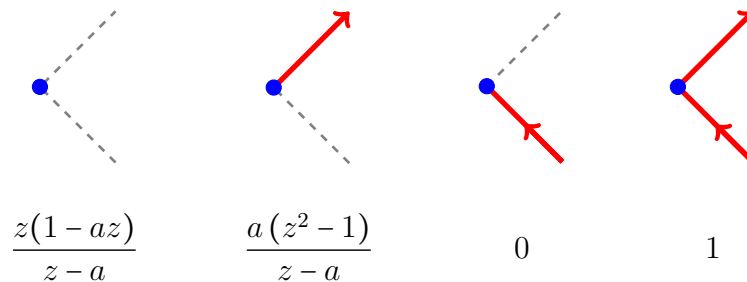
Chapter 5

Half-space Spectral Theory

This chapter will focus on the properties of the symmetric functions from Definition 3.10 in the case where the boundary vertices become upper-triangular. In this specialisation a symmetrisation formula for the partition function F_μ is provided, which is shown to obey a spatial orthogonality relation.

5.1 Eigenvector relation

Let us consider specialising the boundary parameters a, c so that the boundary weights become upper-triangular. This is achieved by specialising one of the boundary parameters, in our case $c \rightarrow \infty$, whilst leaving the parameter a arbitrary. In the limit $c \rightarrow \infty$ the boundary vertices of (2.20) are tabulated below.



In this upper-triangular boundary specialisation, the partition function $F_\mu(z_1, \dots, z_n)$ from (3.30), indexed by a single half-space configuration $\mu \in \mathbb{W}$ and an alphabet $(z_1, \dots, z_n) \in \mathbb{C}^n$, has the following eigenvector property.

Corollary 5.1 (of Theorem 3.14). Let $n, L > 0$ be fixed integers. Fix the alphabets

This chapter will be the basis of a future publication whose main result, Theorem 5.12, was posed as a conjecture in the publication [P1].

$(x_1, \dots, x_L), (z_1, \dots, z_n)$ and assume there exists $\rho > 0$ such that

$$\left| \frac{1 - x_i y_k}{1 - q x_i y_k} \frac{q(1 - z_j / y_k)}{1 - q z_j / y_k} \right| \leq \rho < 1, \quad \left| \frac{1 - x_i y_k}{1 - q x_i y_k} \frac{1 - q z_j y_k}{1 - z_j y_k} \right| \leq \rho < 1, \quad (5.1)$$

for all $1 \leq i \leq L, 1 \leq j \leq n$ and $k \in \mathbb{N}$. Then, in the upper-triangular boundary limit $c \rightarrow \infty$, the partition functions from Definition 3.10 satisfy the following summation identity

$$\sum_{\kappa} G_{\kappa/\mu}(x_1, \dots, x_L) F_{\kappa}(z_1, \dots, z_n) = \prod_{i=1}^L \frac{x_i(1 - a x_i)}{x_i - a} \prod_{i=1}^n \prod_{j=1}^L \left[\frac{x_j - q z_i}{x_j - z_i} \frac{1 - z_i x_j}{1 - q z_i x_j} \right] F_{\mu}(z_1, \dots, z_n). \quad (5.2)$$

Note that we may interpret the half-space configurations $\mu, \kappa \in \mathbb{W}$ as coordinates in some discrete and countable *physical space*, while the alphabets x, z are coordinates in some continuous and uncountable *spectral space*. Corollary 5.1 allows us to interpret the partition function $F_{\mu}(z_1, \dots, z_n)$ as an eigenvector of the matrix $G_{\nu/\mu}(x_1, \dots, x_L)$ where matrix multiplication takes place over the physical space and the eigenvectors are indexed by the uncountable spectral space. Recall also the sum-to-unity property of the function $G_{\nu/\mu}$ from Proposition 3.13. Provided that we have a non-negative specialisation, we may regard the function $G_{\nu/\mu}$ as the discrete-time transition probability of a Markov processes between two physical half-space coordinates in \mathbb{W} .

5.2 Symmetrisation and coordinate Bethe ansatz

For a fixed integer $n > 0$ we denote the group of *signed permutations* of $\{1, \dots, n\}$ by $\mathcal{B}_n = S_n \times \{\pm 1\}^n$. This group is of order $2^n n!$ and is sometimes known as the *hyperoctahedral group*. The group \mathcal{B}_n acts on functions of some alphabet (z_1, \dots, z_n) of length n by permuting the variables as

$$\sigma(f(z_1, \dots, z_n)) = f(z_{\sigma(1)}, \dots, z_{\sigma(n)}),$$

where the negative indices are defined by $z_{-k} := 1/(q z_k)$ for any $1 \leq k \leq n$. Wherever it is convenient, we will use either σ_i or $\sigma(i)$ to denote the image a signed permutation. The symmetric group of permutations, S_n , is contained within \mathcal{B}_n as a subgroup.

5.2.1 Symmetrisation formula for F_{μ}

The main result of this section is Proposition 5.2: a symmetrisation formula for the function which plays the role of eigenvector in Corollary 5.1. Let us define the following *plane wave*

function for $k > 0$

$$\varphi_k(z) = \varphi_k(z|\mathbf{Y}) := \frac{1 - az}{1 - qz^2} \frac{(1 - q)z}{1 - zy_k} \prod_{j=1}^{k-1} \frac{1 - qzy_j}{1 - zy_j}. \quad (5.3)$$

Proposition 5.2. Let $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{W}^m$ be a half-space configuration with m parts and consider the partition function $F_\mu(z_1, \dots, z_n)$ from (3.30) of an alphabet of length $n \geq m$. In the upper-triangular specialisation $c \rightarrow \infty$, the partition function may be expressed as the following symmetrisation formula

$$F_\mu(z_1, \dots, z_n) = (-a)^{n-m} V_{n-m} \prod_{i=1}^n \frac{1 - z_i^2}{z_i - a} \sum_{\sigma \in \mathcal{B}_n} \sigma \left(\prod_{1 \leq i < j \leq n} \left[\frac{z_i - qz_j}{z_i - z_j} \frac{1 - z_i z_j}{1 - qz_i z_j} \right] \prod_{i=1}^m \varphi_{\mu_i}(z_i) \right), \quad (5.4)$$

where φ is given by (5.3) and the overall normalisation factor is

$$V_k := \frac{q^{\binom{k}{2}} (1 - q)^k}{\prod_{i=1}^k [(1 - q^i) (1 + q^{i-1})]}. \quad (5.5)$$

Remark 5.3. By virtue of Corollary 5.1, Proposition 5.2 provides the explicit form of the eigenvectors of the matrix defined by the partition function G . This symmetrisation formula is of the same form that one would expect to derive using the coordinate Bethe ansatz method, where the hyperoctahedral group \mathcal{B}_n emerges rather than the symmetric group S_n due to the presence of the boundary at the origin in the half-space. In the language of Bethe ansatz, the parameters z_1, \dots, z_n are not constrained by the requirement that they are solutions to any Bethe ansatz equations. They are in fact free to take values satisfying the constraints (5.1) since the half-space is infinite volume where there is no quantisation condition. Similar symmetrisation formulas to (5.4) have been derived for integrable models in half-space for models previously using coordinate Bethe ansatz (see for example [TW13]), and for vertex model constructions for BC_n -symmetric Hall–Littlewood polynomials (see for example [WZJ16]).

Before we present the proof of Proposition 5.2, it will be important to outline several key intermediate results. The first is the following symmetrisation identity.

Proposition 5.4. For any fixed integer $n \geq 0$ the following factorisation holds on any alphabet (z_1, \dots, z_n) :

$$\sum_{\sigma \in \mathcal{B}_n} \sigma \left(\prod_{1 \leq i < j \leq n} \left[\frac{z_i - qz_j}{z_i - z_j} \frac{1 - z_i z_j}{1 - qz_i z_j} \right] \right) = \frac{1}{V_n},$$

where the constant V_n is given by (5.5).

Proposition 5.4 will be used extensively in Chapter 6, and so, we will defer the proof until there (see Proposition 6.7). The argument essentially follows by matching with a summation identity

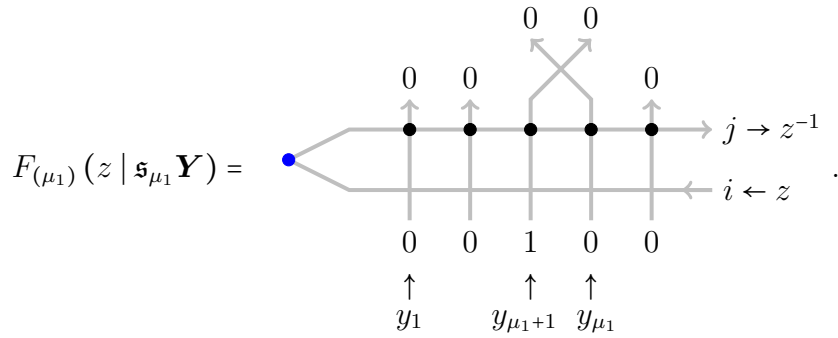
found in [Ven15] which is used to show that a BC_n -symmetric Hall–Littlewood polynomial indexed by an empty partition is equal to 1.

Let us define a *local permutation operator* \mathfrak{s}_i for any $i \in \mathbb{N}$, which acts on the infinite-length alphabet $\mathbf{Y} = (y_1, y_2, \dots)$ by interchanging the order of y_i and y_{i+1} . That is, $\mathfrak{s}_i \mathbf{Y} = (y_1, \dots, y_{i-1}, y_{i+1}, y_i, y_{i+2}, \dots)$.

Lemma 5.5. Let $(\mu_1) \in \mathbb{W}^1$ be a fixed half-space configuration with one part (so that $\mu_1 \in \mathbb{N}$). The one double-row partition function $F_{(\mu_1)}(z)$ obeys the following recursion relation:

$$F_{(\mu_1+1)}(z \mid \mathbf{Y}) = \frac{y_{\mu_1+1} - qy_{\mu_1}}{y_{\mu_1+1} - y_{\mu_1}} F_{(\mu_1)}(z \mid \mathfrak{s}_{\mu_1} \mathbf{Y}) - \frac{y_{\mu_1}(1 - q)}{y_{\mu_1+1} - y_{\mu_1}} F_{(\mu_1)}(z \mid \mathbf{Y}). \quad (5.6)$$

Proof. Let us consider the one-double row partition function $F_{(\mu_1)}(z \mid \mathfrak{s}_{\mu_1} \mathbf{Y})$, onto which we may attach an intertwining vertex to the top of the lattice as:



The intertwining vertex may be attached without changing the value of the partition function since the attached vertex is frozen with weight 1. Application of the Yang–Baxter equation (2.4) leads to the following expansion:

$$\begin{aligned}
 F_{(\mu_1)}(z \mid \mathfrak{s}_{\mu_1} \mathbf{Y}) &= \begin{array}{c} \begin{array}{c} \begin{array}{c} \text{0} \quad \text{0} \quad \text{0} \quad \text{0} \quad \text{0} \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ \text{0} \quad \text{0} \quad \text{1} \quad \text{0} \quad \text{0} \\ \uparrow \quad \quad \uparrow \quad \uparrow \\ y_1 \quad y_{\mu_1+1} \quad y_{\mu_1} \end{array} \\ \begin{array}{c} \text{0} \quad \text{0} \quad \text{0} \quad \text{0} \quad \text{0} \\ \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow \\ \text{0} \quad \text{0} \quad \text{1} \quad \text{0} \quad \text{0} \\ \uparrow \quad \quad \uparrow \quad \uparrow \\ y_1 \quad y_{\mu_1+1} \quad y_{\mu_1} \end{array} \end{array} \\
 &= \frac{y_{\mu_1}(1 - q)}{y_{\mu_1+1} - qy_{\mu_1}} F_{(\mu_1)}(z \mid \mathbf{Y}) + \frac{y_{\mu_1+1} - y_{\mu_1}}{y_{\mu_1+1} - qy_{\mu_1}} F_{(\mu_1+1)}(z \mid \mathbf{Y}), \quad (5.7)
 \end{aligned}$$

where each term on the right-hand side of (5.7) comes from each configuration of the intertwining vertex. The coefficients may be calculated from reading off the weights of the intertwining vertex from the table (2.2). The result (5.6) follows by re-arranging the right-hand side of

(5.7). □

The following collection of properties is central in the proof of Proposition 5.2.

Proposition 5.6. Let $n \geq m \geq 0$ be fixed integers and let $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{W}^m$ be a half-space configuration with m parts. The partition function $F_\mu(z_1, \dots, z_n)$ from Definition 3.10 satisfies the following properties in the upper-triangular specialisation $c \rightarrow \infty$:

- (i) $F_\mu(z_1, \dots, z_n)$ is a meromorphic function in the variable y_{μ_1} . Its singularities are all simple poles and occur at the points $y_{\mu_1} = z_i^{-1}, qz_i$ for $1 \leq i \leq n$.
- (ii) $F_\mu(z_1, \dots, z_n)$ is a symmetric function under permutations of its alphabet (z_1, \dots, z_n) .
- (iii) The following residue evaluation holds:

$$\begin{aligned} \operatorname{Res}_{y_{\mu_1}=z_1^{-1}} F_\mu(z_1, \dots, z_n) &= \frac{1 - az_1}{a - z_1} \frac{(1 - q)(1 - z_1^2)}{1 - qz_1^2} \prod_{i=2}^n \left[\frac{z_1 - qz_i}{z_1 - z_2} \frac{1 - z_1 z_i}{1 - qz_1 z_i} \right] \\ &\quad \times \prod_{j=1}^{\mu_1-1} \frac{1 - qz_1 y_j}{1 - z_1 y_j} F_{(\mu_2, \dots, \mu_m)}(z_2, \dots, z_n). \end{aligned} \quad (5.8)$$

- (iv) The following limit holds

$$\lim_{y_{\mu_1} \rightarrow \infty} F_\mu(z_1, \dots, z_n) = 0.$$

- (v) When the coordinate is empty

$$F_\emptyset(z_1, \dots, z_n) = \prod_{i=1}^n \frac{a(1 - z_i^2)}{a - z_i}. \quad (5.9)$$

- (vi) $F_\mu(z_1, \dots, z_n)$ obeys the following symmetry under the inversion of its first variable $z_1 \mapsto 1/(qz_1)$:

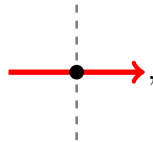
$$\frac{z_1 - a}{1 - z_1^2} \cdot F_\mu(z_1, z_2, \dots, z_n) = \frac{qz_1(aqz_1 - 1)}{1 - q^2 z_1^2} \cdot F_\mu\left(\frac{1}{qz_1}, z_2, \dots, z_n\right). \quad (5.10)$$

Proof. We proceed by demonstrating each of these properties individually where we assume throughout that $\mu = (\mu_1, \dots, \mu_m) \in \mathbb{W}^m$ is a half-space configuration with length $0 \leq m \leq n$.

- (i) Consider the column in the double-row lattice of the partition function $F_\mu(z_1, \dots, z_n)$ from (3.30) associated with μ_1 , which is responsible for the function's y_{μ_1} -dependence. The partition function $F_\mu(z_1, \dots, z_n)$ is a rational function in y_{μ_1} since the weights of the vertex configurations which may appear in the column are themselves rational functions in y_{μ_1} . The lower rows of the partition function is a constructed from the weights in the

table (2.2) with argument z_i/y_{μ_1} , for each $1 \leq i \leq n$, whose weights are either constants or contribute possible simple poles at $y_{\mu_1} = qz_i$. Similarly, the upper rows are constructed from weights in the table (2.5) with argument $z_i y_{\mu_1}$, for each $1 \leq i \leq n$, whose weights have possible simple poles at $y_{\mu_1} = z_i^{-1}$. And so, $F_\mu(z_1, \dots, z_n)$ is a meromorphic function in y_{μ_1} with the appropriate singularities.

- (ii) This property follows directly from Corollary 3.11.
- (iii) We will calculate the residue at $y_{\mu_1} = z_1^{-1}$ explicitly on the partition function (3.30). This residue isolates specific configurations of the dotted vertex in the column associated with y_{μ_1} on the bottom double-row. That is, the evaluation of the residue means that the following vertex configuration will not appear



since its weight is equal to 1 and any lattice configuration containing this vertex configuration is therefore analytic at the point $y_{\mu_1} = z_1^{-1}$. By virtue of the boundary conditions only the following configurations may appear in the dotted vertex in question with the following residues:

$$\text{Res}_{y_{\mu_1}=z_1^{-1}} \begin{array}{c} z_i^{-1} \rightarrow \text{---} \\ \uparrow \\ y_{\mu_1} \end{array} = \text{Res}_{y_{\mu_1}=z_1^{-1}} \begin{array}{c} z_i^{-1} \rightarrow \text{---} \\ \uparrow \\ y_{\mu_1} \end{array} = \frac{q-1}{z_1}. \tag{5.11}$$

This leads to the following calculation on the lattice of the entire partition function:

$$\operatorname{Res}_{y_{\mu_1}=z_1^{-1}} F_{\mu}(z_1, \dots, z_n \mid \mathbf{Y}) = \frac{q-1}{z_1} \cdot \left(\text{Diagram} \right) \quad (5.12)$$

Let us examine the different subsections of the lattice on the right-hand side of (5.12) individually. Using the branching relation of Proposition 3.12, we may rewrite the residue as

$$\operatorname{Res}_{y_{\mu_1}=z_1^{-1}} F_{\mu}(z_1, \dots, z_n \mid \mathbf{Y}) = \sum_{\kappa} F_{\kappa}(z_2, \dots, z_n \mid \tilde{\mathbf{Y}}) \cdot \operatorname{Res}_{y_{\mu_1}=z_1^{-1}} F_{\mu/\kappa}(z_1 \mid \mathbf{Y}),$$

where $\tilde{\mathbf{Y}}$ is the infinite-length alphabet \mathbf{Y} with y_{μ_1} replaced by z_1^{-1} and where the sum over $\kappa \in \mathbb{W}$ is over the state in between the first and second double-rows of (5.12). The section of (5.17) contained within the dotted rectangle may then be recognised as the partition function $F_{\lambda}(z_2, \dots, z_n \mid \tilde{\mathbf{Y}})$.

Note that whenever κ has non-zero length, its largest part satisfies $\kappa_1 \leq \mu_1$. Upon evaluating the residue at $y_{\mu_1} = z_1^{-1}$ and factoring out its value, we may regard the responsible vertex as a factorised bulk vertex (akin to Proposition 2.4). For fixed κ , let us consider the residue of the function $F_{\mu/\kappa}(z_1)$. By virtue of the allowed configurations of the μ_1 -column vertex (5.11), the bottom row can be understood to have the right-boundary conditions of the partition function G defined by (3.29). In fact, this relationship can be made precise by re-normalising the first $\mu_1 - 1$ dotted vertices using (2.6), which yields

$$\operatorname{Res}_{y_{\mu_1}=z_1^{-1}} F_{\mu/\kappa}(z_1 \mid \mathbf{Y}) = \frac{q-1}{z_1} \frac{q(1-z_1^2)}{1-qz_1^2} \prod_{j=1}^{\mu_1-1} \frac{1-y_j z_1}{1-qy_j z_1} \cdot G_{\kappa/(\mu_2, \dots, \mu_m)}(z_1 \mid \tilde{\mathbf{Y}})$$

Note that in the lattice path formulation of the partition function $G_{\kappa/(\mu_2, \dots, \mu_m)}(z_1 \mid \tilde{\mathbf{Y}})$ (see (3.29)), the vertical and horizontal spectral parameters coincide in the upper row of the column associated with μ_1 , which is consistent with the application of the factorisation

property (Proposition 2.4) as above. Putting this all together, we may recognise (5.12) as

$$\begin{aligned} \operatorname{Res}_{y_{\mu_1}=z_1^{-1}} F_{\mu}(z_1, \dots, z_n | \mathbf{Y}) &= \frac{q-1}{z_1} \frac{q(1-z_1^2)}{1-qz_1^2} \prod_{j=1}^{\mu_1-1} \frac{1-y_j z_1}{1-qy_j z_1} \\ &\quad \times \sum_{\kappa} G_{\kappa/(\mu_2, \dots, \mu_m)}(z_1 | \tilde{\mathbf{Y}}) F_{\kappa}(z_2, \dots, z_n | \tilde{\mathbf{Y}}). \end{aligned} \quad (5.13)$$

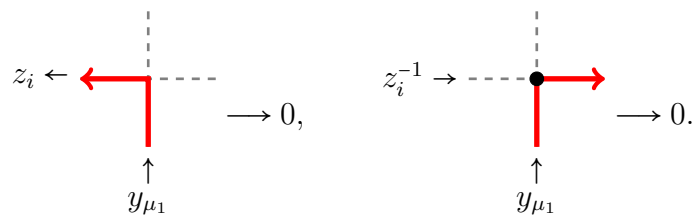
We may then apply the skew-Cauchy identity of Theorem 3.14 to re-write the sum over κ in (5.13) as

$$\begin{aligned} \prod_{i=2}^n \left[\frac{z_1 - qz_i}{z_1 - z_i} \frac{1 - z_1 z_i}{1 - qz_1 z_i} \right] \sum_{\lambda} F_{(\mu_2, \dots, \mu_m)/\lambda}(z_2, \dots, z_n | \tilde{\mathbf{Y}}) G_{\emptyset/\lambda}(z_1 | \tilde{\mathbf{Y}}) \\ = \prod_{i=2}^n \left[\frac{z_1 - qz_i}{z_1 - z_i} \frac{1 - z_1 z_i}{1 - qz_1 z_i} \right] \cdot \frac{z_1(1 - az_1)}{z_1 - a} F_{(\mu_2, \dots, \mu_m)}(z_2, \dots, z_n | \tilde{\mathbf{Y}}), \end{aligned} \quad (5.14)$$

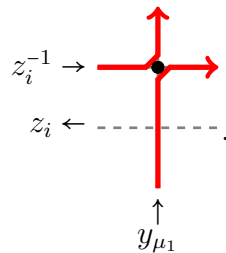
which has been simplified since the only non-zero term in (5.14) is the one due to $\lambda = \emptyset$. The result (5.8) follows by including (5.14) into (5.13).

Note that we have assumed that the z_1, z_2, \dots, z_n parameters are such that convergence conditions of the skew-Cauchy identity (3.32) are satisfied for all $k \in \mathbb{N} \setminus \{\mu_1\}$. It is easily verified that these conditions are satisfied for $k = \mu_1$. The result (5.8) may be understood to hold for all appropriate values of $z_1, z_2, \dots, z_n \in \mathbb{C}$ via analytic continuation.

- (iv) Consider the weights of the column corresponding to y_{μ_1} that makes up the partition function $F_{\mu}(z_1, \dots, z_n)$. Observe that in the limit $y_{\mu_1} \rightarrow \infty$ the following weights vanish in each double-row:



It follows from the boundary conditions that for each double-row the only possible configuration with non-zero weight in the column associated to y_{μ_1} is the following:



However, since the external edges on the top of the lattice in (3.30) must be empty, this configuration may not occur on the top double-row of the lattice and so the whole partition function vanishes.

- (v) Recall the double-row operator construction for $F_\mu(z_1, \dots, z_n)$ from (3.28), which may be simplified for $\mu = \emptyset$ with Proposition 3.3 to yield

$$\langle \emptyset | \dot{B}(z_1) \cdots \dot{B}(z_n) | \emptyset \rangle = \prod_{i=1}^n h(z_i) \cdot \langle \emptyset | \emptyset \rangle = \prod_{i=1}^n \frac{a(1-z_i^2)}{a-z_i},$$

where we have evaluated the function $h(z_i)$ from (2.19) in the upper-triangular specialisation $c \rightarrow \infty$.

- (vi) Using the partition function symmetry and branching relation of Proposition 3.12, we may write

$$F_\mu(z_1, z_2, \dots, z_n) = \sum_{\lambda} F_\lambda(z_1) F_{\mu/\lambda}(z_2, \dots, z_n),$$

so that it suffices to prove the inversion symmetry property on the one double-row partition function $F_\lambda(z_1)$. Since $F_\lambda(z_1)$ vanishes whenever λ has two or more parts, the result follows by considering the empty configuration and all configurations $\lambda \in \mathbb{W}^1$ with one part. Let us first consider $\lambda = \emptyset$, where the required inversion symmetry (5.10) follows immediately from the evaluation of $F_\emptyset(z_1)$ in (5.9). Let us then consider $\lambda = (1)$ (so that there is one part adjacent to the boundary) which may be calculated explicitly as the sum of two possible lattice path configurations as

$$F_{(1)}(z_1) = \frac{z_1(1-q)}{y_1 - qz_1} + \frac{z_1(1-az_1)}{z_1 - a} \frac{y_1 - z_1}{y_1 - qy_1} \frac{1-q}{1-y_1z_1} = \frac{1-z_1^2}{z_1 - a} \frac{z_1(y_1 - a)(1-q)}{(y_1 - qz_1)(1-y_1z_1)}. \quad (5.15)$$

The right-hand side of (5.15) may then be explicitly verified to satisfy the inversion symmetry property (5.10).

In order to show the property for all $\lambda = (\lambda_1) \in \mathbb{W}^1$ with $\lambda_1 \geq 1$, we will proceed via induction on the coordinate λ_1 . Let us fix $\lambda_1 \geq 0$ and assume that $F_{(\lambda_1)}(z_1)$ satisfies the required inversion symmetry (5.10). Recall Lemma 5.5, whereby $F_{(\lambda_1+1)}(z_1 | \mathbf{Y})$ may be written as a linear combination of $F_{(\lambda_1)}(z_1)$ with permuted y alphabets. Importantly, the only z_1 dependence in this expression for $F_{(\lambda_1+1)}(z_1)$ is through the functions $F_{(\lambda_1)}(z_1)$ which satisfies the required inversion symmetry (5.10) because of our induction hypothesis. And so, we may conclude that the property holds on $F_{(\lambda_1+1)}(z_1)$ and all $\lambda \in \mathbb{W}^1$ via induction. \square

We are now ready to proceed with the proof of the symmetrisation formula for F_μ . The proof will proceed along similar lines at the proof of Theorem 4.1.

Proof of Proposition 5.2. Let us define the rational function $\mathfrak{F}_\mu(z_1, \dots, z_n)$ as the right-hand side of (5.4). The argument will proceed by first individually demonstrating that the rational function $\mathfrak{F}_\mu(z_1, \dots, z_n)$ satisfies each of the properties of Proposition 5.6, after which it will be shown that this is sufficient to determine that the rational function is equal to the partition function $F_\mu(z_1, \dots, z_n)$ from Definition 3.10.

- (i) For each term associated with $\sigma \in \mathcal{B}_n$, the only dependence on y_{μ_1} in the symmetrisation formula for $\mathfrak{F}_\mu(z_1, \dots, z_n)$ comes from the single factor of $\varphi_{\mu_1}(z_{\sigma(1)})$. This factor is itself a meromorphic function of y_{μ_1} since it is a rational function. The only singularities of $\varphi_{\mu_1}(z_{\sigma(1)})$ in y_{μ_1} occur at $y_{\mu_1} = z_{\sigma(1)}^{-1}$, which will take only take the required values depending on the signed permutation $\sigma \in \mathcal{B}_n$.
- (ii) Since it is the product of the symmetric product $\prod_i \frac{1-z_i^2}{z_i-a}$ and a symmetrisation over \mathcal{B}_n , $\mathfrak{F}_\mu(z_1, \dots, z_n)$ is a manifestly symmetric function.
- (iii) The only terms within the symmetrisation formula (5.4) for $\mathfrak{F}_\mu(z_1, \dots, z_n)$ which have a singularity at $y_{\mu_1} = z_1^{-1}$ are those which have a factor of $\varphi_{y_{\mu_1}}(z_1)$. These terms are exactly those associated with signed permutations $\sigma \in \mathcal{B}_n$ satisfying $\sigma(1) = +1$. It is, therefore, only these terms within the symmetrisation which contribute to the residue. We calculate their contribution directly as

$$\begin{aligned} \operatorname{Res}_{y_{\mu_1}=z_1^{-1}} \mathfrak{F}_\mu(z_1, \dots, z_n) &= (-a)^{n-m} V_{n-m} \prod_{i=1}^n \frac{1-z_i^2}{z_i-a} \\ &\times \sum_{\substack{\sigma \in \mathcal{B}_n \\ \sigma(1)=+1}} \sigma \left(\prod_{1 \leq i < j \leq n} \left[\frac{z_i - qz_j}{z_i - z_j} \frac{1 - z_i z_j}{1 - qz_i z_j} \right] \prod_{i=2}^m \varphi_{\mu_i}(z_i) \right) \operatorname{Res}_{y_{\mu_1}=z_1^{-1}} \varphi_{y_{\mu_1}}(z_1). \end{aligned} \quad (5.16)$$

Observe that the factor following factor is invariant under $z_j \mapsto 1/(qz_j)$

$$\frac{z_i - qz_j}{z_i - z_j} \frac{1 - z_i z_j}{1 - qz_i z_j}.$$

It follows that

$$\prod_{j=2}^n \left[\frac{z_1 - qz_j}{z_1 - z_j} \frac{1 - z_1 z_j}{1 - qz_1 z_j} \right]$$

is invariant under the action of all $\sigma \in \mathcal{B}_n$ satisfying $\sigma(1) = +1$. And so, all z_1 -dependent factors in the sum of (5.16) may be factored out. This yields:

$$\begin{aligned} \operatorname{Res}_{y_{\mu_1}=z_1^{-1}} \mathfrak{F}_\mu(z_1, \dots, z_n) &= \frac{1-z_1^2}{z_1-a} \prod_{j=2}^n \left[\frac{z_1 - qz_j}{z_1 - z_j} \frac{1 - z_1 z_j}{1 - qz_1 z_j} \right] \operatorname{Res}_{y_{\mu_1}=z_1^{-1}} \varphi_{y_{\mu_1}}(z_1) \\ &\times (-a)^{n-m} V_{n-m} \prod_{i=2}^n \frac{1-z_i^2}{z_i-a} \sum_{\substack{\sigma \in \mathcal{B}_n \\ \sigma(1)=+1}} \sigma \left(\prod_{2 \leq i < j \leq n} \left[\frac{z_i - qz_j}{z_i - z_j} \frac{1 - z_i z_j}{1 - qz_i z_j} \right] \prod_{i=2}^m \varphi_{\mu_i}(z_i) \right), \end{aligned} \quad (5.17)$$

where we may recognise the bottom line of (5.17) as the symmetrisation formula for $\mathfrak{F}_{(\mu_2, \dots, \mu_m)}(z_2, \dots, z_n)$. The required property (5.8) follows by explicitly calculating

$$\operatorname{Res}_{y_{\mu_1}=z_1^{-1}} \varphi_{y_{\mu_1}}(z_1) = -(1-q) \frac{1-az_1}{1-qz_1^2} \prod_{j=1}^{\mu_1-1} \frac{1-qzy_j}{1-zy_j}.$$

- (iv) Along the same lines as the first property, each term in the symmetrisation formula has a factor $\varphi_{y_{\mu_1}}(z_{\sigma(1)})$. It follows from (5.3) that $\lim_{y_{\mu_1} \rightarrow \infty} \varphi_{y_{\mu_1}}(z_{\sigma(1)}) = 0$.
- (v) This property corresponds to $m = 0$, where it follows directly from Proposition 5.4.
- (vi) Consider the following two rational functions:

$$\frac{z_1 - a}{1 - z_1^2} \cdot \mathfrak{F}_{\mu}(z_1, z_2, \dots, z_n), \quad \frac{qz_1(aqz_1 - 1)}{1 - q^2 z_1^2} \cdot \mathfrak{F}_{\mu}\left(\frac{1}{qz_1}, z_2, \dots, z_n\right).$$

Observe that the multiplication by the rational pre-factors on both the functions reduces them both to symmetrisation formulas over \mathcal{B}_n of the same function. The inversion symmetry (5.10) follows immediately.

By virtue of Proposition 5.6, we have now shown that $\mathfrak{F}_{\mu}(z_1, \dots, z_n)$ satisfies the same collection of properties (i)-(vi) as the partition function $F_{\mu}(z_1, \dots, z_n)$. It remains to show that these properties imply that the two rational functions are equal. Let us define the following rational function

$$\mathfrak{Z}_{\mu}(z_1, \dots, z_n) := \mathfrak{F}_{\mu}(z_1, \dots, z_n) - F_{\mu}(z_1, \dots, z_n),$$

which we will show vanishes by induction on the length of the configuration μ . Throughout the remainder of the proof let us fix the difference $n - m \geq 0$, so that the induction will proceed on m, n together.

By construction when $m = 0$ (or $\mu = \emptyset$) we have $\mathfrak{Z}_{\emptyset}(z_1, \dots, z_{n-m}) = 0$. Then, let us assume that for some fixed integer $m \geq 0$ that the function vanishes for all configurations $\mu = (\mu_1, \dots, \mu_m)$ of length m i.e.

$$\mathfrak{Z}_{(\mu_1, \dots, \mu_m)}(z_1, \dots, z_n) = 0. \tag{5.18}$$

Then consider, for some fixed half-space configuration $\kappa = (\kappa_1, \dots, \kappa_{m+1})$ of length $m + 1$, the function $\mathfrak{Z}_{(\kappa_1, \dots, \kappa_{m+1})}(z_1, \dots, z_{n+1})$ which is, as the difference of two appropriate rational functions, a meromorphic function in y_{κ_1} whose singularities are simple poles at the points $y_{\kappa_1} = z_i^{-1}, qz_i$ for all $1 \leq i \leq n + 1$. However, since F_{κ} and \mathfrak{F}_{κ} are both symmetric and share the same residue at $y_{\kappa_1} = z_1^{-1}$, the following residues vanish:

$$\operatorname{Res}_{y_{\kappa_1}=z_i^{-1}} \mathfrak{Z}_{(\kappa_1, \dots, \kappa_{m+1})}(z_1, \dots, z_{n+1}) = 0,$$

for all $1 \leq i \leq n+1$. Using the inversion symmetry (5.10), we may similarly calculate the residue at the point $y_{\kappa_1} = qz_1$:

$$\begin{aligned} \operatorname{Res}_{y_{\kappa_1}=qz_1} F_{(\kappa_1, \dots, \kappa_{m+1})}(z_1, z_2, \dots, z_{n+1}) &= \frac{1 - z_1^2}{z_1 - a} \frac{qz_1(aqz_1 - 1)}{1 - q^2 z_1^2} \cdot \operatorname{Res}_{y_{\kappa_1}=w_1^{-1}} F_{\kappa}(w_1, z_2, \dots, z_{n+1}) \Big|_{w_1 = \frac{1}{qz_1}} \\ &= \frac{z_1(a - qz_1)}{a - z_1} \frac{(1 - q)(1 - z_1^2)}{1 - qz_1^2} \prod_{i=2}^{n+1} \left[\frac{qz_1 - z_i}{z_1 - z_i} \frac{1 - q^2 z_1 z_i}{q(1 - qz_1 z_i)} \right] \\ &\quad \times \prod_{j=1}^{\kappa_1-1} \frac{q(y_j - z_1)}{y_j - qz_1} F_{(\kappa_2, \dots, \kappa_{m+1})}(z_2, \dots, z_{n+1}). \end{aligned}$$

The same calculation may be performed on the symmetrisation formula $\mathfrak{F}_{(\kappa_1, \dots, \kappa_{m+1})}(z_1, \dots, z_n)$. And since we assume that $\mathfrak{F}_{(\kappa_2, \dots, \kappa_{m+1})}(z_2, \dots, z_n) = F_{(\kappa_2, \dots, \kappa_{m+1})}(z_2, \dots, z_n)$, it follows from the symmetry of both functions that

$$\operatorname{Res}_{y_{\kappa_1}=qz_i} \mathfrak{Z}_{(\kappa_1, \dots, \kappa_{m+1})}(z_1, \dots, z_{n+1}) = 0,$$

for all $1 \leq i \leq n+1$. From this we conclude that $\mathfrak{Z}_{(\kappa_1, \dots, \kappa_{m+1})}(z_1, \dots, z_{n+1})$ is an entire function of y_{κ_1} . Using the properties that we showed earlier, it also follows that

$$\lim_{y_{\kappa_1} \rightarrow \infty} \mathfrak{Z}_{(\kappa_1, \dots, \kappa_{m+1})}(z_1, \dots, z_n) = 0.$$

From this, we may conclude that it is a bounded entire function, which is identically zero by virtue of Liouville's Theorem. Thus, we conclude that (5.18) holds on all configurations of length $m+1$ which is the necessary induction step. \square

5.2.2 Symmetrisation formula for G_ν

Much like its dual function, the partition function G_ν also admits a symmetrisation formula in the upper-triangular boundary parameter specialisation $c \rightarrow \infty$. However, this symmetrisation is only over permutations S_n rather than signed permutations \mathcal{B}_n as is the case for F_μ .

Let us define the *dual plane wave* function for $k > 0$

$$\psi_k(z) = \psi_k(z|\mathbf{Y}) := \frac{a - z}{z(1 - az)} \frac{1 - qz^2}{1 - z^2} \frac{(1 - q)y_k}{1 - qzy_k} \prod_{j=1}^{k-1} \frac{1 - zy_j}{1 - qzy_j} \quad (5.19)$$

which will play an important role in the remainder of the chapter. Note that the dual plane wave function (5.19) is distinct from the plane wave function (5.3). We will also define the

following modified version of the dual plane wave function

$$\tilde{\psi}_k(z) = \tilde{\psi}_k(z|\mathbf{Y}) := \frac{1-z^2}{a-z} \frac{(1-q)zy_k}{1-qzy_k} \prod_{j=1}^{k-1} \frac{1-zy_j}{1-qzy_j}. \quad (5.20)$$

Corollary 5.7 (of Theorem 4.1). For a fixed integer n , let $\nu \in \mathbb{W}^n$ be a configuration with n parts and $(x_1, \dots, x_n) \in \mathbb{C}^n$ be an alphabet of length n whose components satisfy the requirements of Theorem 4.1. Then the partition function (3.29) indexed by state ν has the following symmetrisation formula in the upper-triangular boundary parameter specialisation $c \rightarrow \infty$:

$$G_\nu(x_1, \dots, x_n) = a^n \sum_{\sigma \in S_n} \sigma \left(\prod_{1 \leq i < j \leq n} \left[\frac{x_j - qx_i}{x_j - x_i} \frac{1 - x_i x_j}{1 - qx_i x_j} \right] \prod_{i=1}^n \tilde{\psi}_{\nu_i}(x_i) \right).$$

Proof. Consider the subset formula for G_ν given by (4.1). Since the length ν and the alphabet (x_1, \dots, x_n) are of the same length, the outer sum reduces to a single term $K = [n]$ while the sum over permutations S_n remains. The only remaining products outside the symmetrisation are symmetric under the action of all $\sigma \in S_n$, and so they may be brought into the sum. The result follows by noting that $h(x_i) = \frac{a(1-x_i^2)}{x_i-a}$ and identifying the function (5.20). \square

5.3 Orthogonality relations

In this section we will outline orthogonality relations of the functions F_κ and G_ν in the upper-triangular boundary parameter specialisation which have symmetrisation formulas given in Proposition 5.2 and Corollary 5.7 respectively.

5.3.1 Preliminary results

The orthogonality relations we present will involve various multiple-integral expressions with nested complex contours which we now outline.

Definition 5.8. For a fixed alphabet of vertical spectral parameters $\mathbf{Y} = (y_1, y_2, \dots)$, we denote by $\bar{\mathcal{C}}_1, \dots, \bar{\mathcal{C}}_n$ a collection of positively oriented closed complex contours satisfying

- For all $1 \leq i < j \leq n$, we have that $q\bar{\mathcal{C}}_j$ lies completely outside the interior of $\bar{\mathcal{C}}_i$, where $q\bar{\mathcal{C}}_j$ denotes the image of $\bar{\mathcal{C}}_j$ under multiplication by q . In addition, if 1 is within the interior of $\bar{\mathcal{C}}_i$ we also require that $\bar{\mathcal{C}}_i$ is completely contained in the interior of $\bar{\mathcal{C}}_j$.
- For all $k \in \mathbb{N}$, the contour $\bar{\mathcal{C}}_i$ surrounds all points y_k^{-1} and does not surround the points $qy_k^{-1}, q^{-1}y_k, q^{-1}y_k^{-1}, a$ for all $k \in \mathbb{N}$.

We note that the contours $\bar{\mathcal{C}}_i$ of Definition 5.8 are nested in the same way as those \mathcal{C}_i of Definition 4.3 except that they surround the points $w_i = y_k^{-1}$ for all $1 \leq i \leq n$ and $k \in \mathbb{N}$. As such, many of the properties of the contours of Definition 4.3 carry over to those of Definition 5.8. In particular, whenever the point 1 is not contained within the interior of the innermost contour $\bar{\mathcal{C}}_1$, then all contours of an appropriate multiple integral expression may be deformed to coincide by $\mathcal{C}_j \mapsto \mathcal{C}$ without changing the value of the integrals, where \mathcal{C} is a small contour surrounding the points y_j^{-1} for all $j \in \mathbb{N}$.

Definition 5.9. Let $H(w_1, \dots, w_n)$ be a rational function in n variables which is analytic at the points $w_i = w_j^{-1}$ for all $1 \leq i \neq j \leq n$ and has possible simple poles at the points $w_k = y_j^{-1}$ for all $1 \leq k \leq n$ and $j \in \mathbb{N}$. We call H an *admissible function* if the specialisation $H(w_1, \dots, w_n) \Big|_{w_\ell = w_k^{-1}}$ has removable singularities at $w_k = y_j^{-1}$ for all $1 \leq k, \ell \leq n$ and $j \in \mathbb{N}$.

Note that there is no requirement on a rational function being symmetric in order to be to be admissible. The following result will be useful for the symmetrisation and manipulation of multiple-integral expressions whose contours are arranged such that they are nested according to Definition 5.8.

Lemma 5.10. Let H be an admissible rational function in n variables. Then the following multiple contour integral expressions are equal:

$$\begin{aligned} & \oint_{\bar{\mathcal{C}}_1} \frac{dw_1}{2\pi i} \dots \oint_{\bar{\mathcal{C}}_n} \frac{dw_n}{2\pi i} \prod_{1 \leq i < j \leq n} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] H(w_1, \dots, w_n) \\ &= \frac{1}{n!} \oint_{\mathcal{C}} \frac{dw_1}{2\pi i} \dots \oint_{\mathcal{C}} \frac{dw_n}{2\pi i} \prod_{1 \leq i < j \leq n} \frac{1 - qw_i w_j}{1 - w_i w_j} \sum_{\sigma \in S_n} \sigma \left(\prod_{1 \leq i < j \leq n} \frac{w_i - w_j}{w_i - qw_j} \cdot H(w_1, \dots, w_n) \right), \end{aligned} \quad (5.21)$$

where the contours $\bar{\mathcal{C}}_1, \dots, \bar{\mathcal{C}}_n$ on the left-hand side satisfy the conditions of Definition 5.8. The contours on the right-hand side all coincide, where \mathcal{C} is a contour which surrounds the points y_j^{-1} for all $j \in \mathbb{N}$ whilst not surrounding any other singularities of the symmetrisation inside its integrand.

Lemma 5.10 essentially amounts to the ability to symmetrise multiple contour integral expressions of certain integrands whose contours are initially arranged according to Definition 5.8. The fact that these contour expressions may be symmetrised is not immediately obvious. This is largely due to the factor of $\prod_{i < j} (1 - w_i w_j)$ which appears in the integrand of the left-hand side of (5.21) which contribute singularities when the w -variables are all in neighbourhood of $1 \in \mathbb{C}$. These types of factors are usually not present in the studies of orthogonality in full-space models (see for example [TW08, BCPS15]), and it is essentially a feature of coordinate Bethe ansatz scattering factors in half-space models. The appearance of these factors expressions of half-space models may be found in, for example [TW13, He24].

Proof. As outlined in Figure 4.1, when the point $1 \in \mathbb{C}$ is not contained within the interior of $\bar{\mathcal{C}}_k$ for all $1 \leq k \leq n$, the contours of each w_k -variable on the left-hand side of (5.21) may be deformed to coincide $\bar{\mathcal{C}}_k \mapsto \mathcal{C}$ without crossing any singularities of the integrand, after which, the symmetrisation may be performed freely.

So let us assume, without loss of generality, that the point $1 \in \mathbb{C}$ is contained within the interior of the innermost contour $\bar{\mathcal{C}}_1$ arranged according to Definition 5.8. Let $1 < k \leq n$ be a fixed integer and let us consider the w_k -integral on the left-hand side of (5.21) under the deformation of the contour $\bar{\mathcal{C}}_k \mapsto \bar{\mathcal{C}}_{k-1}$. The only possible singularity which may be crossed under this deformation are the simple poles at the points $w_k = w_{k-1}^{-1}$. We will ultimately show that this may be regarded as a removable singularity in w_k once all integrations are performed. Let us then freely evaluate the residue of the left-hand side at this point

$$\begin{aligned} & \operatorname{Res}_{w_k=w_{k-1}^{-1}} \prod_{1 \leq i < j \leq n} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] H(w_1, \dots, w_n) \\ &= \frac{(1-q)(1-w_{k-1}^2)}{w_{k-1}(w_{k-1}^2 - q)} \prod_{\substack{1 \leq i < j \leq n \\ (i,j) \neq (k-1,k)}} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] \Bigg|_{w_k=w_{k-1}^{-1}} H(w_1, \dots, w_{k-1}, w_{k-1}^{-1}, \dots, w_n). \end{aligned} \quad (5.22)$$

By virtue of our assumption that the function H is admissible, the right-hand side of (5.22) has removable singularities at the points $w_{k-1} = y_j^{-1}$ for all $j \in \mathbb{N}$. And so, the only w_{k-1} singularities enclosed by the $\bar{\mathcal{C}}_{k-1}$ contour are those at the points $w_{k-1} = w_\ell^{-1}$ for each $1 \leq \ell < k-1$. From here, we will show that the following integral vanishes

$$\oint_{\bar{\mathcal{C}}_{k-1}} \frac{dw_{k-1}}{2\pi i} \operatorname{Res}_{w_k=w_{k-1}^{-1}} \prod_{1 \leq i < j \leq n} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] H(w_1, \dots, w_n) = 0, \quad (5.23)$$

for each $1 < k \leq n$ by describing a series of sequential contour manipulations. The fact that the w_{k-1} -integral (5.23) vanishes is sufficient to allow for the required contour deformation of $\bar{\mathcal{C}}_k \mapsto \bar{\mathcal{C}}_{k-1}$ for the w_k -integral.

The first step is the deformation of the w_2 -integral from $\bar{\mathcal{C}}_2 \mapsto \bar{\mathcal{C}}_1$, which implies (5.23) for $k = 2$ since the resulting w_1 -integral surrounds no singularities. Let us assume that for $k > 2$ we have been able to deform all contours $\bar{\mathcal{C}}_{k-1} \mapsto \bar{\mathcal{C}}_1$. Then we may freely deform the w_k -integral contour from $\bar{\mathcal{C}}_k \mapsto \bar{\mathcal{C}}_2$ without crossing any singularities of w_k . Now the poles at $w_k = w_\ell^{-1}$ for $1 \leq \ell < k$ may be accounted for by deforming the w_k -integration from $\bar{\mathcal{C}}_2 \mapsto \bar{\mathcal{C}}_1$. These singularities are all understood to be removable by virtue of (5.23) for $k = 2$. It follows via induction that all contours may be deformed so that they coincide as $\bar{\mathcal{C}}_k \mapsto \bar{\mathcal{C}}_1 =: \mathcal{C}$ to match the contour arrangement on the right-hand side of (5.21).

Since all contours coincide, the symmetrisation may take place which recovers the right-hand

side of (5.21). This completes the argument. \square

Lemma 5.11. For any $\kappa \in \mathbb{W}$, the partition function $F_\kappa(w_1, \dots, w_n)$ is an admissible rational function according to Definition 5.9.

Proof. It suffices to consider F_κ under the specialisation $w_2 = w_1^{-1}$ using the branching rule (Proposition 3.12) and the symmetry (Corollary 3.11):

$$F_\kappa(w_1, w_1^{-1}, w_3, \dots, w_n) = \sum_\lambda F_\lambda(w_1, w_1^{-1}) F_{\kappa/\lambda}(w_3, \dots, w_n).$$

Lemma 5.11 will follow by showing that $F_\lambda(w_1, w_1^{-1})$ has vanishing residue at $w_1 = y_j^{-1}$ for all $j \in \mathbb{N}$.

Let us remark that whenever $\lambda = \emptyset$, the partition function $F_\emptyset(w_1, w_1^{-1})$ may be evaluated using (5.9), which is analytic at $w_1 = y_j^{-1}$ for all $j \in \mathbb{N}$. And so, we only need to show that $F_\lambda(w_1, w_1^{-1})$ has vanishing residues at $w_1 = y_j^{-1}$ for all $j \in \mathbb{N}$. To this end, let us define the following partition function for fixed $\lambda \neq \emptyset$:

The diagram shows a partition function $f_\lambda(w_1) :=$ represented by a grid of nodes and edges. There are two horizontal rows of nodes. The top row has nodes labeled $1 \rightarrow w_1$ and $0 \leftarrow w_1^{-1}$. The bottom row has nodes labeled $1 \rightarrow w_1^{-1}$ and $0 \leftarrow w_1$. Vertical edges connect corresponding nodes in the two rows, with labels $\eta_1^\lambda, \eta_2^\lambda, \dots, \eta_N^\lambda$ placed between them. Below these labels are the variables y_1, y_2, \dots, y_N . Above the top row, five vertical arrows point upwards to the label 0 . The entire diagram is labeled (5.24).

which implicitly depends on some integer $N \geq \lambda_1$. We may then write

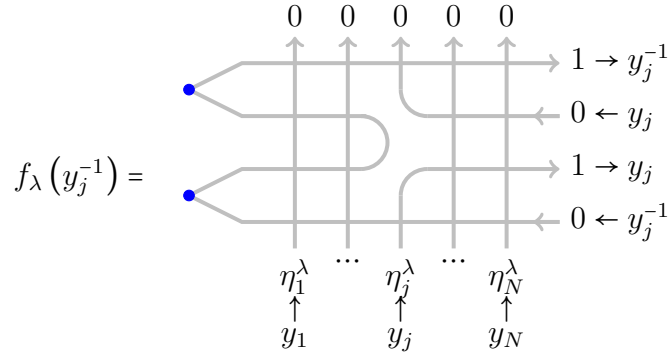
$$F_\lambda(w_1, w_1^{-1}) = f_\lambda(w_1) \cdot \prod_{i=1}^N \left[\frac{1 - qw_1 y_i}{1 - w_1 y_i} \frac{w_1 - qy_i}{w_1 - y_i} \right],$$

which accounts for the change in the normalisation of the vertices from replacing the dotted vertices with stochastic vertices for both double-rows according to (2.6). Importantly, the function $f_\lambda(w_1)$ defined by (5.24) is analytic at $w_1 = y_j^{-1}$. This means that we may calculate the required residue as

$$\operatorname{Res}_{w_1=y_j^{-1}} F_\lambda(w_1, w_1^{-1}) = f_\lambda(y_j^{-1}) \cdot \operatorname{Res}_{w_1=y_j^{-1}} \prod_{i=1}^N \left[\frac{1 - qw_1 y_i}{1 - w_1 y_i} \frac{w_1 - qy_i}{w_1 - y_i} \right]. \quad (5.25)$$

The residue on the right-hand side of (5.25) is non-zero for all $1 \leq j \leq N$. And so, we will proceed by showing that $f_\lambda(y_j^{-1}) = 0$ for all $1 \leq j \leq N$. Let us then consider this partition function

(5.24), evaluated at $w_1 = y_j^{-1}$ for $1 \leq j \leq \lambda_1$, which may be written using the factorisation property (Proposition 2.4) as



This may then be simplified using the unitarity properties of the R -matrix and K -matrix (Propositions 2.6 and 2.19) to yield

At this point we may again use the unitarity of the R -matrix and the Yang–Baxter equation (2.4) to find

The product in (5.27) comes from identifying the first $j - 1$ columns of (5.26) after applying unitarity. Now observe that all vertices on the upper row of the right-hand side of (5.27) are frozen. Evaluating these frozen vertices and removing them from the lattice yields the

following expression:

$$f_\lambda(y_j^{-1}) = \prod_{i=1}^{j-1} \delta_{\eta_i^\lambda, 0} \cdot \frac{1 - y_j^2}{1 - qy_j^2} \begin{array}{c} \begin{array}{ccc} & 0 & 0 & 1 \\ & \uparrow & \uparrow & \uparrow \\ \eta_j^\lambda & \dots & \eta_N^\lambda & \\ \uparrow & & \uparrow & \\ y_j & & y_N & \end{array} \\ \begin{array}{c} \text{---} \rightarrow 1 \rightarrow y_j \\ \text{---} \leftarrow 0 \leftarrow y_j^{-1} \end{array} \end{array} \quad (5.28)$$

Finally, observe that the right-most vertex in (5.28) has no possible allowed configuration, and so, we conclude that the entire partition function $f_\lambda(y_j^{-1}) = 0$. This completes the argument. \square

5.3.2 Spatial orthogonality

Theorem 5.12. Let $n \geq m \geq 0$ be fixed integers and let $\nu \in \mathbb{W}^n, \kappa \in \mathbb{W}^m$ be half-space configurations with n and m parts respectively. Then the following *spatial orthogonality* result holds:

$$\delta_{\nu, \kappa} = \frac{1}{(1-q)^n} \oint_{\bar{\mathcal{C}}_1} \frac{dw_1}{2\pi i} \dots \oint_{\bar{\mathcal{C}}_n} \frac{dw_n}{2\pi i} \prod_{1 \leq i < j \leq n} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] \prod_{i=1}^n \psi_{\nu_i}(w_i) F_\kappa(w_1, \dots, w_n), \quad (5.29)$$

where the contours $\bar{\mathcal{C}}_1, \dots, \bar{\mathcal{C}}_n$ are defined according to Definition 5.8 so that they all surround the points $w_i = y_j^{-1}$, for $1 \leq i \leq n$ and $j \in \mathbb{N}$ and no other singularities of the integrand.

The orthogonality relation (5.29) consists of series of contour integrals whose contours are nested in the complex plane according to Definition 5.8. The following Corollary is a symmetrised version of the same orthogonality relation.

Corollary 5.13. For fixed integers $n \geq m \geq 0$, the following symmetrised spatial orthogonality property holds as a consequence of Theorem 5.12:

$$\delta_{\nu, \kappa} = \frac{1}{(1-q)^n a^n n!} \oint_{\mathcal{C}} \frac{dw_1}{2\pi i} \dots \oint_{\mathcal{C}} \frac{dw_n}{2\pi i} \prod_{1 \leq i \neq j \leq n} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] \\ \times \prod_{i=1}^n \left[\left(\frac{w_i - a}{w_i (1 - w_i^2)} \right)^2 \frac{1 - qw_i^2}{1 - aw_i} \right] G_\nu(w_1, \dots, w_n) F_\kappa(w_1, \dots, w_n),$$

where the contour \mathcal{C} is a small contour surrounding the points $w_i = y_j^{-1}$ for all $1 \leq i \leq n$ and $j \in \mathbb{N}$.

Proof. Firstly, observe that the rational function $\prod_i \psi_{\nu_i}(w_i) F_{\kappa}(w_1, \dots, w_n)$ satisfies the requirements being an admissible function according to Definition 5.9. This can be seen simply by Lemma 5.11 and where it is clear that multiplication by $\prod_i \psi_{\nu_i}(w_i)$ does not affect the admissibility. The argument then follows by freely deforming the contour integrals of (5.29) to coincide and subsequently symmetrising according to Lemma 5.10. The partition function G_{ν} can be seen in the factorised integrand by noting the symmetrisation formula of Corollary 5.7. \square

Let us define the following product of partition functions

$$\chi(\nu, \kappa \mid \mathbf{Y}) := G_{\nu}(w_1, \dots, w_n \mid \mathbf{Y}) \cdot F_{\kappa}(w_1, \dots, w_n \mid \mathbf{Y}). \quad (5.30)$$

Let us also recall the local permutation operator \mathfrak{s}_i , defined for $i \in \mathbb{N}$ and its action on the infinite-length y alphabet by swapping the order of y_i and y_{i+1} as: $\mathfrak{s}_i \mathbf{Y} = (y_1, \dots, y_{i-1}, y_{i+1}, y_i, \dots)$. Let us also define its action on half-space configurations $\nu \in \mathbb{W}$ by interchanging the occupations of the two-sites i and $i+1$. That is, if $(\eta_i^{\nu}, \eta_{i+1}^{\nu}) = (j, k)$, then $\mathfrak{s}_i \nu \in \mathbb{W}$ is a half-space configuration defined by the occupations $(\eta_i^{\mathfrak{s}_i \nu}, \eta_{i+1}^{\mathfrak{s}_i \nu}) = (k, j)$ with all other $\eta_{\ell}^{\nu} = \eta_{\ell}^{\mathfrak{s}_i \nu}$ for all $\ell \in \mathbb{N} \setminus \{i, i+1\}$.

Lemma 5.14. Let $n \geq m \geq 1$ be fixed integers. Consider two half-space configurations $\kappa \in \mathbb{W}^m, \nu \in \mathbb{W}^n$ with m and n particles respectively. For any integer $i \in \mathbb{N}$, whenever $(\eta_i^{\kappa}, \eta_{i+1}^{\kappa}) = (0, 1)$ the following relation holds:

$$\begin{aligned} & \frac{q(y_{i+1} - y_i)}{y_{i+1} - qy_i} \chi(\nu, \mathfrak{s}_i \kappa \mid \mathbf{Y}) + \frac{(1-q)y_{i+1}}{y_{i+1} - qy_i} \chi(\nu, \kappa \mid \mathbf{Y}) \\ &= \begin{cases} \chi(\nu, \kappa \mid \mathfrak{s}_i \mathbf{Y}), & \text{if } \eta_i^{\nu} = \eta_{i+1}^{\nu} \\ \frac{(1-q)y_i}{y_{i+1} - qy_i} \chi(\nu, \kappa \mid \mathfrak{s}_i \mathbf{Y}) + \frac{q(y_{i+1} - y_i)}{y_{i+1} - qy_i} \chi(\mathfrak{s}_i \nu, \kappa \mid \mathfrak{s}_i \mathbf{Y}), & \text{if } (\eta_i^{\nu}, \eta_{i+1}^{\nu}) = (1, 0) \\ \frac{(1-q)y_{i+1}}{y_{i+1} - qy_i} \chi(\nu, \kappa \mid \mathfrak{s}_i \mathbf{Y}) + \frac{y_{i+1} - y_i}{y_{i+1} - qy_i} \chi(\mathfrak{s}_i \nu, \kappa \mid \mathfrak{s}_i \mathbf{Y}), & \text{if } (\eta_i^{\nu}, \eta_{i+1}^{\nu}) = (0, 1) \end{cases} \quad (5.31) \end{aligned}$$

Proof. Recall the partition function expression for the function $G_{\nu}(w_1, \dots, w_n)$ given in Lemma 3.20. In our case, where $\nu \in \mathbb{W}^n$ is a half-space configuration with n parts and we consider an alphabet of length n , particle conservation enforces the fact that every boundary vertex in (3.43) is frozen in the configuration which creates a path at each vertex. In turn, this freezes the entire triangular section of the lattice, which may be evaluated in the upper-triangular

specialisation as:

Observe that whenever $\nu \in \mathbb{W}^n$ is a half-space configuration with exactly n parts then we may express

$$G_\nu(w_1, \dots, w_n | \mathbf{Y}) = \frac{1 - w_i w_j}{1 - q w_i w_j} \prod_{i=1}^n \frac{a(1 - w_i^2)}{a - w_i} \cdot \Gamma_\nu(w_1, \dots, w_n | \mathbf{Y}),$$

where, following the freezing of the triangular section of the lattice of (3.43), Γ_ν may be realised as the following partition function⁴

We may deduce by Lemma 3.20 and Corollary 5.7 that Γ_ν has the following symmetrisation formula:

$$\Gamma_\nu(w_1, \dots, w_n) = \sum_{\sigma \in S_n} \sigma \left(\prod_{1 \leq i < j \leq n} \frac{x_j - q x_i}{x_j - x_i} \prod_{i=1}^n \left[\frac{(1 - q) z y_{\nu_i}}{1 - q z y_{\nu_i}} \prod_{j=1}^{\nu_i - 1} \frac{1 - z y_j}{1 - q z y_j} \right] \right).$$

We may then identify the quantity (5.30) as

$$\chi(\nu, \kappa | \mathbf{Y}) = \prod_{1 \leq i < j \leq n} \frac{1 - w_i w_j}{1 - q w_i w_j} \prod_{i=1}^n \frac{a(1 - w_i^2)}{a - w_i} \cdot \tilde{\chi}(\nu, \kappa | \mathbf{Y}),$$

⁴These boundary conditions have been considered extensively in the context of full-space models where they are typically denoted by F_κ , see for example [Bor17, BP18] in the context of the higher-spin vertex model. The particular case of the fundamental six-vertex model is addressed directly in [GGW24].

where $\tilde{\chi}(\nu, \kappa) = \Gamma_\nu(w_1, \dots, w_n) \cdot F_\kappa(w_1, \dots, w_n)$. And so, it suffices to prove the relation (5.31) on the quantity $\tilde{\chi}(\nu, \kappa)$, which we observe may be expressed as a single partition function

$$\tilde{\chi}(\nu, \kappa | \mathbf{Y}) = \begin{array}{c} \eta_1^\nu \quad \eta_2^\nu \quad \dots \quad \dots \quad \dots \\ \begin{array}{c} 1 \longrightarrow 0 \rightarrow w_n \\ 1 \longrightarrow 0 \rightarrow w_1 \\ 1 \longrightarrow 0 \rightarrow w_1 \\ \bullet \longrightarrow 1 \rightarrow w_n \\ \bullet \longleftarrow 0 \leftarrow w_n^{-1} \\ \bullet \longrightarrow 1 \\ \bullet \longleftarrow 0 \\ \bullet \longrightarrow 1 \rightarrow w_1^{-1} \\ \bullet \longleftarrow 0 \leftarrow w_1 \end{array} \\ \eta_1^\kappa \quad \eta_2^\kappa \quad \dots \quad \dots \quad \dots \\ \uparrow \quad \uparrow \\ y_1 \quad y_2 \end{array} \quad . \quad (5.32)$$

Note that it is only possible to attach these partition functions since the vertical edges between the double and single row partition functions are all frozen with no occupations.

Let us fix some $\kappa \in \mathbb{W}$ with some $i \in \mathbb{N}$ such that $(\eta_i^\kappa, \eta_{i+1}^\kappa) = (0, 1)$. Consider now the lattice (5.32) with an intertwining vertex attached to the bottom of the lattice between the i and $(i+1)$ -th columns. Application of the Yang–Baxter equation (2.4) yields the following equality:

$$\begin{array}{c} \eta_1^\nu \quad \dots \quad \eta_i^\nu \quad \eta_{i+1}^\nu \quad \dots \\ \begin{array}{c} 1 \longrightarrow 0 \\ 1 \longrightarrow 0 \\ 1 \longrightarrow 0 \\ \bullet \longrightarrow 1 \\ \bullet \longleftarrow 0 \\ \bullet \longrightarrow 1 \\ \bullet \longleftarrow 0 \\ \bullet \longrightarrow 1 \\ \bullet \longleftarrow 0 \end{array} \\ \eta_1^\kappa \quad \dots \quad \eta_i^\kappa \quad \eta_{i+1}^\kappa \quad \dots \\ \uparrow \quad \dots \quad \uparrow \quad \uparrow \\ y_1 \quad \dots \quad 0 \quad 1 \quad \dots \\ \uparrow \quad \uparrow \\ y_{i+1} \quad y_i \end{array} = \begin{array}{c} \eta_i^\nu \quad \eta_{i+1}^\nu \\ \begin{array}{c} \eta_1^\nu \quad \dots \quad \eta_i^\nu \quad \eta_{i+1}^\nu \quad \dots \\ 1 \longrightarrow 0 \\ 1 \longrightarrow 0 \\ 1 \longrightarrow 0 \\ \bullet \longrightarrow 1 \\ \bullet \longleftarrow 0 \\ \bullet \longrightarrow 1 \\ \bullet \longleftarrow 0 \\ \bullet \longrightarrow 1 \\ \bullet \longleftarrow 0 \end{array} \\ \eta_1^\kappa \quad \dots \quad \eta_i^\kappa \quad \eta_{i+1}^\kappa \quad \dots \\ \uparrow \quad \dots \quad \uparrow \quad \uparrow \\ y_1 \quad \dots \quad 0 \quad 1 \quad \dots \\ \uparrow \quad \uparrow \\ y_{i+1} \quad y_i \end{array} \quad . \quad (5.33)$$

We will recover the relation (5.31) by expanding the possible configurations of the intertwining vertex on both sides of (5.33). Firstly, the left-hand side of (5.33) may be expanded as

$$\begin{array}{c} \nearrow \\ \uparrow \quad \uparrow \\ y_{i+1} \quad y_i \end{array} \cdot \tilde{\chi}(\nu, \mathfrak{s}_i \kappa | \mathbf{Y}) \quad + \quad \begin{array}{c} \nearrow \\ \uparrow \quad \uparrow \\ y_{i+1} \quad y_i \end{array} \cdot \tilde{\chi}(\nu, \kappa | \mathbf{Y}), \quad (5.34)$$

which agrees with the left-hand side of (5.31). The right-hand side of (5.33) may be expanded at the top of its lattice in a similar manner to (5.34). This expansion ultimately yields the right-hand side of (5.31), depending on the local configuration of $(\eta_i^\nu, \eta_{i+1}^\nu)$, however we omit the details. \square

We are now ready to present the proof of the spatial orthogonality result.

Proof of Theorem 5.12. Let us begin by noting that for any half-space configurations $\nu, \kappa \in \mathbb{W}$, that the Kronecker delta function $\delta_{\nu, \kappa}$ obeys the same relation (5.31) as $\chi(\nu, \kappa | \mathbf{Y})$ from Lemma 5.14. In order to see this, we may consider the left-hand side of (5.31) for the local occupations of $(\eta_i^\kappa, \eta_{i+1}^\kappa) = (0, 1)$, while varying $(\eta_i^\nu, \eta_{i+1}^\nu)$, which is:

$$\frac{q(y_{i+1} - y_i)}{y_{i+1} - qy_i} \delta_{(\eta_i^\nu, \eta_{i+1}^\nu), (1,0)} + \frac{(1-q)y_{i+1}}{y_{i+1} - qy_i} \delta_{(\eta_i^\nu, \eta_{i+1}^\nu), (0,1)} = \begin{cases} 0, & \text{if } \eta_i^\nu = \eta_{i+1}^\nu \\ \frac{q(y_{i+1} - y_i)}{y_{i+1} - qy_i}, & \text{if } (\eta_i^\nu, \eta_{i+1}^\nu) = (1, 0), \\ \frac{(1-q)y_{i+1}}{y_{i+1} - qy_i}, & \text{if } (\eta_i^\nu, \eta_{i+1}^\nu) = (0, 1) \end{cases} \quad (5.35)$$

which agrees the right-hand side of (5.31) with $\chi(\nu, \kappa)$ replaced by $\delta_{(\eta_i^\nu, \eta_{i+1}^\nu), (\eta_i^\kappa, \eta_{i+1}^\kappa)}$. It follows that the local relation (5.35) may be extended to the global relation on $\delta_{\nu, \kappa}$.

Throughout this proof, we let $n \geq 1$ be a fixed integer and assume that $\nu \in \mathbb{W}^n$ is a half-space configuration with n parts. We then denote by $\mathcal{J}_{\nu, \kappa}$ the multiple integral expression on the right-hand side of (5.29), where we will aim to show that $\mathcal{J}_{\nu, \kappa} = \delta_{\nu, \kappa}$ for all $\nu \in \mathbb{W}^n$ and $\kappa \in \mathbb{W}^m$ for any integer $0 \leq m \leq n$. In the case of $m = 0$ (or $\kappa = \emptyset$), the fact that $\mathcal{J}_{\nu, \emptyset} = \delta_{\nu, \emptyset} = 0$ follows by considering the expression for F_\emptyset in (5.9). This is because the integrand of $\mathcal{J}_{\nu, \emptyset}$ has no singularities at the points $w_i = y_j^{-1}$ for all $1 \leq i \leq n$ and $j \in \mathbb{N}$.

In much the same way as the proof of Corollary 5.13, we note that we may symmetrise the integrand of the right-hand side of (5.29). This leads to the following integral expression

$$\begin{aligned} \mathcal{J}_{\nu, \kappa} = & \frac{1}{(1-q)^n a^n n!} \oint_{\mathcal{C}} \frac{dw_1}{2\pi i} \cdots \oint_{\mathcal{C}} \frac{dw_n}{2\pi i} \prod_{1 \leq i \neq j \leq n} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] \\ & \times \prod_{i=1}^n \left[\left(\frac{w_i - a}{w_i (1 - w_i^2)} \right)^2 \frac{1 - qw_i^2}{1 - aw_i} \right] \chi(\nu, \kappa | w_1, \dots, w_n | \mathbf{Y}), \quad (5.36) \end{aligned}$$

whose contours coincide and surround singularities at $w_i = y_j^{-1}$ for all $1 \leq i \leq n$ and $j \in \mathbb{N}$. We note that the w_1, \dots, w_n parameters in (5.30) are the integration variables of (5.36), and so, we add them to the notation of the χ here. The fact that $\chi(\nu, \kappa | w_1, \dots, w_n | \mathbf{Y})$ is a factor of the integrand of (5.36) endows the quantity $\mathcal{J}_{\nu, \kappa}$ with the property that it also satisfies the relation (5.31). And so, since both $\delta_{\nu, \kappa}$ and $\mathcal{J}_{\nu, \kappa}$ both satisfy the same relation, we may use this to compare the quantities under varying the states ν, κ .

In particular, we may begin by fixing $\kappa = (m, \dots, 1)$, so that all of its m parts are adjacent to the boundary, and deduce the fact that $\delta_{\nu, \kappa} = \mathcal{J}_{\nu, \kappa}$ for all $\nu \in \mathbb{W}^n, \kappa \in \mathbb{W}^m$ via the relation (5.31).

So then let us consider the integral expression for $\mathcal{J}_{\nu, (m, \dots, 1)}$. By virtue of Lemma 5.10, we may un-symmetrise the integral expression (5.36) yielding the expression on the right-hand side of (5.29) with $\kappa = (m, \dots, 1)$. From this, we may expand the symmetrisation formula of Proposition 5.2 to yield

$$\mathcal{J}_{\nu, (m, \dots, 1)} = (-a)^{n-m} V_{n-m} \sum_{\sigma \in \mathcal{B}_n} \mathcal{I}_{\nu, (m, \dots, 1)}(\sigma) \quad (5.37)$$

where

$$\begin{aligned} \mathcal{I}_{\nu, (m, \dots, 1)}(\sigma) := & \frac{1}{(1-q)^n} \oint_{\bar{\mathcal{C}}_1} \frac{dw_1}{2\pi i} \cdots \oint_{\bar{\mathcal{C}}_n} \frac{dw_n}{2\pi i} \\ & \times \prod_{1 \leq i < j \leq n} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \frac{w_{\sigma(i)} - qw_{\sigma(j)}}{w_{\sigma(i)} - w_{\sigma(j)}} \frac{1 - w_{\sigma(i)} w_{\sigma(j)}}{1 - qw_{\sigma(i)} w_{\sigma(j)}} \right] \\ & \times \prod_{i=1}^n \psi_{\nu_i}(w_i) \prod_{i=1}^m \varphi_{m-i+1}(w_{\sigma(i)}). \quad (5.38) \end{aligned}$$

Recall that the contours $\bar{\mathcal{C}}_1, \dots, \bar{\mathcal{C}}_n$ of (5.38) are arranged according to Definition 5.8, whereby $\bar{\mathcal{C}}_1$ is the inner-most contour, contained either on or within the interior of all other contours. This means that we may, in practice, calculate the quantity (5.38) by evaluating the w_1 -integral first by having its contour surrounding only the singularities at $w_1 = y_j^{-1}$ for all $j \in \mathbb{N}$ using the residue theorem.

For fixed signed permutation $\sigma \in \mathcal{B}_n$, let us denote the component $k \in \{1, \dots, n\}$ such that

either $\sigma(k) = +1$ or $\sigma(k) = -1$ (equivalently $k = \sigma^{-1}(\pm 1)$). Let us define the following quantity:

$$\tilde{\varphi}_j(w) := \begin{cases} \varphi_{m-j+1}(w) & \text{if } j \leq m \\ 1 & \text{if } j > m \end{cases}.$$

Then, since they are responsible for the w_1 -singularities contained within the interior of the $\bar{\mathcal{C}}_1$ contour, we calculate the w_1 -dependent plane wave functions of (5.38) as

$$\begin{aligned} & \frac{1 - w_1^2}{w_1 - a} \psi_{\nu_1}(w_1) \tilde{\varphi}_k(w_{\sigma(k)}) \\ &= \begin{cases} \frac{(1-q)y_{\nu_1}}{1 - qw_1 y_{\nu_1}} \frac{1-q}{1 - w_1 y_{m-k+1}} \prod_{j=1}^{\nu_1-1} \frac{1 - w_1 y_j}{1 - qw_1 y_j} \prod_{j=1}^{m-k} \frac{1 - qw_1 y_j}{1 - w_1 y_j} & \text{if } k \leq m \text{ and } \sigma(k) = +1 \\ \frac{a - qw_1}{1 - aw_1} \frac{(1-q)y_{\nu_1}}{1 - qw_1 y_{\nu_1}} \frac{1-q}{y_{m-k+1} - qw_1} \prod_{j=1}^{\nu_1-1} \frac{1 - w_1 y_j}{1 - qw_1 y_j} \prod_{j=1}^{m-k} \frac{q(y_j - w_1)}{y_j - qw_1} & \text{if } k \leq m \text{ and } \sigma(k) = -1. \\ \frac{1 - qw_1^2}{w_1(aw_1 - 1)} \frac{(1-q)y_{\nu_1}}{1 - qw_1 y_{\nu_1}} \prod_{j=1}^{\nu_1-1} \frac{1 - w_1 y_j}{1 - qw_1 y_j} & \text{if } k > m \end{cases} \end{aligned} \quad (5.39)$$

We will now make some key observations. Firstly, we may immediately see that the bottom two cases on the right-hand side of (5.39) are analytic at all points $w_1 = y_j^{-1}$ for $j \in \mathbb{N}$. Additionally, observe that since ν is half-space configuration with n parts, then its largest part must satisfy $\nu_1 \geq n$. Consider the top case in (5.39), where the only way by which there will be a singularity at a point $w_1 = y_j^{-1}$ is if $m - k + 1 \geq \nu_1$. The only way by which this inequality may be satisfied is if all of $m = n, k = 1$ and $\nu_1 = n$ hold simultaneously. Note that whenever $\nu_1 = n$ for $\nu \in \mathbb{W}^n$, since the parts are ordered ($\nu_1 > \dots > \nu_n \geq 1$), the only possible configuration satisfies $\nu_i = n - i + 1$ for all $1 \leq i \leq n$. Putting this all together, we may now immediately conclude that $\mathcal{I}_{\nu, (m, \dots, 1)}(\sigma)$ vanishes for all $\sigma \in \mathcal{B}_n$ unless $m = n$ and $\nu = (n, \dots, 1)$. And so, we may conclude that $\mathcal{J}_{\nu, (m, \dots, 1)} = 0$ whenever $m < n$ or $\nu \neq (n, \dots, 1)$.

All that remains to show is that $\mathcal{J}_{(n, \dots, 1), (n, \dots, 1)} = 1$, which will ultimately follow by showing that the identity signed permutation $\sigma = \text{id} \in \mathcal{B}_n$ is responsible for the only non-zero contribution to (5.37). Recall above where we determined that we must have $k = \sigma^{-1}(\pm 1) = 1$, in $\mathcal{I}_{(n, \dots, 1), (n, \dots, 1)}(\sigma)$ is only non-zero whenever the first part of σ is $\sigma(1) = +1$.

And so, we consider the quantity $\mathcal{I}_{(n, \dots, 1), (n, \dots, 1)}(\sigma)$ in the case $\sigma(1) = +1$, where we may evaluate the w_1 -integral just by taking the residue the simple pole at $w_1 = y_n^{-1}$. Let us consider the remaining factors of (5.38) after the evaluation of the w_1 -integral. Firstly, the scattering factors

under the specialisation $w_1 = y_n^{-1}$:

$$\prod_{1 \leq i < j \leq n} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \frac{w_{\sigma(i)} - qw_{\sigma(j)}}{w_{\sigma(i)} - w_{\sigma(j)}} \frac{1 - w_{\sigma(i)} w_{\sigma(j)}}{1 - qw_{\sigma(i)} w_{\sigma(j)}} \right] \Big|_{w_1 = y_n^{-1}}$$

are analytic at $w_k = y_n^{-1}$ for all $2 \leq k \leq n$ for all signed permutations $\sigma \in \mathcal{B}_n$. So then, the singularities enclosed by the contour of the subsequent w_2 -integral are due to the factors

$$\frac{1 - w_2^2}{w_2 - a} \psi_{n-1}(w_1) \varphi_{n-\sigma^{-1}(\pm 2)+1}(w_{\pm 2}).$$

We may conclude, by a similar analysis to (5.39), that there is only a non-zero contribution to the w_2 -integral whenever $\sigma(2) = +2$. It follows, by induction for instance, that the only non-zero quantity $\mathcal{I}_{(n, \dots, 1), (n, \dots, 1)}(\sigma)$ is from the signed permutation satisfying $\sigma(k) = +k$ for all k , which is exactly the identity permutation.

And so, it suffices to consider

$$\mathcal{J}_{(n, \dots, 1), (n, \dots, 1)} = \mathcal{I}_{(n, \dots, 1), (n, \dots, 1)}(\text{id}),$$

which is expressed explicitly, by using (5.39) and noticing that the scattering factors of (5.38) all cancel, as:

$$\mathcal{I}_{(n, \dots, 1), (n, \dots, 1)}(\text{id}) = (1 - q)^n \oint_{\bar{\mathcal{C}}_1} \frac{dw_1}{2\pi i} \dots \oint_{\bar{\mathcal{C}}_n} \frac{dw_n}{2\pi i} \prod_{i=1}^n \frac{y_i}{(1 - w_i y_{n-i+1})(1 - qw_i y_{n-i+1})}. \quad (5.40)$$

Notice that for each w_i -variable, the only singularity contained within the interior of the $\bar{\mathcal{C}}_i$ -contour in (5.40) is the simple pole at $w_i = y_{n-i+1}^{-1}$. It follows by a straightforward residue calculus computation that $\mathcal{I}_{(n, \dots, 1), (n, \dots, 1)}(\text{id}) = 1$.

And so, we conclude that $\mathcal{J}_{\nu, (m, \dots, 1)} = \delta_{\nu, (m, \dots, 1)}$ for all $\nu \in \mathbb{W}^n$. It follows by the relation of Lemma 5.14 that $\mathcal{J}_{\nu, \kappa} = \delta_{\nu, \kappa}$ for all $\kappa \in \mathbb{W}^m$ for all $0 \leq m \leq n$, completing the argument. \square

5.3.3 Integral formula for $G_{\nu/\mu}$

An immediate application of the spatial originality relation of Theorem 5.12 is the following explicit multiple contour integral expression for the partition function $G_{\nu/\mu}$ from Definition 3.10 indexed by two half-space configurations.

Theorem 5.15. For fixed integers $n \geq m \geq 1$, let $\nu, \mu \in \mathbb{W}$ be half-space configurations $\nu = (\nu_1, \dots, \nu_n)$ and $\mu = (\mu_1, \dots, \mu_m)$. The partition function (3.29), in the upper-triangular boundary specialisation $c \rightarrow \infty$, indexed by the skew configuration ν/μ is given by the following

integral formula

$$G_{\nu/\mu}(x_1, \dots, x_L) = \frac{1}{(1-q)^n} \prod_{i=1}^L \frac{x_i(1-ax_i)}{x_i-a} \oint_{\bar{\mathcal{C}}_1} \frac{dw_1}{2\pi i} \dots \oint_{\bar{\mathcal{C}}_n} \frac{dw_n}{2\pi i} \prod_{i=1}^L \prod_{j=1}^n \left[\frac{x_i - qw_j}{x_i - w_j} \frac{1 - w_j x_i}{1 - qw_j x_i} \right] \\ \times \prod_{1 \leq i < j \leq n} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] \prod_{i=1}^n \psi_{\nu_i}(w_i) F_{\mu}(w_1, \dots, w_n), \quad (5.41)$$

where the contours $\bar{\mathcal{C}}_1, \dots, \bar{\mathcal{C}}_n$ are arranged according to the Definition 5.8 so that all contours surround and enclose the points $w_i = y_j^{-1}$ for all $1 \leq i \leq n$ and $j \in \mathbb{N}$.

Because of the sum-to-unity property of the partition function G (Proposition 3.13), the integral formula (5.41) has the interpretation of a transition probability of a discrete-time Markov process (assuming that we have a non-negative specialisation of $G_{\nu/\mu}$). Theorem 5.15 may be regarded as an extension of Theorem 4.4 from empty to a generic initial condition in the upper-triangular boundary specialisation.

Proof. Consider the integrand of the right-hand side of (5.41), where we may recognise the right-hand side of the eigenvector relation (5.2). Observe that the convergence condition (5.1) may be satisfied by both considering an alphabet x_1, \dots, x_L where $\frac{1-x_i y_k}{1-q x_i y_k}$ is sufficiently small, and also by suitably deforming the integration contours so that the factors $(1 - w_i/y_k)$ and $(1 - qw_i y_k)$ are sufficiently small for each $1 \leq i \leq n$ and $k \in \mathbb{N}$. Once these conditions are satisfied, we may apply the eigenvector relation of Corollary 5.1 so that the right-hand side of (5.41) becomes

$$\frac{1}{(1-q)^n} \oint_{\bar{\mathcal{C}}_1} \frac{dw_1}{2\pi i} \dots \oint_{\bar{\mathcal{C}}_n} \frac{dw_n}{2\pi i} \prod_{1 \leq i < j \leq n} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] \prod_{i=1}^n \psi_{\nu_i}(w_i) \\ \times \sum_{\kappa} G_{\kappa/\nu}(x_1, \dots, x_L) F_{\kappa}(w_1, \dots, w_n). \quad (5.42)$$

Now, using linearity of the integral, we may expand the right-hand side of (5.42), where the spatial orthogonality result of Theorem 5.12 yields $G_{\nu/\mu}(x_1, \dots, x_L)$ which completes the argument. \square

Chapter 6

Asymmetric Simple Exclusion Process in Half-space

6.1 Model outline

We consider the *asymmetric simple exclusion process* (ASEP) in half-space $\mathbb{N} := \mathbb{Z}_{>0}$ with open boundary conditions, and with finitely many particles. This is a continuous time Markov process with state space given by

$$\mathbb{W} = \{S \subset \mathbb{N} : S \text{ is finite}\},$$

(note that \mathbb{W} is countable). We index these half-space configurations by $x = (x_1, \dots, x_N) \in \mathbb{W}$ satisfying $x_1 > \dots > x_N \geq 1$, and we think of the state $X_t = (X_t(1), \dots, X_t(N))$ of the process at time t as the positions (in decreasing order) of N particles present at that time t .

The dynamics of the process are as follows: a particle at site $x \geq 1$ jumps to $x + 1$ at rate 1 and to $x - 1$ (assuming $x \geq 2$) at rate $q \geq 0$, with jumps being permitted only if the target site is empty and, additionally, at rate $\alpha \geq 0$ new particles enter the system at site 1 whenever this site is empty and exit the system through site one at rate $\gamma \geq 0$ (see Figure 6.1). More precisely, the evolution of the system can be separated into *bulk dynamics*, with transition rates

$$\begin{aligned} x \mapsto (x_1, \dots, x_i + 1, \dots, x_N) & \quad \text{at rate } 1 \quad \text{if } x_{i-1} > x_i + 1, \\ x \mapsto (x_1, \dots, x_i - 1, \dots, x_N) & \quad \text{at rate } q \quad \text{if } x_{i+1} < x_i - 1 \quad (x_i \geq 2), \end{aligned}$$

This chapter is based on the preprint [P2], except for Sections 6.2 and 6.3.1 which are based on the publication [P1]. Minor modifications have been made for overall coherence and so that the thesis references its other chapters where relevant.

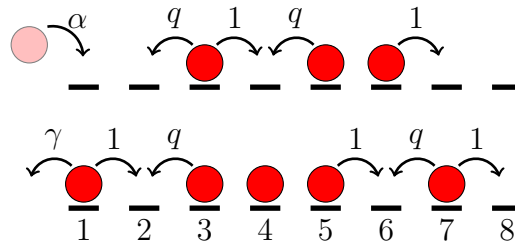


Figure 6.1: Dynamics of the ASEP in half-space \mathbb{N} with generic open boundaries.

and *boundary dynamics*, with transition rates

$$\begin{aligned} (x_1, \dots, x_N) &\mapsto (x_1, \dots, x_N, 1) && \text{at rate } \alpha && \text{if } N = 0 \text{ or } x_N > 1, \\ (x_1, \dots, x_N) &\mapsto (x_1, \dots, x_{N-1}) && \text{at rate } \gamma && \text{if } x_N = 1. \end{aligned}$$

In this text we consider ASEP with generic open boundary conditions at the origin. Note that the number of particles $|X_t|$ is a random variable which evolves with t .

The ASEP Markov generator \mathcal{L} acts on functions $f : \mathbb{W} \rightarrow \mathbb{C}$ as

$$\begin{aligned} \mathcal{L}f(x) &= \alpha(1 - \eta_1)(f(x \cup \{1\}) - f(x)) + \gamma\eta_1(f(x \setminus \{1\}) - f(x)) \\ &\quad + \sum_{s=1}^{\infty} \eta_s(1 - \eta_{s+1})(f(x^{s,s+1}) - f(x)) + \sum_{s=2}^{\infty} q\eta_s(1 - \eta_{s-1})(f(x^{s,s-1}) - f(x)), \end{aligned} \quad (6.1)$$

where $x^{s,s\pm 1} \in \mathbb{W}$ is the configuration obtained by the interchange of occupations at sites s and $s \pm 1$, and where we have used the notation η_s as the occupation of x at site $s \in \mathbb{N}$:

$$\eta_s = \begin{cases} 1 & \text{if } x_j = s \text{ for some } j, \\ 0 & \text{otherwise.} \end{cases}$$

Let us define an inner product with an orthogonal basis indexed by half-space configurations \mathbb{W} by $\langle y|x \rangle = \delta_{y,x}$. With this inner product, the *transition matrix elements* of the Markov generator \mathcal{L} are expressed as $\langle y|\mathcal{L}|x \rangle$. That is, when $y \neq x$ the matrix element $\langle y|\mathcal{L}|x \rangle$ is equal to the possible single-particle transition rates to go from state y to state x (if there is no possible single particle transition from y to x then the matrix element is zero), while the diagonal terms are equally defined as $\langle y|\mathcal{L}|y \rangle = -\sum_{x' \neq y} \langle y|\mathcal{L}|x' \rangle$.

With this notation, the action of \mathcal{L} on $f : \mathbb{W} \rightarrow \mathbb{C}$ can be expressed as

$$\mathcal{L}f(y) = \sum_z \langle x|\mathcal{L}|z \rangle f(z).$$

6.1.1 ASEP transition probability

In this section we present the transition probability for the half-space ASEP with generic initial conditions. We denote the transition probability by

$$\mathbb{P}_t(y \rightarrow x) := \mathbb{P}(X_t = x | X_0 = y). \quad (6.2)$$

The transition probabilities $\mathbb{P}_t(y \rightarrow x)$ can be formulated as the solution of the *master equation*, or *Kolmogorov forward equation*

$$\frac{d}{dt} \mathbb{P}_t(y \rightarrow x) = \sum_{x'} \mathbb{P}_t(y \rightarrow x') \langle x' | \mathcal{L} | x \rangle, \quad (6.3)$$

subject to the initial condition

$$\mathbb{P}_0(y \rightarrow x) = \delta_{y,x}.$$

Explicit integral formula expressions for transition probabilities have proven to be fruitful for obtaining exact asymptotic results for the ASEP under various boundary and initial conditions conditions. Notably for the full-space \mathbb{Z} (i.e. no boundary), solutions of the master equation were obtained for TASEP ($q = 0$) [Sch97b] and for generic bulk rates [TW08] using coordinate Bethe ansatz techniques. In half-space, explicit expressions were similarly obtained for closed boundary conditions ($\alpha = \gamma = 0$) [TW13].

The transition probability (6.2) is equivalently formulated as the unique⁵ solution of the *Kolmogorov backward equation*

$$\frac{d}{dt} \mathbb{P}_t(y \rightarrow x) = \sum_{y'} \langle y | \mathcal{L} | y' \rangle \mathbb{P}_t(y' \rightarrow x), \quad (6.4)$$

with the same initial condition. Here we will employ (6.4) rather than (6.3) to derive a formula for $\mathbb{P}_t(y \rightarrow x)$.

6.2 Reduction from six-vertex model

In this section we outline a method to obtain solutions to the half-line ASEP equations of motion (6.3) via a reduction of the symmetric function (3.27). An explicit evaluation is provided for generic open boundary conditions for the case of an empty initial configuration of

⁵Since our state space \mathbb{W} is countable, the solution of the backward Kolmogorov equation is unique whenever the process is not explosive (see e.g. Theorem 2.33 in [Lig10]); non-explosivity holds in our case because the sequence of jump times is clearly dominated stochastically from below by a sequence of independent exponential random variables τ_k with parameters $|X_0| + k$, which satisfy $\sum_{k \geq 1} \tau_k = \infty$ almost surely.

particles in Theorem 4.4. This is extended to arbitrary initial conditions with upper-triangular boundary conditions in Theorem 5.15.

Let $\mu, \nu \in \mathbb{W}$ be arbitrary particle configurations. The transition probability can be expressed as a formal solution to the evolution equation in terms of the Markov generator. This is given as⁶

$$\mathbb{P}_t(\mu \rightarrow \nu) = \langle \mu | \exp(t\mathcal{L}) | \nu \rangle, \quad (6.5)$$

where the exponential is regarded as the formal operator exponential. We now proceed by outlining the specialisation required to recover the ASEP transition probability from the half-space six-vertex model.

Proposition 6.1. Fix configurations $\mu, \nu \in \mathbb{W}$ and time $t \geq 0$. The ASEP half-line transition probability is recovered as the limit of the symmetric function

$$\mathbb{P}_t(\mu \rightarrow \nu) = \lim_{L \rightarrow \infty} G_{\nu/\mu}(x_1, \dots, x_L | \mathbf{Y}) \Big|_{y_j=1, x_i=1-(1-q)t/(2L)}, \quad (6.6)$$

where we have specified the spectral parameters $x_i = 1 - (1 - q)t/(2L)$ and vertical parameters $y_j = 1$ for all $1 \leq i \leq L$ and $j \in \mathbb{N}$ prior to taking the limit $L \rightarrow \infty$. The limit (6.6) holds provided that we choose boundary Markov rates as

$$\alpha = \frac{ac(1-q)}{(1-a)(1-c)}, \quad \gamma = -\frac{1-q}{(1-a)(1-c)},$$

where we are free to choose a, c so that both $\alpha, \gamma \geq 0$ while we restrict $q \geq 0$.

Let us comment briefly on the sensibility of the specialisation (6.6) for fixed $L \in \mathbb{N}$ and $0 < q < 1$. Much of the analysis of the partition function $G_{\nu/\mu}(x_1, \dots, x_L)$ from Chapters 3 and 4 stems from the symmetry of the function itself. This in turn depends on the fact that the parameters satisfy the constraints (3.13). Let us consider fixed vertical spectral parameters $y_j = 1$ for all $j \in \mathbb{N}$ and inhomogeneous $x_i = 1 - (1 - q)\epsilon_i$ for all $1 \leq i \leq L$ where the $\epsilon_i \ll 1$ are all small. Let us consider then $\epsilon_i \sim \epsilon_j$ for all $1 \leq i \neq j \leq L$, where as each $\epsilon_i \rightarrow 0$ we have

$$\left| \frac{1 - x_i y_k}{1 - q x_i y_k} \frac{q(1 - x_j / y_k)}{1 - q x_j / y_k} \right| \Big|_{y_k=1, x_i=1-(1-q)\epsilon_i, x_j=1-(1-q)\epsilon_j} = q\epsilon_i \epsilon_j + O(\epsilon_i^3),$$

which may be easily made to satisfy constraint (3.13). In particular, this implies that under the specialisation, $G_{\nu/\mu}(x_1, \dots, x_L)$ becomes a symmetric function of the appropriate $\epsilon_1, \dots, \epsilon_L \ll 1$ parameters which means that we may use many of the results from earlier in the thesis.


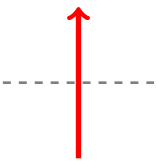
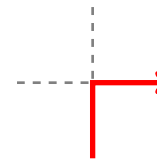
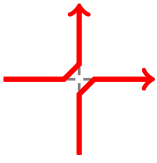
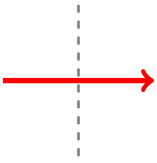
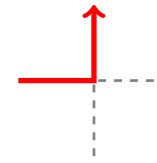
⁶We will denote half-space configurations by the letters μ, ν rather than x, y locally in this section to better match the conventions of Chapter 3, wherein $x_i, y_j \in \mathbb{C}$ denote spectral parameters rather than particle positions.

We will first demonstrate a degeneration of the double-row.

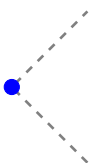
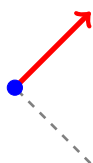

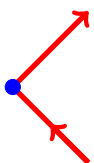
Lemma 6.2. Fix $\epsilon > 0$ let $x = 1 - (1 - q)\epsilon$ and $y_j = 1$ for all $j \in \mathbb{N}$. Then the double-row operator (3.2) is given by

$$A(x|\mathbf{Y}) \Big|_{x=1-(1-q)\epsilon, y_j=1} = 1 + 2\epsilon\mathcal{L} + O(\epsilon^2). \tag{6.7}$$

Proof. To demonstrate this, we choose $x_i = 1 - (1 - q)\epsilon$, $y_j = 1$ and observe that the weights from (2.2) with horizontal parameters x_i^{-1} and vertical parameters y_j acquire the following form.

		
1	$q\epsilon + O(\epsilon^2)$	$1 - q\epsilon + O(\epsilon^2)$
(6.8)		
		
1	$\epsilon + O(\epsilon^2)$	$1 - \epsilon + O(\epsilon^2)$

This table represents the weights of the upper row in the double row of (3.1). The weights of the lower row can be obtained by rotating these vertices. The boundary weights have a similar form.

			
$1 - 2\alpha\epsilon + O(\epsilon^2)$	$2\alpha\epsilon + O(\epsilon^2)$	$2\gamma\epsilon + O(\epsilon^2)$	$1 - 2\gamma\epsilon + O(\epsilon^2)$
(6.9)			

The double-row operator (6.7) can be determined by calculating for specific configurations $\mu, \nu \in \mathbb{W}$ the partition function using the weights in (6.8),(6.9) and matching with the action of the ASEP generator (6.1). □

Proof of Proposition 6.1. Using the double-row operator definition of the partition (3.27) and the scaling (6.7) with $\epsilon = t/(2L)$, we can arrive at the following expression for the partition function

$$G_{\nu/\mu}(x_1, \dots, x_L|\mathbf{Y}) \Big|_{y_j=1, x_i=1-(1-q)t/(2L)} = \langle \mu | \left(1 + \frac{t}{L}\mathcal{L} + O\left(\frac{1}{L^2}\right) \right)^L | \nu \rangle.$$

In order to take the limit we use the definition of the operator exponential. This yields

$$\lim_{L \rightarrow \infty} \left(1 + \frac{t}{L} \mathcal{L} + O\left(\frac{1}{L^2}\right) \right)^L = \lim_{L \rightarrow \infty} \left(1 + \frac{t}{L} \mathcal{L} \right)^L = \exp(t\mathcal{L}).$$

This recovers the formal solution for ASEP transition probability (6.5). \square

6.3 Integral formula for transition probability

Recall that for a fixed integer $N > 0$, $\mathcal{B}_N = S_N \times \{\pm 1\}^N$ denotes the group of signed permutations of $\{1, \dots, N\}$. The group \mathcal{B}_N acts on functions of some alphabet (w_1, \dots, w_N) of length N by permuting the variables as

$$\sigma(f(w_1, \dots, w_N)) = f(w_{\sigma(1)}, \dots, w_{\sigma(N)}),$$

where the negative indices are defined by $w_{-k} := 1/(qw_k)$ for any $1 \leq k \leq N$.

In order to state the transition probability as a complex contour integral we must explicitly outline the domain of integration.

Definition 6.3. For a fixed integer N , let $\mathcal{D}_1, \dots, \mathcal{D}_N$ be a collection of positively oriented closed complex integration contours satisfying:

- For all $1 \leq i \leq N$, each contour \mathcal{D}_i encloses the point 1 and does not enclose the points $0, q^{-1}$ and $(1 - q - \alpha)/\alpha$.
- For all $1 \leq i < j \leq N$, each \mathcal{D}_i is completely contained within the interior of each \mathcal{D}_j . Additionally, $q\mathcal{D}_j$ lies completely outside the interior of \mathcal{D}_i , where $q\mathcal{D}_j$ denotes the image of \mathcal{D}_j under multiplication by q . This also implies that no part of \mathcal{D}_j is contained within the interior of $q^{-1}\mathcal{D}_i$.
- For all $1 \leq k, \ell \leq N$, the contours \mathcal{D}_k and $q^{-1}\mathcal{D}_\ell^{-1}$ do not intersect and have completely disjoint interiors. The contour \mathcal{D}_ℓ^{-1} is the contour which traverses the points which are the reciprocal of those on \mathcal{D}_ℓ , and whose orientation is fixed so that $q^{-1}\mathcal{D}_\ell^{-1}$ encloses the point q^{-1} .

A depiction of nested contours which satisfy the restrictions of Definition 6.3 is given in Figure 6.2.

We now present the main result of this section: an explicit multiple integral formula for the transition probability of the half-space ASEP.

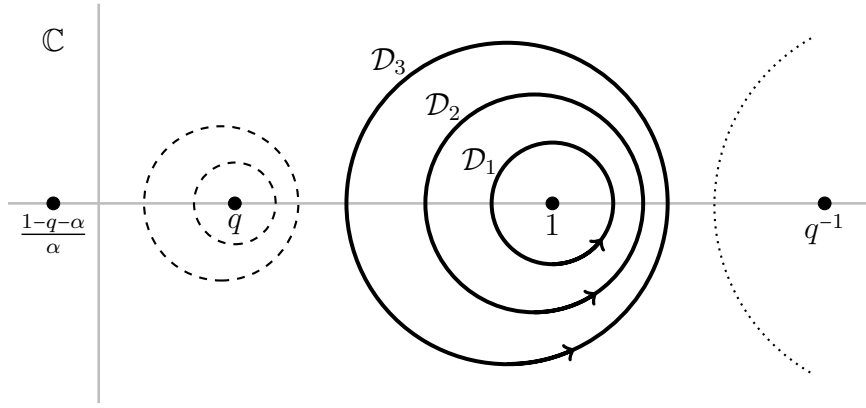


Figure 6.2: Examples of nested complex contours $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ satisfying the restrictions of Definition 6.3 for some $0 < q < 1$. The contours $q\mathcal{D}_1, q\mathcal{D}_2$ depicted with dashed lines while part of the contour $q^{-1}\mathcal{D}_3^{-1}$ is depicted with a dotted line. This example depicts parameters satisfying $\alpha > 1 - q$, however, this is not a requirement.

Theorem 6.4. Let $N \geq M \geq 0$ be fixed integers and let $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_M)$ be ordered coordinates of particles in the half-space ASEP. The transition probability between y and x for the half-space ASEP under the specialisation $\gamma = 0$ is given by

$$\begin{aligned} \mathbb{P}_t(y \rightarrow x) &= V_{N-M} \alpha^{N-M} e^{-\alpha t} \oint_{\mathcal{D}_1} \frac{dw_1}{2\pi i} \dots \oint_{\mathcal{D}_N} \frac{dw_N}{2\pi i} \prod_{1 \leq i < j \leq N} \left[\frac{w_j - w_i}{qw_j - w_i} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] \\ &\quad \times \prod_{i=1}^N \left[\frac{1 - qw_i^2}{w_i(q + \alpha - 1 - \alpha w_i)(1 - qw_i)} \left(\frac{1 - w_i}{1 - qw_i} \right)^{x_i - 1} \exp\left(\frac{(1 - q)^2 w_i t}{(1 - w_i)(1 - qw_i)} \right) \right] \\ &\quad \times \sum_{\sigma \in \mathcal{B}_N} \sigma \left(\prod_{1 \leq i < j \leq N} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \prod_{i=1}^M \left[\frac{1 - q - \alpha + \alpha w_i}{1 - qw_i^2} \frac{(1 - q)w_i}{1 - w_i} \left(\frac{1 - qw_i}{1 - w_i} \right)^{y_i - 1} \right] \right), \quad (6.10) \end{aligned}$$

where the contours surround singularities at $w_i = 1$ and are defined according to Definition 6.3. The overall normalisation is given by the following q -dependent factor

$$V_k = \frac{q^{\binom{k}{2}} (1 - q)^k}{\prod_{i=1}^k [(1 - q^i)(1 + q^{i-1})]}. \quad (6.11)$$

We will present two proofs of Theorem 6.4. The first proof follows from the reduction of the stochastic six-vertex model (Proposition 6.1) applied to the explicit integral formula obtained for the partition function $G_{\nu/\mu}$ in Theorem 5.15. The second, alternative proof uses completely independent techniques for the benefit readers not familiar with the technicalities of vertex models developed in Chapters 2-5. For the purposes of the overall narrative of this thesis, comments will be made within the alternative proof, where appropriate, to explain connections with the stochastic six-vertex model machinery presented earlier.

6.3.1 Proof using reduction from six-vertex model

First proof of Theorem 6.4. Note that in this proof we will use the six-vertex model conventions of Chapters 3-5, whereby $\mu, \nu \in \mathbb{W}$ are half-space configurations while $x_i, y_j \in \mathbb{C}$ are spectral parameters. And so, we will proceed to prove the integral formula of Theorem 6.4 for $\mathbb{P}_t(\mu \rightarrow \nu)$.

Consider the integral formula for $G_{\nu/\mu}(x_1, \dots, x_L)$ from (5.41) with contours $\{\bar{\mathcal{C}}_1, \dots, \bar{\mathcal{C}}_n\}$ taken to surround 1 as well as the points x_1, \dots, x_L . The conditions of the contours from Definition 5.8 also mean that the contours must be nested whilst they do not intersect.

Upon substitution in (5.41) and setting $x_i = 1 - (1 - q)t/(2L)$, the limit $L \rightarrow \infty$ can be taken on the integrand in a straightforward manner. We have also calculated under the ASEP limit

$$\lim_{L \rightarrow \infty} \prod_{j=1}^L \left[\frac{qw_i - x_j}{w_i - x_j} \frac{1 - w_i x_j}{1 - qw_i x_j} \right]_{x_j = 1 - (1-q)t/(2L)} = \exp\left(\frac{(1-q)^2 w_i t}{(1-w_i)(1-qw_i)} \right).$$

In order to obtain the result of the theorem, the $L \rightarrow \infty$ limit must also be simultaneously applied to the contours. This will deform the contours $\bar{\mathcal{C}}_i \mapsto \mathcal{D}_i$ using the contours from Definitions 5.8 and 6.3. This deformation occurs without crossing over any other singularities of the integrand. Note that in this limit we have written $a = \alpha/(\alpha + q - 1)$ for $\alpha + q \neq 1$. The symmetrisation on the last line of the integrand of (6.10) may be identified with the symmetrisation of the function F_μ from (5.4) whose spectral parameters are fixed at $y_j = 1$ for all $j \in \mathbb{N}$. This yields the result. \square

6.3.2 Alternative proof

The alternative proof of this result is based on studying the following function, which encapsulates many of the essential algebraic properties of the half-space ASEP⁷:

$$\mathcal{F}_y(w_1, \dots, w_N) := V_{N-M} \alpha^{N-M} \sum_{\sigma \in \mathcal{B}_N} \sigma \left(\prod_{1 \leq i < j \leq N} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \prod_{i=1}^M \varphi_{y_i}(w_i) \right), \quad (6.12)$$

where

$$\varphi_y(w) := \frac{1 - q - \alpha + \alpha w}{1 - qw^2} \frac{(1 - q)w}{1 - w} \left(\frac{1 - qw}{1 - w} \right)^{y-1}.$$

For consistency with what follows, we define $\mathcal{F}_y(w) = 0$ whenever $M > N$. Our formula (6.10) for the transition probability may then be expressed in terms of \mathcal{F}_y :

⁷This function is equivalent, up to multiplication by an overall rational coefficient, the symmetrisation formula for the partition function F_μ of Proposition 5.2 with all vertical spectral parameters set to 1

$$\begin{aligned} \mathbb{P}_t(y \rightarrow x) &= e^{-\alpha t} \oint_{\mathcal{D}_1} \frac{dw_1}{2\pi i} \cdots \oint_{\mathcal{D}_N} \frac{dw_N}{2\pi i} \prod_{1 \leq i < j \leq N} \left[\frac{w_j - w_i}{qw_j - w_i} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] \\ &\times \prod_{i=1}^N \left[\frac{1 - qw_i^2}{w_i(q + \alpha - 1 - \alpha w_i)(1 - qw_i)} \left(\frac{1 - w_i}{1 - qw_i} \right)^{x_i - 1} \exp\left(\frac{(1 - q)^2 w_i t}{(1 - w_i)(1 - qw_i)} \right) \right] \mathcal{F}_y(w_1, \dots, w_N). \end{aligned} \quad (6.13)$$

The case of empty initial conditions is recovered from (6.13) as the special case $M = 0$ or $y = \emptyset$, following from the fact that

$$\mathcal{F}_\emptyset(w_1, \dots, w_N) = \alpha^N,$$

which is proven in Section 6.4.1. Note, in particular, that this special case has a completely factorised integrand.

The next two results encapsulate the main identities needed in the proof of Theorem 6.4, which follow from properties of the function (6.12). Their proofs are very technical, and we defer them to Section 6.4.

The first of the two results shows that the function $\mathcal{F}_y(w_1, \dots, w_N)$ from (6.12) is an eigenvector of the ASEP Markov generator, and is equivalent to diagonalising the generator using coordinate Bethe ansatz on the Kolmogorov backward equation (6.4). The proof of this result appears in Section 6.4.2. This result is equivalent to the eigenvector relation of Corollary 5.1 in the ASEP limit as outlined in Section 6.2.

Lemma 6.5. For fixed integers $0 \leq M \leq N$, let $y = (y_1, \dots, y_M)$ be a half-space particle configuration and let $(w_1, \dots, w_N) \in \mathbb{C}^N$ be a fixed alphabet. Then the function (6.12) has the following eigenvector relation with the half-space ASEP Markov generator:

$$\sum_{y'} \langle y | \mathcal{L} | y' \rangle \mathcal{F}_{y'}(w_1, \dots, w_N) = \left(-\alpha + \sum_{i=1}^N \frac{(1 - q)^2 w_i}{(1 - w_i)(1 - qw_i)} \right) \mathcal{F}_y(w_1, \dots, w_N).$$

The second intermediate result needed in the proof of Theorem 6.4 corresponds the initial condition of (6.10). The proof of this result appears in Section 6.4.3. This initial condition is equivalent to a homogeneous version of the spatial orthogonality property Theorem 5.12 at the level of the six-vertex model. We now present it for the half-space ASEP.

Lemma 6.6. For fixed integers $N \geq M \geq 0$ let $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_M)$ be half-space configurations. The following orthogonality holds:

$$\begin{aligned} &\oint_{\mathcal{D}_1} \frac{dw_1}{2\pi i} \cdots \oint_{\mathcal{D}_N} \frac{dw_N}{2\pi i} \prod_{1 \leq i < j \leq N} \left[\frac{w_j - w_i}{qw_j - w_i} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] \\ &\times \prod_{i=1}^N \left[\frac{1 - qw_i^2}{w_i(q + \alpha - 1 - \alpha w_i)(1 - qw_i)} \left(\frac{1 - w_i}{1 - qw_i} \right)^{x_i - 1} \right] \mathcal{F}_y(w_1, \dots, w_N) = \delta_{x,y}. \end{aligned}$$

We can now use these two results to prove Theorem 6.4.

Second proof of Theorem 6.4. Consider the multiple contour integral expression on the right hand side of (6.13), which we want to show is equal to the half-space ASEP transition probability. This integral expression has the following factor in the integrand:

$$g_t(y) = e^{-\alpha t} \prod_{i=1}^N \exp\left(\frac{(1-q)^2 w_i t}{(1-w_i)(1-qw_i)}\right) \mathcal{F}_y(w_1, \dots, w_N).$$

Expanding the exponential and using Lemma 6.5 shows that

$$g_t(y) = \sum_{m=0}^{\infty} \frac{t^m}{m!} \sum_{y^{(1)}, \dots, y^{(m)}} \langle y | \mathcal{L} | y^{(1)} \rangle \langle y^{(1)} | \mathcal{L} | y^{(2)} \rangle \dots \langle y^{(m-1)} | \mathcal{L} | y^{(m)} \rangle \mathcal{F}_{y^{(m)}}(w_1, \dots, w_N)$$

whence it follows that g_t satisfies

$$\frac{d}{dt} g_t(y) = \sum_{y'} \langle y | \mathcal{L} | y' \rangle g_t(y').$$

Letting $p_t(y, x)$ denote the right hand side of (6.13), this shows that

$$\frac{d}{dt} p_t(y, x) = \sum_{y'} \langle y | \mathcal{L} | y' \rangle p_t(y', x).$$

Hence $p_t(y, x)$ solves the Kolmogorov backward equation (6.4). Lemma 6.6 yields the initial condition $p_0(y, x) = \delta_{x,y}$, finishing the proof. \square

6.4 Details of the alternative ASEP transition probability proof

In this section we present the details of the alternative proof of Theorem 6.4 which were omitted from Section 6.3, namely the proofs of the eigenvalue relation Lemma 6.5 and the initial condition Lemma 6.6.

6.4.1 Preliminary results

Before proceeding to the proofs of the intermediate results of Lemmas 6.5 and 6.6, it is useful to present some initial properties of the function (6.12). The following is a restatement of Proposition 5.4

Proposition 6.7. For any fixed integer $N \geq 0$ the following factorisation holds on any alphabet (w_1, \dots, w_N) :

$$\sum_{\sigma \in \mathcal{B}_N} \sigma \left(\prod_{1 \leq i < j \leq N} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \right) = \frac{1}{V_N}, \quad (6.14)$$

where the constant V_N is given by (6.11).

Proof. Let us define the functions

$$f(w_1, \dots, w_N) := \prod_{1 \leq i < j \leq N} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right], \quad g(z_1, \dots, z_N) := \prod_{1 \leq i < j \leq N} \left[\frac{z_i - qz_j}{z_i - z_j} \frac{1 - qz_i^{-1} z_j^{-1}}{1 - z_i^{-1} z_j^{-1}} \right],$$

so that $f(w_1, \dots, w_N)$ is the summand of the symmetrisation (6.14). Let us also consider the alphabet (z_1, \dots, z_N) defined by $z_i = q^{1/2} w_i$, so that $f(w_1, \dots, w_N) = q^{-\binom{N}{2}} g(z_1, \dots, z_N)$. The standard identification for negative indices under the action of \mathcal{B}_N is $w_{-i} = 1/(qw_i)$ which may be written in terms of z_i as $w_{-i} = q^{-1/2} z_i^{-1}$. And so, on the z -alphabet we make the identification $z_{-i} = z_i^{-1}$. It follows that the symmetrisation (6.14) may be evaluated as

$$\sum_{\sigma \in \mathcal{B}_N} \sigma(f(w_1, \dots, w_N)) = q^{-\binom{N}{2}} \sum_{\sigma \in \mathcal{B}_N} \sigma(g(z_1, \dots, z_N)),$$

where the action of the signed permutations $\sigma \in \mathcal{B}_N$ on the z -alphabet now uses $z_{-k} = z_k^{-1}$ for negative indices. The right hand side symmetrisation is equivalent to a known identity from [Ven15]:

$$\sum_{\sigma \in \mathcal{B}_N} \sigma(g(z_1, \dots, z_N)) = \prod_{i=1}^N \frac{(1 - q^i)(1 + q^{i-1})}{1 - q},$$

which follows from the fact that a BC_N -symmetric Hall–Littlewood polynomial indexed by the empty partition is equal to 1, i.e. $K_{\emptyset}(z_1^{\pm}, \dots, z_N^{\pm}) = 1$, with boundary parameters $t_0 = t_1 = t_2 = t_3 = 0$ and $a = -b = 1$. Application of this identity yields the result (6.14). \square

Observe that the scattering has the following invariance under $w_j \mapsto w_{-j} = 1/(qw_j)$:

$$\left. \frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right|_{w_j \mapsto w_{-j}} = \frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j}.$$

This observation allows for the symmetrisation identity of Proposition 6.7 has the following extension to a partial symmetrisation:

$$\sum_{\sigma \in \tilde{\mathcal{B}}_{N-M}} \sigma \left(\prod_{1 \leq i < j \leq N} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \right) = \frac{1}{V_{N-M}} \prod_{i=1}^M \prod_{j=i+1}^N \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right]. \quad (6.15)$$

for any fixed integer $0 \leq M < N$ where $\tilde{\mathcal{B}}_{N-M} \cong \mathcal{B}_{N-M}$ is the subgroup consisting of all signed

permutations which act on w by leaving w_1, \dots, w_M fixed whilst permuting w_{M+1}, \dots, w_N . While Proposition 6.7 implies that whenever indexed by an empty state $\mathcal{F}_\emptyset(w_1, \dots, w_N) = \alpha^N$, the following result allows for an alternative expression of the function (6.12) using a partial symmetrisation.

Proposition 6.8. Let $N \geq M \geq 1$ be fixed integers and let $y = (y_1, \dots, y_M)$ be a half-space configuration. Then, the function (6.12) has the following alternative expression:

$$\begin{aligned} \mathcal{F}_y(w_1, \dots, w_N) &= \alpha^{N-M} \sum_{\substack{T \subseteq \{1, \dots, N\} \\ |T|=M}} \sum_{\sigma \in \mathcal{B}_M} \prod_{1 \leq i < j \leq M} \left[\frac{w_{T_{\sigma(i)}} - qw_{T_{\sigma(j)}}}{w_{T_{\sigma(i)}} - w_{T_{\sigma(j)}}} \frac{1 - w_{T_{\sigma(i)}} w_{T_{\sigma(j)}}}{1 - qw_{T_{\sigma(i)}} w_{T_{\sigma(j)}}} \right] \prod_{i=1}^M \varphi_{y_i}(w_{T_{\sigma(i)}}) \\ &\quad \times \prod_{i=1}^M \prod_{\substack{j=1 \\ j \notin T}}^N \left[\frac{w_{T_{\sigma(i)}} - qw_j}{w_{T_{\sigma(i)}} - w_j} \frac{1 - w_{T_{\sigma(i)}} w_j}{1 - qw_{T_{\sigma(i)}} w_j} \right]. \end{aligned} \quad (6.16)$$

Proof. Consider the symmetrisation formula (6.12) and let us decompose the sum over \mathcal{B}_N as

$$\begin{aligned} \mathcal{F}_y(w_1, \dots, w_N) &= V_{N-M} \alpha^{N-M} \\ &\quad \times \sum_{\substack{T \subseteq \{1, \dots, N\} \\ |T|=M}} \sum_{\sigma \in \mathcal{B}_M} \sum_{\rho \in \mathcal{B}_{N-M}} \prod_{1 \leq i < j \leq M} \left[\frac{w_{T_{\sigma(i)}} - qw_{T_{\sigma(j)}}}{w_{T_{\sigma(i)}} - w_{T_{\sigma(j)}}} \frac{1 - w_{T_{\sigma(i)}} w_{T_{\sigma(j)}}}{1 - qw_{T_{\sigma(i)}} w_{T_{\sigma(j)}}} \right] \prod_{i=1}^M \varphi_{y_i}(w_{T_{\sigma(i)}}) \\ &\quad \times \prod_{i=1}^M \prod_{j=1}^{N-M} \left[\frac{w_{T_{\sigma(i)}} - qw_{T_{\rho(j)}}}{w_{T_{\sigma(i)}} - w_{T_{\rho(j)}}} \frac{1 - w_{T_{\sigma(i)}} w_{T_{\rho(j)}}}{1 - qw_{T_{\sigma(i)}} w_{T_{\rho(j)}}} \right] \prod_{1 \leq i < j \leq N-M} \left[\frac{w_{T_{\rho(i)}} - qw_{T_{\rho(j)}}}{w_{T_{\rho(i)}} - w_{T_{\rho(j)}}} \frac{1 - w_{T_{\rho(i)}} w_{T_{\rho(j)}}}{1 - qw_{T_{\rho(i)}} w_{T_{\rho(j)}}} \right]. \end{aligned} \quad (6.17)$$

We will proceed by explicitly evaluating the sum over $\rho \in \mathcal{B}_{N-M}$. Now, observe that, for fixed T , the first factor on the bottom line of (6.17) is invariant under the choice of $\rho \in \mathcal{B}_{N-M}$. Consequently, we may evaluate the sum over ρ using Proposition 6.7 as

$$\sum_{\rho \in \mathcal{B}_{N-M}} \prod_{1 \leq i < j \leq N-M} \left[\frac{w_{T_{\rho(i)}} - qw_{T_{\rho(j)}}}{w_{T_{\rho(i)}} - w_{T_{\rho(j)}}} \frac{1 - w_{T_{\rho(i)}} w_{T_{\rho(j)}}}{1 - qw_{T_{\rho(i)}} w_{T_{\rho(j)}}} \right] = \frac{1}{V_{N-M}},$$

which yields the desired result (6.16) once substituted into (6.17). \square

Proposition 6.9. Let $N \geq M \geq 1$ be fixed integers and let $w = (w_1, \dots, w_N)$ be a fixed alphabet and let $y_1, \dots, y_M \in \mathbb{Z}$ be fixed so that $y = (y_1, \dots, y_M)$ is a collection of integers⁸. The function (6.12) satisfies the following recursion relation for all $k \in \{1, \dots, N\}$:

$$\mathcal{F}_y(w_1, \dots, w_N) \Big|_{w_k=0} = \alpha \mathcal{F}_y(w_1, \dots, w_{k-1}, w_{k+1}, \dots, w_N).$$

⁸Typically we consider the case where $y_1 \geq \dots \geq y_M \geq 1$ so that y is a half-space configuration. However we will need the more general statement in the proof of Lemma 6.5 later on.

Proof. Observe that, whenever $w_k = 0$, the following property holds for any $y \in \mathbb{Z}$

$$\varphi_y(w_k) \Big|_{w_k=0} = \varphi_y \left(\frac{1}{qw_k} \right) \Big|_{w_k=0} = 0.$$

By setting $w_k = 0$ within the the symmetrisation formula (6.12) observe that terms corresponding to $\sigma(\ell) = \pm k$ for any $1 \leq \ell \leq M$ vanish. Now observe the following property for any $\sigma \in \mathcal{B}_N$ which satisfies $\sigma(\ell) = \pm k$:

$$\begin{aligned} & \prod_{1 \leq i \leq j \leq N} \left[\frac{w_{\sigma(i)} - qw_{\sigma(j)}}{w_{\sigma(i)} - w_{\sigma(j)}} \frac{1 - w_{\sigma(i)}w_{\sigma(j)}}{1 - qw_{\sigma(i)}w_{\sigma(j)}} \right] \Big|_{w_k=0} \\ &= \prod_{\substack{1 \leq i \leq j \leq N \\ i, j \neq \ell}} \left[\frac{w_{\sigma(i)} - qw_{\sigma(j)}}{w_{\sigma(i)} - w_{\sigma(j)}} \frac{1 - w_{\sigma(i)}w_{\sigma(j)}}{1 - qw_{\sigma(i)}w_{\sigma(j)}} \right] \times \begin{cases} q^{N-\ell} & \text{if } \sigma(\ell) = +k \\ q^{-(N-\ell)} & \text{if } \sigma(\ell) = -k \end{cases}. \end{aligned}$$

The application of this specialisation to the symmetrisation formula (6.12) yields:

$$\begin{aligned} \mathcal{F}_y(w_1, \dots, w_N) \Big|_{w_k=0} &= V_{N-M} \alpha^{N-M} \sum_{\ell=M+1}^N q^{N-\ell} \sum_{\substack{\sigma \in \mathcal{B}_N \\ \sigma(\ell)=+k}} \sigma \left(\prod_{\substack{1 \leq i \leq j \leq N \\ i, j \neq \ell}} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \prod_{i=1}^M \varphi_{y_i}(w_i) \right) \\ &+ V_{N-M} \alpha^{N-M} \sum_{\ell=M+1}^N q^{-(N-\ell)} \sum_{\substack{\sigma \in \mathcal{B}_N \\ \sigma(\ell)=-k}} \sigma \left(\prod_{\substack{1 \leq i \leq j \leq N \\ i, j \neq \ell}} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \prod_{i=1}^M \varphi_{y_i}(w_i) \right). \end{aligned} \quad (6.18)$$

Now, for each fixed value of ℓ , both restricted sums over signed permutations can be identified as being proportional to $\mathcal{F}_y(w_1, \dots, w_{k-1}, w_{k+1}, \dots, w_N)$. We emphasise that this identification is independent of ℓ so that we may re-write (6.18) by factoring out the \mathcal{F}_y function as

$$\mathcal{F}_y(w_1, \dots, w_N) \Big|_{w_k=0} = \frac{V_{N-M}}{V_{N-M-1}} \sum_{\ell=M+1}^N (q^{N-\ell} + q^{-(N-\ell)}) \cdot \alpha \mathcal{F}_y(w_1, \dots, w_{k-1}, w_{k+1}, \dots, w_N).$$

Finally, Proposition (6.9) follows by noting the following property of the normalisation factor (6.11)

$$\frac{V_{m-1}}{V_m} = \sum_{j=0}^{m-1} (q^j + q^{-j}),$$

which holds for any integer $m > 0$. □

Proposition 6.10. Let $N \geq M \geq 0$ be fixed integers and let $w = (w_1, \dots, w_N)$ be a fixed alphabet and let $y_1, \dots, y_M \in \mathbb{Z}$ be fixed so that $y = (y_1, \dots, y_M)$ is a collection of integers. Then the function $\mathcal{F}_y(w)$, given by the symmetrisation formula (6.12), is a meromorphic function in each variable w_1, \dots, w_N whose singularities are all located at the points $w_i = 1, q^{-1}$ for each $1 \leq i \leq N$.

Proof. Let us consider the function $\mathcal{F}_y(w_1, \dots, w_N)$ as a function of the variable w_k for some fixed integer $1 \leq k \leq N$. Excluding poles at $w_k = 1, q^{-1}$, by observing the individual terms of symmetrisation formula (6.12), the other possible singularities are at the points $w_k = w_j, 1/(qw_j)$ for integers $j \neq k$ and also at $w_k = \pm q^{-1/2}$. We will proceed by showing that these singularities are removable, so that the function $\mathcal{F}_y(w)$ may be defined to be analytic at those points. Firstly, the simple poles at $w_k = w_j, 1/(qw_j)$ are removable by virtue of the symmetrisation over \mathcal{B}_N and is therefore symmetric under the action of any signed permutation.

Let us now consider the apparent singularities at $w_k = \pm q^{-1/2}$, which are due to factors $(1 - qw_k^2)$ which appear in the denominator of $\varphi_{y_\ell}(w_k)$ and $\varphi_{y_\ell}\left(\frac{1}{qw_k}\right)$ for some integer $1 \leq \ell \leq M$. Let us consider the terms in (6.12) associated with signed permutations $\sigma, \sigma' \in \mathcal{B}_N$ satisfying $\sigma(\ell) = +k$ and $\sigma'(\ell) = -k$ but otherwise $\sigma(i) = \sigma'(i)$ for all other integers $i \neq \ell$. It is sufficient to consider the w_k -dependent factors of the sum of these two terms:

$$\begin{aligned} & \prod_{i=1}^{\ell-1} \left[\frac{w_k - qw_{\sigma(i)}}{w_k - w_{\sigma(i)}} \frac{1 - w_k w_{\sigma(i)}}{1 - qw_k w_{\sigma(i)}} \right] \prod_{i=\ell+1}^N \left[\frac{w_{\sigma(i)} - qw_k}{w_{\sigma(i)} - w_k} \frac{1 - w_k w_{\sigma(i)}}{1 - qw_k w_{\sigma(i)}} \right] \varphi_{y_\ell}(w_k) \\ & + \prod_{i=1}^{\ell-1} \left[\frac{qw_k - w_{\sigma(i)}}{q(w_k - w_{\sigma(i)})} \frac{1 - w_k w_{\sigma(i)}}{1 - qw_k w_{\sigma(i)}} \right] \prod_{i=\ell+1}^N \left[\frac{w_{\sigma(i)} - qw_k}{w_{\sigma(i)} - w_k} \frac{1 - w_k w_{\sigma(i)}}{1 - qw_k w_{\sigma(i)}} \right] \varphi_{y_\ell}\left(\frac{1}{qw_k}\right), \end{aligned} \quad (6.19)$$

where overall shared factors are neglected. Now, observe that each of these terms both contain the factor $\frac{w_\ell}{1 - qw_\ell^2}$ which changes sign under $w_\ell \mapsto 1/(qw_\ell)$. This means we may write (6.19) as

$$\frac{w_k}{1 - qw_k^2} \left(h(w_k) - h\left(\frac{1}{qw_k}\right) \right) \prod_{i=\ell+1}^N \left[\frac{w_{\sigma(i)} - qw_k}{w_{\sigma(i)} - w_k} \frac{1 - w_k w_{\sigma(i)}}{1 - qw_k w_{\sigma(i)}} \right], \quad (6.20)$$

where the following function is analytic at $w_k = \pm q^{-1/2}$:

$$h(w_k) = \frac{(1-q)(1-q-\alpha+\alpha w_k)}{(1-w_k)} \left(\frac{1-qw_k}{1-w_k} \right)^{y_\ell} \prod_{i=1}^{\ell-1} \left[\frac{w_k - qw_{\sigma(i)}}{w_k - w_{\sigma(i)}} \frac{1 - w_k w_{\sigma(i)}}{1 - qw_k w_{\sigma(i)}} \right].$$

From here, by showing that it has vanishing residue, we observe that the singularities in expression (6.20) at the points $w_k = \pm q^{-1/2}$ are removable. \square

The following evaluation of the function (6.12) is particularly useful.

Lemma 6.11. Let $N > 0$ be a fixed integer and let $w = (w_1, \dots, w_N)$ be a fixed alphabet. The function (6.12) indexed by the half-space configuration $y = (1)$ is explicitly given by:

$$\mathcal{F}_{(1)}(w_1, \dots, w_N) = \alpha^{N-1} \sum_{k=1}^N \frac{(1-q)^2 w_k}{(1-w_k)(1-qw_k)}. \quad (6.21)$$

Proof. Let us define the function $P(w_1, \dots, w_N)$ as the right hand side of (6.21). Then observe

that

$$P(w_1, \dots, w_N) \cdot \prod_{i=1}^N [(1 - w_i)(1 - qw_i)] \quad (6.22)$$

is clearly a symmetric polynomial of maximal degree 2 in each variable w_1, \dots, w_N . As such, the polynomial (6.22) is completely determined by three independent specialisations. The following are easily verifiable for each $k \in \{1, \dots, N\}$:

(i) The residue at $w_k = 1$:

$$\operatorname{Res}_{w_k=1} P(w_1, \dots, w_N) = \alpha^{N-1}(q - 1).$$

(ii) The residue at $w_k = q^{-1}$:

$$\operatorname{Res}_{w_k=q^{-1}} P(w_1, \dots, w_N) = \alpha^{N-1}(q^{-1} - 1).$$

(iii) The recurrence at $w_k = 0$:

$$P(w_1, \dots, w_N) \Big|_{w_k=0} = \alpha P(w_1, \dots, w_{k-1}, w_{k+1}, \dots, w_N).$$

Lemma 6.11 follows by showing that each of these properties holds for the symmetrisation formula (6.12) for the function $\mathcal{F}_{(1)}(w)$. But first we must show that

$$\mathcal{F}_{(1)}(w_1, \dots, w_N) \cdot \prod_{i=1}^N [(1 - w_i)(1 - qw_i)] \quad (6.23)$$

is a polynomial in each w -variable of maximal degree 2. By virtue of Proposition 6.10, the function (6.23) is a meromorphic function whose only possible poles occur at $w_k = 1, q^{-1}$ for each integer $1 \leq k \leq N$. By observing individual terms in its symmetrisation formula (6.12), we observe that $\mathcal{F}_{(1)}(w)$ has at most simple poles at $w_k = 1, q^{-1}$, so that (6.23) is indeed a polynomial in w_k . And so, we may determine its degree by investigating the behavior as $w_k \rightarrow \infty$ term-wise for each individual signed permutation $\sigma \in \mathcal{B}_N$. It is easily verifiable that for each $\sigma \in \mathcal{B}_N$:

$$\prod_{i=1}^N [(1 - w_i)(1 - qw_i)] \cdot \sigma \left(\prod_{1 \leq i < j \leq N} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \varphi_1(w_1) \right) = \mathcal{O}(w_k^2)$$

as $w_k \rightarrow \infty$ for each integer $1 \leq k \leq N$. So we conclude that (6.23) has maximal degree 2.

Let us now show that $\mathcal{F}_{(1)}(w_1, \dots, w_N)$ satisfies the required properties (i)–(iii). In what follows, let $k \in \{1, \dots, N\}$ be fixed positive integer.

(i) Observe that, in the symmetrisation formula (6.12), the only terms with singularities at $w_k = 1$ are those where $\varphi_y(w_k)$ appears. In our case, these terms will correspond with signed permutations satisfying $\sigma(1) = +k$ where there is a simple pole at $w_k = 1$. Then observe that the scattering factor has the following specialisation:

$$\left. \frac{w_k - qw_j}{w_k - w_j} \frac{1 - w_k w_j}{1 - qw_k w_j} \right|_{w_k=1} = 1.$$

This means that for individual signed permutations satisfying $\sigma(1) = +k$, we may evaluate the residue of the simple pole at $w_k = 1$ by evaluating

$$\operatorname{Res}_{w_k=1} \varphi_1(w_k) = q - 1.$$

Meanwhile, on the symmetrisation formula we have

$$\begin{aligned} \operatorname{Res}_{w_k=1} \mathcal{F}_{(1)}(w_1, \dots, w_N) &= \alpha^{N-1} V_{N-1} \sum_{\substack{\sigma \in \mathcal{B}_N \\ \sigma(1)=k}} \prod_{2 \leq i \leq j \leq N} \sigma \left(\left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \right) \operatorname{Res}_{w_k=1} \varphi_1(w_k) \\ &= \alpha^{N-1} (q - 1), \end{aligned}$$

where we have used Proposition 6.7 to simplify the last line. This is the desired result.

(ii) It follows by a similar argument that

$$\operatorname{Res}_{w_k=q^{-1}} \mathcal{F}_{(1)}(w_1, \dots, w_N) = \alpha^{N-1} (q^{-1} - 1),$$

where the residue isolates signed permutations satisfying $\sigma(1) = -k$.

(iii) This property follows directly as a consequence of Proposition 6.9. \square

6.4.2 Proof of eigenvector property

Let us recall the result of Lemma 6.5 where, for a fixed half-space configuration with M particles y , the following holds on any alphabet $w = (w_1, \dots, w_N)$:

$$\sum_{y' \in \mathbb{W}} \langle y | \mathcal{L} | y' \rangle \mathcal{F}_{y'}(w_1, \dots, w_N) = \left(-\alpha + \sum_{i=1}^N \frac{(1-q)^2 w_i}{(1-w_i)(1-qw_i)} \right) \mathcal{F}_y(w_1, \dots, w_N). \quad (6.24)$$

So that the function $\mathcal{F}_y(w)$ from (6.12) may be regarded as an eigenvector of the transition matrix of the half-space ASEP with $\gamma = 0$.

Proof of Lemma 6.5. Let us first examine the left hand side of (6.24) more explicitly. Let us

denote $\mathcal{F}_y(w_1, \dots, w_N) = f(y)$ for some fixed half-space configuration $y = (y_1, \dots, y_M)$. The left hand side of (6.24) can be written as the following *free-evolution equation*:

$$\sum_{y' \in \mathbb{W}} \langle y | \mathcal{L} | y' \rangle f(y') = q \sum_{i=1}^M f(y_1, \dots, y_i - 1, \dots, y_M) + \sum_{i=1}^M f(y_1, \dots, y_i + 1, \dots, y_M) + \alpha f(y_1, \dots, y_M, 1) - (M(1+q) + \alpha) f(y_1, \dots, y_M), \quad (6.25)$$

whose right hand side is subject to the following scattering conditions:

(i) *Bulk scattering*: for each $1 < i \leq M$, whenever $y_{i-1} = y_i + 1$:

$$qf(y_1, \dots, y_i, y_i, \dots, y_M) + f(y_1, \dots, y_i + 1, y_i + 1, \dots, y_M) = (1+q)f(y_1, \dots, y_i + 1, y_i, \dots, y_M). \quad (6.26)$$

(ii) *Boundary scattering*: whenever $y_M = 1$:

$$qf(y_1, \dots, y_{M-1}, 0) + \alpha f(y_1, \dots, y_{M-1}, 1, 1) = (\alpha + q)f(y_1, \dots, y_{M-1}, 1). \quad (6.27)$$

Observe that both scattering conditions are relations between functions indexed by both *physical* and *nonphysical* half-space configurations. For a fixed $y \in \mathbb{W}$, we may apply each of the scattering conditions to the right hand side of (6.25) to remove any nonphysical configurations. In this way, the bulk scattering (6.26) enforces the particle exclusion condition and the boundary scattering (6.27) enforces the boundary dynamics at $\gamma = 0$. One can readily check that this is consistent with the multiplication by the transition matrix on the left hand side, thus yielding the time-evolution of the half-space ASEP.

We first prove that the function (6.12) satisfies the free-evolution equation (6.25) before showing both scattering conditions (6.26), (6.27).

(a) *Free-evolution equation*: We begin by showing that $f(y) = \mathcal{F}_y(w)$ satisfies the following:

$$\left(-\alpha + \sum_{i=1}^N \frac{(1-q)^2 w_i}{(1-w_i)(1-qw_i)} \right) f(y_1, \dots, y_M) = \alpha f(y_1, \dots, y_M, 1) + q \sum_{i=1}^M f(y_1, \dots, y_i - 1, \dots, y_M) + \sum_{i=1}^M f(y_1, \dots, y_i + 1, \dots, y_M) - (M(1+q) + \alpha) f(y_1, \dots, y_M). \quad (6.28)$$

Firstly, we may immediately eliminate terms $-\alpha f(y)$ from each side of (6.28). Then, on what remains, consider the term corresponding to $\sigma \in \mathcal{B}_N$ within the symmetrisation formula (6.12)

individually within the remaining terms of (6.28). The following property will be useful:

$$q\varphi_{y_i-1}(w_i) + \varphi_{y_i+1}(w_i) - (1+q)\varphi_{y_i}(w_i) = \frac{(1-q)^2 w_i}{(1-w_i)(1-qw_i)} \varphi_{y_i}(w_i).$$

Application of this property within the the right hand side of (6.28) yields the following

$$\begin{aligned} & \sum_{k=1}^N \frac{(1-q)^2 w_k}{(1-w_k)(1-qw_k)} \sum_{\sigma \in \mathcal{B}_N} \sigma \left(\prod_{1 \leq i < j \leq N} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \prod_{i=1}^M \varphi_{y_i}(w_i) \right) \\ &= \sum_{\sigma \in \mathcal{B}_N} \sigma \left(\prod_{1 \leq i < j \leq N} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \prod_{i=1}^M \varphi_{y_i}(w_i) \left(\frac{V_{N-M-1}}{V_{N-M}} \varphi_1(w_{M+1}) + \sum_{k=1}^M \frac{(1-q)^2 w_k}{(1-w_k)(1-qw_k)} \right) \right). \end{aligned}$$

By re-arrangement, we can see that (6.28) holds if the following vanishes under symmetrisation:

$$\prod_{1 \leq i < j \leq N} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \prod_{i=1}^M \varphi_{y_i}(w_i) \left(\frac{V_{N-M-1}}{V_{N-M}} \varphi_1(w_{M+1}) - \sum_{k=M+1}^N \frac{(1-q)^2 w_k}{(1-w_k)(1-qw_k)} \right). \quad (6.29)$$

Observe, as a consequence of Lemma 6.11, that (6.29) vanishes under symmetrisation of the variables w_{M+1}, \dots, w_N whilst leaving w_1, \dots, w_M fixed and the result follows.

(bi) *Bulk scattering*: Consider, for some fixed alphabet $w = (w_1, \dots, w_N)$ and configuration $y = (y_1, \dots, y_M)$ with $y_{k-1} = y_k + 1$ for some $1 < k \leq M$, the following

$$\begin{aligned} & q\mathcal{F}_{(y_1, \dots, y_k, y_k, \dots, y_M)}(w) + \mathcal{F}_{(y_1, \dots, y_{k-1}, y_k+1, y_k+1, \dots, y_M)}(w) - (1+q)\mathcal{F}_{(y_1, \dots, y_{k-1}, y_k, \dots, y_M)}(w) \\ &= V_{N-M} \alpha^{N-M} \sum_{\sigma \in \mathcal{B}_N} \sigma \left(\prod_{\substack{1 \leq i < j \leq N \\ (i,j) \neq (k-1,k)}} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \prod_{i=1}^M \varphi_{y_i}(w_i) \right. \\ & \quad \left. \times \frac{w_{k-1} - qw_k}{w_{k-1} - w_k} \left[q + \frac{1 - qw_{k-1}}{1 - w_{k-1}} \frac{1 - qw_k}{1 - w_k} - (1+q) \frac{1 - qw_{k-1}}{1 - w_{k-1}} \right] \right). \quad (6.30) \end{aligned}$$

The top-line of the summand on the right hand side of (6.30) is symmetric under the interchange of w_{k-1} and w_k . Whereas the bottom line may be simplified as

$$\frac{1-q}{(1-w_{k-1})(1-w_k)} \frac{(w_{k-1} - qw_k)(w_k - qw_{k-1})}{w_{k-1} - w_k},$$

which is skew-symmetric under the same interchange. Therefore, the sum over all permutations $\sigma \in S_N$ on the right hand side of (6.30) vanishes. The bulk scattering condition (6.26) follows as a result.

(bii) *Boundary scattering*: Let us, for a fixed alphabet $w = (w_1, \dots, w_N)$, consider the following

$$q\mathcal{F}_{(y_1, \dots, y_{M-1}, 0)}(w) + \alpha\mathcal{F}_{(y_1, \dots, y_{M-1}, 1, 1)}(w) - (q+\alpha)\mathcal{F}_{(y_1, \dots, y_{M-1}, 1)}(w), \quad (6.31)$$

which will ultimately be shown to vanish. Let us first consider (6.31) in the case $N = M$ separately. It is convenient to treat this case separately since the function $\mathcal{F}_y(w)$ from (6.12) vanishes whenever the half-space configuration y has more parts than its alphabet w . This means that the middle term in (6.31) does not appear whenever $N = M$. In this case, the symmetrisation formula (6.12) can be compared term-wise, where the following function which appears in the summand

$$q\varphi_0(w_N) - (q + \alpha)\varphi_1(w_N), \quad (6.32)$$

while all other factors in the summand are invariant under the action of $w_N \mapsto 1/(qw_N)$. Now note the property whereby

$$\varphi_1(w) + \varphi_1\left(\frac{1}{qw}\right) = \frac{q}{q + \alpha} \left(\varphi_0(w) + \varphi_0\left(\frac{1}{qw}\right) \right) = \frac{(1 - q)^2 w}{(1 - w)(1 - qw)}.$$

This property may be used to show that (6.32) changes sign under $w_N \mapsto 1/(qw_N)$. And so, whenever $N = M$, (6.31) vanishes.

Now let us consider (6.31) for $M < N$. We will ultimately show that the following function

$$\begin{aligned} f(w_M, \dots, w_N) &= \prod_{M \leq i < j \leq N} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \\ &\times V_{N-M} \left(q\varphi_0(w_M) + \frac{V_{N-M-1}}{V_{N-M}} \varphi_1(w_M) \varphi_1(w_{M+1}) - (q + \alpha)\varphi_1(w_M) \right), \end{aligned} \quad (6.33)$$

vanishes when summed over signed permutations of the sub-alphabet (w_M, \dots, w_N) . The function $f(w_M, \dots, w_N)$ is part of the summand of the combined symmetrisation formulas for the functions in (6.31). The fact that the vanishing partial symmetrisation enforces the vanishing of the fully-symmetrised functions in (6.31) follows in along the same lines as the partial symmetrisation formula (6.15). The vanishing of the function (6.33) under symmetrisation is equivalent to the following condition in the function (6.12) for any $N \geq 2$:

$$\frac{q}{q + \alpha} \mathcal{F}_{(0)}(w_1, \dots, w_N) + \frac{\alpha}{q + \alpha} \mathcal{F}_{(1,1)}(w_1, \dots, w_N) = \mathcal{F}_{(1)}(w_1, \dots, w_N). \quad (6.34)$$

Let us then denote the left hand side of (6.34) as the function $\mathcal{G}(w_1, \dots, w_N)$, which is clearly a symmetric function. Moreover, we claim that the function defined by

$$\mathcal{G}(w_1, \dots, w_N) \cdot \prod_{i=1}^N [(1 - w_i)(1 - qw_i)]$$

is a symmetric polynomial of maximal degree 2 in each of the variables w_1, \dots, w_N . This claim follows from the observation that the function (6.23) is also a polynomial of maximal degree 2 when the configuration (1) is replaced by either (0) or (1, 1) by following an analogous argument which we do not explicitly present.

Before we proceed further, let us come up with an alternative expression for the function $\mathcal{G}(w_1, \dots, w_N)$. First, by invoking the alternative expression (6.16), the function $\mathcal{F}_{(0)}(w_1, \dots, w_N)$ has the following expression:

$$\mathcal{F}_{(0)}(w_1, \dots, w_N) = \alpha^{N-1} \sum_{1 \leq |i| \leq N} \frac{1-w_i}{1-qw_i} \varphi_1(w_i) \prod_{\substack{j=1 \\ j \neq |i|}}^N \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1-w_i w_j}{1-qw_i w_j} \right], \quad (6.35)$$

where the sum over i is over $\{-N, \dots, 1\} \cup \{1, \dots, N\}$ where $w_{-k} = 1/(qw_k)$ for each $k > 0$. Similarly, we may apply the expression (6.16) to express

$$\begin{aligned} \mathcal{F}_{(1,1)}(w_1, \dots, w_N) &= \sum_{1 \leq |i| \leq N} \varphi_1(w_i) \prod_{\substack{j=1 \\ j \neq |i|}}^N \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1-w_i w_j}{1-qw_i w_j} \right] \\ &\quad \times \alpha^{N-2} \sum_{\substack{1 \leq |k| \leq N \\ |k| \neq |i|}} \varphi_1(w_k) \prod_{\substack{j=1 \\ j \neq |i|, j \neq |k|}}^N \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1-w_i w_j}{1-qw_i w_j} \right], \end{aligned} \quad (6.36)$$

where we may recognise the bottom line of (6.36) as $\mathcal{F}_{(1)}(w_1, \dots, w_{|i|-1}, w_{|i|+1}, \dots, w_N)$. Then using Lemma 6.11, we may write (6.36) as

$$\mathcal{F}_{(1,1)}(w_1, \dots, w_N) = \alpha^{N-2} \sum_{1 \leq |i| \leq N} \varphi_1(w_i) \left(\sum_{\substack{j=1 \\ j \neq |i|}}^N \frac{(1-q)^2 w_j}{(1-w_j)(1-qw_j)} \right) \prod_{\substack{j=1 \\ j \neq |i|}}^N \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1-w_i w_j}{1-qw_i w_j} \right]. \quad (6.37)$$

Now, we claim that the function $\mathcal{G}(w) = (q\mathcal{F}_{(0)}(w) + \alpha\mathcal{F}_{(1,1)}(w))/(q + \alpha)$ can be written using expressions (6.35) and (6.37) to yield:

$$\begin{aligned} &\mathcal{G}(w_1, \dots, w_N) \\ &= \frac{\alpha^{N-1}}{q + \alpha} \sum_{1 \leq |i| \leq N} \varphi_1(w_i) \left(\frac{q-w_i}{1-w_i} + \sum_{j=1}^N \frac{(1-q)^2 w_j}{(1-w_j)(1-qw_j)} \right) \prod_{\substack{j=1 \\ j \neq |i|}}^N \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1-w_i w_j}{1-qw_i w_j} \right], \end{aligned} \quad (6.38)$$

where, again, the sum over i is over $\{-N, \dots, 1\} \cup \{1, \dots, N\}$ where $w_{-k} = 1/(qw_k)$ for each $k > 0$. Now to complete the argument to prove the identity (6.34), we show that (6.38) constitutes a suitable expression on which to demonstrate the properties (i)–(iii) from the proof of Lemma 6.11.

(I) The residue at $w_k = 1$:

$$\operatorname{Res}_{w_k=1} \mathcal{G}(w_1, \dots, w_N) = \alpha^{N-1} (q-1).$$

To see this property let us expand (6.38) to separate the positive and negative sum over i :

$$\begin{aligned}
& \mathcal{G}(w_1, \dots, w_N) \\
&= \frac{\alpha^{N-1}}{q + \alpha} \sum_{i=1}^N \varphi_1(w_i) \left(\frac{q(1-w_i)}{1-qw_i} + \sum_{\substack{j=1 \\ j \neq i}}^N \frac{(1-q)^2 w_j}{(1-w_j)(1-qw_j)} \right) \prod_{\substack{j=1 \\ j \neq i}}^N \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \\
&+ \frac{\alpha^{N-1}}{q + \alpha} \sum_{i=1}^N \varphi_1\left(\frac{1}{qw_i}\right) \left(\frac{1 - qw_i}{1 - w_i} + \sum_{\substack{j=1 \\ j \neq i}}^N \frac{(1-q)^2 w_j}{(1-w_j)(1-qw_j)} \right) \prod_{\substack{j=1 \\ j \neq i}}^N \left[\frac{qw_i - w_j}{w_i - w_j} \frac{1 - q^2 w_i w_j}{q(1 - qw_i w_j)} \right]. \quad (6.39)
\end{aligned}$$

Now, recall that $\varphi_1(w_i)$ has a simple pole at $w_i = 1$ while $\varphi_1\left(\frac{1}{qw_i}\right)$ is analytic at $w_i = 1$. Hence, we may recognise that each term in both sums in (6.39) has a simple pole at the point $w_k = 1$ for each integer $1 \leq k \leq N$. So evaluating the residue of $\mathcal{G}(w_1, \dots, w_N)$ at $w_k = 1$ amounts to calculating this residue on each term in both sums. This yields:

$$\begin{aligned}
\text{Res}_{w_k=1} \mathcal{G}(w_1, \dots, w_N) &= \frac{\alpha^{N-1}(q-1)}{q + \alpha} \sum_{\substack{j=1 \\ j \neq k}}^N \frac{(1-q)^2 w_j}{(1-w_j)(1-qw_j)} + \alpha^{N-1}(q-1) \prod_{\substack{j=1 \\ j \neq k}}^N \left[\frac{q - w_j}{1 - w_j} \frac{1 - q^2 w_j}{q(1 - qw_j)} \right] \\
&+ \frac{\alpha^{N-1}(q-1)}{q + \alpha} \sum_{\substack{i=1 \\ i \neq k}}^N \varphi_1(w_i) \frac{q - w_i}{1 - qw_i} \prod_{\substack{j=1 \\ j \neq i, j \neq k}}^N \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \\
&+ \frac{\alpha^{N-1}(q-1)}{q + \alpha} \sum_{\substack{i=1 \\ i \neq k}}^N \varphi_1\left(\frac{1}{qw_i}\right) \frac{1 - q^2 w_i}{q(1 - w_i)} \prod_{\substack{j=1 \\ j \neq i, j \neq k}}^N \left[\frac{qw_i - w_j}{w_i - w_j} \frac{1 - q^2 w_i w_j}{q(1 - qw_i w_j)} \right]. \quad (6.40)
\end{aligned}$$

The first two terms in (6.40) are due to the residues of the terms in each sum of (6.39) corresponding to $i = k$, while the remaining sums correspond to the terms in (6.39) which are due to the terms corresponding with $i \neq k$.

For some arbitrary integer $1 \leq \ell \leq N$ satisfying $\ell \neq k$, let us denote the right hand side of (6.40) as a function of w_ℓ by $g(w_\ell)$. From here, we claim that $g(w_\ell)$ is in fact a constant. In order to demonstrate this, we first show that $g(w_\ell)$ is an entire function of w_ℓ . The possible singularities are at the points $w_\ell = w_j, 1/(qw_j)$ for $j \neq \ell, j \neq k$ and also $w_\ell = 1, q^{-1}, \pm q^{-1/2}$. However, by virtue of Proposition 6.10, $\mathcal{G}(w_1, \dots, w_N)$ is a meromorphic function in w_ℓ whose only singularities occur at $w_\ell = 1, q^{-1}$. It then follows that the singularities at $w_\ell = w_j, 1/(qw_j)$ present in $g(w_\ell)$ are removable.

It remains to show that the singularities at $w_\ell = 1, q^{-1}$ are removable. Let us first consider the apparent singularities of (6.40) at $w_\ell = 1$, where each term has a simple pole. And so, neglecting the overall constants, we compute the residue of the first two terms on the right hand side as

$$\begin{aligned} \operatorname{Res}_{w_\ell=1} \left\{ \sum_{\substack{j=1 \\ j \neq k}}^N \frac{(1-q)^2 w_j}{(1-w_j)(1-qw_j)} + (q+\alpha) \prod_{\substack{j=1 \\ j \neq k}}^N \left[\frac{q-w_j}{1-w_j} \frac{1-q^2 w_j}{q(1-qw_j)} \right] \right\} \\ = q-1 + \frac{(q+\alpha)(1-q^2)}{q} \prod_{\substack{j=1 \\ j \neq k, j \neq \ell}}^N \left[\frac{q-w_j}{1-w_j} \frac{1-q^2 w_j}{q(1-qw_j)} \right]. \quad (6.41) \end{aligned}$$

A simple computation shows the contribution of the same residue on the two sums over i in (6.40) exactly cancels (6.41). From this we may conclude

$$\operatorname{Res}_{w_\ell=1} \operatorname{Res}_{w_k=1} \mathcal{G}(w_1, \dots, w_N) = 0$$

so that the $g(w_\ell)$ is analytic at $w_\ell = 1$. Although we do not present it, an analogous computation also shows that $g(w_\ell)$ is also analytic at $w_\ell = q^{-1}$, and thus we conclude that it is an entire function of w_ℓ .

By looking at each individual rational factor in the terms of the right hand side of (6.40), it is straightforward to see that the $g(w_\ell)$ is bounded in the limit $w_\ell \rightarrow \infty$.

And so, we may conclude that $g(w_\ell)$ is a bounded entire function in w_ℓ . Since ℓ was arbitrary, we may conclude that it is a bounded entire function in the remaining $w_1, \dots, w_{k-1}, w_{k+1}, \dots, w_N$ alphabet, and by Liouville's theorem, it is a constant. As such, we may evaluate the right hand side of (6.40) by taking any specialisation of the remaining w -alphabet. It is convenient to choose $w_1 = \dots = w_{k-1} = w_{k+1} = \dots = w_N = 0$ so that all terms in (6.40) vanish except the second one which yields the desired result:

$$\operatorname{Res}_{w_k=1} \mathcal{G}(w_1, \dots, w_N) = \alpha^{N-1} (q-1) \prod_{\substack{j=1 \\ j \neq k}}^N \left[\frac{q-w_j}{1-w_j} \frac{1-q^2 w_j}{q(1-qw_j)} \right] \Bigg|_{w_1=\dots=w_N=0} = \alpha^{N-1} (q-1).$$

(II) The residue at $w_k = q^{-1}$:

$$\operatorname{Res}_{w_k=q^{-1}} \mathcal{G}(w_1, \dots, w_N) = \alpha^{N-1} (q^{-1} - 1),$$

which follows by an analogous argument to the one used to calculate the residue at $w_k = 1$.

(III) The function \mathcal{G} inherits the property

$$\mathcal{G}(w_1, \dots, w_N) \Big|_{w_k=0} = \alpha \mathcal{G}(w_1, \dots, w_{k-1}, w_{k+1}, \dots, w_N)$$

from Proposition 6.9. □

6.4.3 Proof of initial condition

Let us recall the result of Lemma 6.6, where the following orthogonality statement on the function $\mathcal{F}_y(w)$ from (6.12)

$$\oint_{\mathcal{D}_1} \frac{dw_1}{2\pi i} \dots \oint_{\mathcal{D}_N} \frac{dw_N}{2\pi i} \prod_{1 \leq i < j \leq N} \left[\frac{w_j - w_i}{qw_j - w_i} \frac{1 - qw_i w_j}{1 - w_i w_j} \right] \\ \times \prod_{i=1}^N \left[\frac{1 - qw_i^2}{w_i(q + \alpha - 1 - \alpha w_i)(1 - qw_i)} \left(\frac{1 - w_i}{1 - qw_i} \right)^{x_i - 1} \right] \mathcal{F}_y(w_1, \dots, w_N) = \delta_{x,y}, \quad (6.42)$$

serves as the initial condition of the half-space ASEP transition probability (6.10). The integration contours of (6.42) satisfy the requirements of Definition 6.3.

Proof of Lemma 6.6. The proof of this result is rather technical, so we will proceed in several steps. For some $\sigma \in \mathcal{B}_N$ and half-space configurations $x = (x_1, \dots, x_N), y = (y_1, \dots, y_M)$, let us define the following integral expression:

$$\mathcal{I}_{x,y}(\sigma) := \oint_{\mathcal{D}_1} \frac{dw_1}{2\pi i} \dots \oint_{\mathcal{D}_N} \frac{dw_N}{2\pi i} \prod_{1 \leq i < j \leq N} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \frac{w_{\sigma(i)} - qw_{\sigma(j)}}{w_{\sigma(i)} - w_{\sigma(j)}} \frac{1 - w_{\sigma(i)} w_{\sigma(j)}}{1 - qw_{\sigma(i)} w_{\sigma(j)}} \right] \\ \times \prod_{i=1}^N \psi_{x_i}(w_i) \prod_{i=1}^M \varphi_{y_i}(w_{\sigma(i)}), \quad (6.43)$$

where the function defined by

$$\psi_x(w) := \frac{1 - qw^2}{w(q + \alpha - 1 - \alpha w)(1 - qw)} \left(\frac{1 - w}{1 - qw} \right)^{x-1}$$

is analytic at $w = 1$ for all integers $x > 0$. It is important to emphasise that the contours in the integral (6.43) are defined according to Definition 6.3 and do not depend on which signed permutation is being considering. The orthogonality statement (6.42) can then be restated as:

$$V_{N-M} \alpha^{N-M} \sum_{\sigma \in \mathcal{B}_N} \mathcal{I}_{x,y}(\sigma) = \delta_{x,y},$$

where in particular, the constants outside the sum reduce to 1 whenever $M = N$. Our argument is then split into the following claims which together imply the result.

- (i) For any configurations with $0 \leq M \leq N$, $\mathcal{I}_{x,y}(\sigma) = 0$ for any $\sigma \in \mathcal{B}_N \setminus S_N$.
- (ii) For any $\sigma \in S_N$, $\mathcal{I}_{x,y}(\sigma) = 0$ whenever $M < N$.

(iii) Whenever $M = N$ the following summation vanishes:

$$\sum_{\sigma \in S_N \setminus \{\text{id}\}} \mathcal{I}_{x,y}(\sigma) = 0.$$

(iv) When $\sigma = \text{id}$ and $M = N$, the integral may be evaluated $\mathcal{I}_{x,y}(\text{id}) = \delta_{x,y}$.

We will proceed by proving these claims individually. Before proceeding, it is useful to remark on the strategy within the proof regarding the order of integration of the contour alignments of Definition 6.3. The multiple integral expression (6.43) may be evaluated by successively taking residues at $w_1 = 1$, followed by $w_2 = 1$, and so on. However, we may switch the order of integration by deforming the contour of the w_2 -integral from \mathcal{D}_2 to lie completely within the interior of \mathcal{D}_1 . Due to the residue theorem, this deformation comes at the expense of the evaluation of residues due to possible poles at the points $w_2 = qw_1, w_1^{-1}$, or any other so-called *dynamic poles* that are crossed by this process. Due to the scattering factor in the integrand, the dynamic poles which are crossed will depend on the signed permutation σ .

In order to proceed, the following expressions will prove to be useful

$$\begin{aligned} \psi_{x_i}(w_i) \varphi_{y_{\sigma^{-1}(i)}}(w_i) &= (q-1) \frac{1}{1-w_i} \frac{1}{1-qw_i} \left(\frac{1-w_i}{1-qw_i} \right)^{x_i - y_{\sigma^{-1}(i)}}, \\ \psi_{x_i}(w_i) \varphi_{y_{\sigma^{-1}(i)}} \left(\frac{1}{qw_i} \right) &= (q-1) \frac{\alpha + qw_i(1-q-\alpha)}{1-q-\alpha+\alpha w_i} \frac{q^{y_{\sigma^{-1}(i)}-1}}{(1-w_i)^2} \left(\frac{1-w_i}{1-qw_i} \right)^{x_i + y_{\sigma^{-1}(i)}}. \end{aligned}$$

In particular, since all individual coordinates are positive, the second expression has a removable singularity at $w_i = 1$. Additionally, whenever $\sigma \in S_N$ is an element of the symmetric group (that is, a signed permutation without any sign flips) then the scattering factor in the integrand of (6.43) may be presented as product over the inversions of σ :

$$\prod_{1 \leq i < j \leq N} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \frac{w_{\sigma(j)} - qw_{\sigma(i)}}{w_{\sigma(i)} - w_{\sigma(j)}} \frac{1 - w_{\sigma(i)} w_{\sigma(j)}}{1 - qw_{\sigma(i)} w_{\sigma(j)}} \right] = (-1)^{|\sigma|} \prod_{\substack{1 \leq i < j \leq N \\ \sigma(i) > \sigma(j)}} \frac{w_{\sigma(i)} - qw_{\sigma(j)}}{w_{\sigma(j)} - qw_{\sigma(i)}}.$$

We now proceed with the proof of (i)–(iv).

(i) Let σ be a fixed signed permutation which contains at least one sign-flip, so that $\sigma \notin S_N$. Let $k \in \{1, \dots, N\}$ be the fixed integer that corresponds to the left-most sign flip within σ . That is, let $\ell \in \{1, \dots, N\}$ be the index satisfying $\sigma_\ell = -k$, so that we may represent the signed permutation as

$$\sigma = (\sigma_1, \dots, \sigma_{\ell-1}, -k, \sigma_{\ell+1}, \dots, \sigma_N),$$

where $\sigma_1, \dots, \sigma_{\ell-1} > 0$. We now show that $\mathcal{I}_{x,y}(\sigma) = 0$. First consider the integral expression (6.43) for a signed permutation with $k = 1$. The \mathcal{D}_1 contour is the inner-most contour according

to Definition 6.3, and so, we may evaluate the $\mathcal{I}_{x,y}(\sigma)$ by taking the residue at $w_1 = 1$. The w_1 -dependence of the integrand is given by

$$\prod_{j=2}^N \left[\frac{w_1 - w_j}{w_1 - qw_j} \frac{1 - qw_1 w_j}{1 - w_1 w_j} \right] \prod_{i=1}^{\ell-1} \frac{w_{\sigma_i} - w_1^{-1}}{w_{\sigma_i} - w_1^{-1}/q} \prod_{j=\ell+1}^N \frac{w_1^{-1}/q - qw_{\sigma_j}}{w_1/q - w_{\sigma_j}} \\ \times \prod_{i \neq \ell} \frac{1 - w_{\sigma_i} w_1^{-1}/q}{1 - w_{\sigma_i} w_1^{-1}} \begin{cases} \psi_{x_1}(w_1) \varphi_{y_\ell} \left(\frac{1}{qw_1} \right) & \text{if } \ell \leq M \\ \psi_{x_1}(w_1) & \text{if } \ell > M \end{cases},$$

which is analytic at $w_1 = 1$. We thus conclude that when $k = 1$ the integral expression (6.43) vanishes.

Now let us assume that $k > 1$. We aim to evaluate the w_k -integral in $\mathcal{I}_{x,y}(\sigma)$ by first deforming its contour from \mathcal{D}_k to be contained within the interior of \mathcal{D}_1 , and consequently also within the interiors of $\mathcal{D}_2, \dots, \mathcal{D}_{k-1}$. The possible dynamic poles of w_k in the integrand of (6.43) are listed below. All of these poles are either simple poles or removable singularities.

- $w_k = q^{-1}w_i$ for all $i < k$
- $w_k = qw_j$ for all $j > k$
- $w_k = w_i^{-1}$ for all $i \neq k$
- $w_k = q^{-1}w_{\sigma(i)}^{-1}$ for all $i < \ell$ (where $\sigma(i) > 0$)
- $w_k = w_{|\sigma(j)|}, q^{-1}w_{|\sigma(j)|}^{-1}$ for all $j > \ell$
- $w_k = w_{\sigma(i)}$ for all $i \neq \ell$ whenever $\sigma(i) > 0$

By virtue of the contour alignment in Definition 6.3 and the numerator of the integrand of (6.43), the only possible singularities crossed by deforming the contour of the w_k -integral are those at $w_k = w_i^{\pm 1}$ for all $i < k$. Note that the factor $\prod_{i \neq k} (w_i - w_k)$ in the numerator of the integrand implies that poles of the form $w_k = w_i$ are removable, so that we only need to consider the effect of poles of the form $w_k = w_i^{-1}$. We will now show that the contribution of these residues vanishes for all $1 \leq i < k$ so that the contour of the w_k -integral can be deformed to lie arbitrarily close to 1.

We first detail how the singularities $w_k = w_i^{-1}$ appear in the integrand of (6.43). For a fixed $\sigma \in \mathcal{B}_N$ whose left-most sign flip is $\sigma_\ell = -k$, consider all possible values $p \in \{1, \dots, k-1\}$ such that $p \neq \sigma_1, \dots, \sigma_{\ell-1}$. These possible values of p precisely correspond to the simple poles of the form $w_k = w_p^{-1}$ that are present in the integrand. For each such p , we also define $m = \sigma^{-1}(\pm p)$, which is the index that maps to either $+p$ or $-p$ under σ . We note that due to the restrictions on the values of p , it follows that $m > \ell$. The residue of all simple poles $w_k = w_p^{-1}$ ultimately do not contribute to the evaluation of $\mathcal{I}_{x,y}(\sigma)$ so that we may deform the contour of the w_k -integral.

If no such p exists then the prescribed deformation can be performed freely without crossing any singularities.

Denote by A the set of all possible values of p as described above. In order to show that $\mathcal{I}_{x,y}(\sigma) = 0$, we will have to show that the any contributions due to residues at $w_k = w_p^{-1}$ vanish for all possible combinations of p . More explicitly for any non-empty $\{p_1, \dots, p_n\} \subseteq A$ with $p_1 > \dots > p_n$, we will show that the following combination of successive residues vanishes:

$$\oint_{\tilde{\mathcal{D}}} \frac{dw_{p_n}}{2\pi i} \operatorname{Res}_{w_k=w_{p_1}^{-1}} \operatorname{Res}_{w_{p_1}=w_{p_2}^{-1}} \cdots \operatorname{Res}_{w_{p_{n-1}}=w_{p_n}^{-1}} g_{x,y}(w_1, \dots, w_N) = 0, \quad (6.44)$$

where $g_{x,y}$ is the integrand of (6.43) and $\tilde{\mathcal{D}}$ is a small circular complex contour, enclosing the point 1 with an arbitrarily small radius. For any value p as described above, consider the following residue on the w_k, w_p -dependent factors of $g_{x,y}$:

$$\operatorname{Res}_{w_k=w_p^{-1}} \prod_{1 \leq i < j \leq N} \left[\frac{w_i - w_j}{w_i - qw_j} \frac{1 - qw_i w_j}{1 - w_i w_j} \frac{w_{\sigma_i} - qw_{\sigma_j}}{w_{\sigma_i} - w_{\sigma_j}} \frac{1 - w_{\sigma_i} w_{\sigma_j}}{1 - qw_{\sigma_i} w_{\sigma_j}} \right] \psi_{x_k}(w_k) \psi_{x_p}(w_p) \tilde{\varphi}_\ell \left(\frac{1}{qw_k} \right) \tilde{\varphi}_m(w_{\pm p}), \quad (6.45)$$

where

$$\tilde{\varphi}_\ell(z) = \begin{cases} \varphi_{y_\ell}(z) & \text{if } \ell \leq M \\ 1 & \text{if } \ell > M \end{cases}.$$

Should there be a simple pole at $w_k = w_p^{-1}$, then (6.45) can be evaluated which yields a factor of the form:

$$\psi_{x_k} \left(\frac{1}{w_p} \right) \psi_{x_p}(w_p) \tilde{\varphi}_\ell \left(\frac{w_p}{q} \right) \tilde{\varphi}_m(w_{\pm p}) \sim \begin{cases} (1 - w_p)^{x_k + x_p - 2} & \text{if } \ell, m > M \\ (1 - w_p)^{x_k + x_p + y_\ell - 3} & \text{if } \ell \leq M, m > M \\ (1 - w_p)^{x_k + x_p + (y_\ell - y_m) - 3} & \text{if } \ell, m < M, \sigma(m) = +p \\ (1 - w_p)^{x_k + x_p + y_\ell + y_m - 4} & \text{if } \ell, m < M, \sigma(m) = -p \end{cases}.$$

We note here that since $x_i, y_j \geq 1$ for all $1 \leq i \leq N, 1 \leq j \leq M$, all contributions are of the form $(1 - w_p)^b$ for some non-negative integer $b \geq 0$. The third item can be seen to have a non-negative exponent since $m > \ell$, it follows that $y_\ell - y_m \geq 1$. For each successive residue evaluation in the chain $p_1 > \dots > p_n$ in (6.44), we obtain a factor of the same form. Evaluating all the residues leads to a factor $(1 - w_{p_n})^{b_1 + \dots + b_n}$ in the w_{p_n} integral of (6.44) which is analytic at $w_{p_n} = 1$ since all $b_i \geq 0$. Therefore, the w_{p_n} integral of (6.44) vanishes. Finally, since this is true for all non-empty sets A , the multiple residues at the points $w_k = w_{p_i}^{-1}$ do not contribute to the evaluation of (6.43). Thus we can deform the contour of the w_k -integral to lie closely around $w_k = 1$ without picking up additional residues. The integration over w_k then vanishes and we thus conclude that $\mathcal{I}_{x,y}(\sigma) = 0$ whenever $\sigma \in \mathcal{B}_N \setminus S_N$.

(ii) For a fixed $\sigma \in S_N$ and configurations x, y satisfying $M < N$, we will prove here that the multiple integral expression (6.43) vanishes. Let us define $k \in \{1, \dots, N\}$ such that $\sigma_N = k$.

Our proof strategy will be similar to the previous part, whereby we aim to deform the contour of the w_k -integral from \mathcal{D}_k to be arbitrarily close to 1. Since $M < N$, the integrand of (6.43) does not contain a φ function with w_k as an argument. Hence, due to the form of the ψ function, the integrand is analytic at $w_k = 1$. With this in mind, consider the w_k -dependent factors in the integrand:

$$(-1)^{|\sigma|} \prod_{\substack{1 \leq i < N \\ \sigma(i) > k}} \frac{w_{\sigma(i)} - qw_k}{w_k - qw_{\sigma(i)}}.$$

From this expression we can conclude that the only possible singularities are of the form $w_k = qw_{\sigma(i)}$ for $\sigma(i) > k$. The alignment of the contours outlined in Definition 6.3 ensures that these singularities are not enclosed by the \mathcal{D}_k contour so that the integrand is an analytic function of w_k both on and within the interior of \mathcal{D}_k . As a result, the integral (6.43) vanishes.

(iii) For any fixed permutation $\sigma \in S_N$ and configurations x, y both containing $M = N$ particles, the integral expression (6.43) can be written as

$$\begin{aligned} \mathcal{I}_{x,y}(\sigma) &= (q-1)^N \oint_{\tilde{\mathcal{D}}} \frac{dw_1}{2\pi i} \dots \oint_{\tilde{\mathcal{D}}} \frac{dw_N}{2\pi i} (-1)^{|\sigma|} \prod_{\substack{1 \leq i < j \leq N \\ \sigma_i > \sigma_j}} \frac{w_{\sigma_i} - qw_{\sigma_j}}{w_{\sigma_j} - qw_{\sigma_i}} \\ &\quad \times \prod_{i=1}^N \left[\frac{1}{1-w_i} \frac{1}{1-qw_i} \left(\frac{1-w_i}{1-qw_i} \right)^{x_i-y_{\sigma^{-1}(i)}} \right]. \end{aligned} \quad (6.46)$$

We emphasise here that in the absence of sign flips in the signed permutation, the contours have been deformed to coincide, $\mathcal{D}_i \mapsto \tilde{\mathcal{D}}$, where $\tilde{\mathcal{D}}$ is a positively-oriented contour enclosing 1 with an arbitrarily small radius. From here, we claim that the expression (6.46) has the desired property

$$\sum_{\sigma \in S_N \setminus \{\text{id}\}} \mathcal{I}_{x,y}(\sigma) = 0, \quad (6.47)$$

which is equivalent to a claim within the proof of Theorem 2.1 of [TW08]. We do not replicate the argument needed to prove the statement (6.47) due to its technical nature. However, the precise details of the matching with [TW08] are provided explicitly in Appendix C.

(iv) Consider finally the integral associated with the identity signed permutation $\text{id} \in \mathcal{B}_N$ for half-space configurations x, y with $M = N$:

$$\mathcal{I}_{x,y}(\text{id}) = (q-1)^N \oint_{\mathcal{D}_1} \frac{dw_1}{2\pi i} \dots \oint_{\mathcal{D}_N} \frac{dw_N}{2\pi i} \prod_{i=1}^N \left[\frac{1}{1-w_i} \frac{1}{1-qw_i} \left(\frac{1-w_i}{1-qw_i} \right)^{x_i-y_i} \right].$$

This integral is readily evaluated by taking successively taking residues at $w_i = 1$ which yields $\mathcal{I}_{x,y}(\text{id}) = \delta_{x,y}$ by the residue theorem. \square

6.5 Pfaffian formulas for the half-space TASEP

6.5.1 Schütz-type Pfaffian transition probability

This section will be devoted to explicit Pfaffian expressions for the transition probabilities of half-space TASEP, that is, the totally asymmetric version of half-space ASEP where particles only jump to the right. The TASEP transition probability is obtained by taking the limit $q \rightarrow 0$ within the integral formula of Theorem 6.4. We will present the general expression for this result under both empty and general deterministic initial conditions.

Definition 6.12. Let us denote by β a positively oriented closed complex integration contour which is sufficiently large so that it surrounds the points $0, 1, \alpha$ and $1 - \alpha$.

Empty initial conditions

Theorem 6.13. Let N be a fixed integer and let $x = (x_1, \dots, x_N)$ be the ordered coordinates of half-space TASEP particles. Then the half-line open TASEP has transition probability from an empty state to x given by the following: when N is even

$$\mathbb{P}_t(\emptyset \rightarrow x) = (-1)^{\binom{N}{2}} e^{-\alpha t} \text{Pf}[\mathbf{Q}], \quad (6.48)$$

and when N is odd the underlying matrix is augmented with an addition row and column

$$\mathbb{P}_t(\emptyset \rightarrow x) = (-1)^{\binom{N}{2}} e^{-\alpha t} \text{Pf} \begin{bmatrix} \mathbf{Q} & \mathbf{p} \\ -\mathbf{p}^T & 0 \end{bmatrix}.$$

Here, \mathbf{Q} is a skew-symmetric $N \times N$ -dimensional matrix and \mathbf{p} is a column vector of length N whose entries are given by

$$\begin{aligned} [\mathbf{Q}]_{i,j} &:= Q_{i,j}(x_{N-i+1}, x_{N-j+1}), \\ [\mathbf{p}]_i &:= p_i(x_{N-i+1}), \end{aligned}$$

for all $1 \leq i, j \leq N$. We define the integral kernels

$$Q_{i,j}(x, y) = \alpha^2 \oint_{\beta} \frac{du}{2\pi i} \oint_{\beta} \frac{dw}{2\pi i} \frac{u-w}{1-u-w} \frac{w^{i-x} e^{t(w-1)}}{(w-\alpha)(w-1)^i} \frac{w^{j-y} e^{t(u-1)}}{(u-\alpha)(u-1)^j}, \quad (6.49)$$

and

$$p_i(x) = \oint_{\beta} \frac{dw}{2\pi i} \frac{w^{i-x} e^{t(w-1)}}{(w-\alpha)(w-1)^i}, \quad (6.50)$$

whose contours surround poles at $w, v = 1, 0, \alpha, 1 - \alpha$ whilst omitting all other singularities of the integrand.

Theorem 6.13 is a special case of Theorem 6.17 which appears later in the text. As such, we only present the proof of the later result. The skew-symmetric nature of the Pfaffian kernel is assured by the following property of (6.49)

$$Q_{k,\ell}(x, y) = -Q_{\ell,k}(y, x).$$

In particular, this implies that the function $Q_{k,k}$ is anti-symmetric.

Remark 6.14. We note here the resemblance of the transition probability (6.48) with its Pfaffian kernel to Schütz's solution of the full-line TASEP [Sch97b] presented with a determinantal kernel.

Remark 6.15. The w and u contours of (6.49) can be deformed to coincide with each other despite the apparent singularity due to the factor of $1 - u - w$ appearing in the denominator of the integrand. Without loss of generality, assume we perform the w integration first. Then the pole at $w = 1 - u$ can be understood to be a removable singularity since the remaining u -integrand has vanishing residue at infinity.

Remark 6.16. Our overall probability measure for the transition probability of Theorem 6.13 is over all numbers of particles in the system. That is, for all $t \geq 0$ we have the normalisation

$$\sum_{N=0}^{\infty} \sum_{1 \leq x_N < \dots < x_1} \mathbb{P}_t(\emptyset \rightarrow (x_1, \dots, x_N)) = 1.$$

General initial conditions

Theorem 6.17. For fixed integers $N \geq M \geq 0$, let $x = (x_1, \dots, x_N)$ be the ordered coordinates of TASEP particles at time t and let $y = (y_1, \dots, y_M)$ denote the ordered initial coordinates of the particles. In the case $y_M > N - M + 1$, the half-line open TASEP has transition probability from an empty state to x given by the following: when $N + M$ is even

$$\mathbb{P}_t(y \rightarrow x) = (-1)^{\binom{N}{2}} e^{-\alpha t} \text{Pf} \begin{bmatrix} Q & U \\ -U^T & 0 \end{bmatrix}, \quad (6.51)$$

when $N + M$ is odd

$$\mathbb{P}_t(y \rightarrow x) = (-1)^{\binom{N}{2}} e^{-\alpha t} \text{Pf} \begin{bmatrix} Q & p & U \\ -p^T & 0 & 0 \\ -U^T & 0 & 0 \end{bmatrix}. \quad (6.52)$$

Here, \mathbf{Q} is a skew-symmetric $N \times N$ -dimensional matrix and \mathbf{p} is a column vector of length N which are the same as those in Theorem 6.13, while \mathbf{U} is a $N \times M$ -dimensional matrix whose entries are given by

$$\begin{aligned} [\mathbf{Q}]_{i,j} &:= Q_{i,j}(x_{N-i+1}, x_{N-j+1}), \\ [\mathbf{p}]_i &:= p_i(x_{N-i+1}), \\ [\mathbf{U}]_{i,k} &:= U_{i-k}(x_{N-i+1} - y_{M-k+1}), \end{aligned}$$

for all $1 \leq i, j \leq N$ and $1 \leq k \leq M$. The functions $Q_{i,j}$ and p_i are defined by (6.49) and (6.50) respectively while we define the integral kernel

$$U_k(z) := \oint_{\gamma} \frac{dw}{2\pi i} \frac{(w-1)^{N-M-k} e^{t(w-1)}}{w^{z-k+N-M+1}}, \quad (6.53)$$

where the contour γ encloses the (possible) pole at the origin.

Remark 6.18. The restriction $y_M > N - M + 1$ on the initial conditions is technical in nature, and arises in the proof of the limit stated in Lemma 6.20 below. While this limit could, in principle, be computed without the restriction, the relatively simple Pfaffian structure obtained in the lemma appears not to persist beyond this case, so we do not attempt a generalisation here. On the other hand, and despite arising in a purely technical manner, the restriction to $y_M > N - M + 1$ admits a clear physical interpretation: it is precisely the assumption ensuring that

$$\mathbb{P}(|X_t| = N \mid X_0 = y) = \mathbb{P}(|X_t| = N - M \mid X_0 = \emptyset).$$

In other words, given M particles initially in the system at positions y , the probability that exactly $N - M$ additional particles enter the system is independent of the specific configuration y provided that it satisfies the constraint $y_M > N - M + 1$.

Before proving Theorem 6.17, let us show how it recovers Schütz's classical determinantal formula for the full-space TASEP.

Corollary 6.19 (Schütz's formula). For fixed integer N , let $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_N)$ be fixed configurations of the full-space TASEP. The transition probability for the full-space TASEP is given by the following determinantal expression

$$\lim_{\alpha \rightarrow 0} \mathbb{P}_t(y \rightarrow x) = \det [U_{i-j}(x_{N-i+1} - y_{N-j+1})]_{1 \leq i, j \leq N}. \quad (6.54)$$

The expression (6.54) exactly coincides with Schütz's determinantal expression for the full-space TASEP transition probability from [Sch97b].

Proof. Since $N = M$, we consider the even version of the transition probability (6.51) where

x, y are half-space coordinates satisfying the conditions of Theorem 6.17. Using Proposition A.1 observe that this Pfaffian reduces to a determinant, so that the transition probability evaluates as

$$\mathbb{P}_t(y \rightarrow x) = e^{-\alpha t} \det[U_{i-j}(x_{N-i+1} - y_{N-j+1})]_{1 \leq i, j \leq N}.$$

In particular, we note this expression is completely independent of the kernel $Q_{k,\ell}$ so that the determinant is independent of α . The result follows by setting $\alpha = 0$ and extending the coordinates x, y to become coordinates of the full-space TASEP by simultaneously shifting $x \mapsto x - (c, \dots, c)$ and $y \mapsto y - (c, \dots, c)$ for any chosen constant integer $c \geq 0$. \square

The key to the proof of Theorem 6.17 is the following result:

Lemma 6.20. Let (w_1, \dots, w_N) be an alphabet of length N and let $y = (y_1, \dots, y_M)$ be a half-space ASEP particle configuration of length $M \leq N$ satisfying $y_M > N - M + 1$. Then the function (6.12) has the following Pfaffian limit: when $N + M$ is even

$$\lim_{q \rightarrow 0} \mathcal{F}_y(w_1, \dots, w_N) = (-1)^{\binom{M}{2}} \alpha^{N-M} \prod_{1 \leq i < j \leq N} \frac{1 - w_i w_j}{w_i - w_j} \text{Pf} \begin{bmatrix} S(w_i, w_j)_{1 \leq i, j \leq N} & g_{k, y_k}(w_i)_{\substack{1 \leq i \leq N \\ 1 \leq k \leq M}} \\ -g_{k, y_k}(w_j)_{\substack{1 \leq k \leq M \\ 1 \leq j \leq N}} & 0 \end{bmatrix}, \quad (6.55)$$

and when $N + M$ is odd

$$\begin{aligned} \lim_{q \rightarrow 0} \mathcal{F}_y(w_1, \dots, w_N) &= (-1)^{\binom{M}{2}} \alpha^{N-M} \\ &\times \prod_{1 \leq i < j \leq N} \frac{1 - w_i w_j}{w_i - w_j} \text{Pf} \begin{bmatrix} S(w_i, w_j)_{1 \leq i, j \leq N} & \tilde{\mathbf{1}}_N & g_{k, y_k}(w_i)_{\substack{1 \leq i \leq N \\ 1 \leq k \leq M}} \\ -\tilde{\mathbf{1}}_N^T & 0 & 0 \\ -g_{k, y_k}(w_j)_{\substack{1 \leq k \leq M \\ 1 \leq j \leq N}} & 0 & 0 \end{bmatrix}. \end{aligned} \quad (6.56)$$

Here the skew-symmetric function S is the kernel of the Stembridge Pfaffian (see Lemma A.2)

$$S(v, w) = \frac{v - w}{1 - vw}$$

while the function $g_{k,y}$ for $y > N - i + 1$ is defined by

$$g_{k,y}(w) = \frac{(1 - \alpha + \alpha w) w^{N-k+1}}{(1 - w)^y}.$$

Proof. Recall the symmetrisation formula (6.12). Let us define the function $\tilde{\mathcal{F}}_y$ as the sym-

metrisation formula (6.12) where the overall constant V_{N-M} is replaced by

$$\begin{cases} \frac{1}{2}q^{\binom{N-M}{2}} & \text{if } N > M \\ 1 & \text{otherwise} \end{cases},$$

so that $\lim_{q \rightarrow 0} \tilde{\mathcal{F}}_y(w) = \lim_{q \rightarrow 0} \mathcal{F}_y(w)$ for all appropriate alphabets $w = (w_1, \dots, w_N)$. We will now consider the $q \rightarrow 0$ limit on the function $\tilde{\mathcal{F}}_y$ in the case $N > M$ only. The $N = M$ case follows by a very similar argument. For $N > M$, the symmetrisation formula becomes:

$$\tilde{\mathcal{F}}_y(w_1, \dots, w_N) := \frac{1}{2}q^{\binom{N-M}{2}}\alpha^{N-M} \sum_{\sigma \in \mathcal{B}_N} \sigma \left(\prod_{1 \leq i < j \leq N} \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \prod_{i=1}^M \varphi_{y_i}(w_i) \right). \quad (6.57)$$

We first separate this symmetrisation over signed permutations \mathcal{B}_N into a sum over the symmetric group S_N and a sum over signs. Observe that the factor

$$\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j}$$

in (6.57) is invariant under $w_j \rightarrow 1/qw_j$. By separating the product over i into $1 \leq i \leq M$ and $M+1 \leq i \leq N$ we may re-write (6.57) as

$$\begin{aligned} \tilde{\mathcal{F}}_y(w_1, \dots, w_N) &= \frac{1}{2}q^{\binom{N-M}{2}}\alpha^{N-M} \sum_{\{\epsilon_i = \pm 1\}} \sum_{\sigma \in S_N} \left(\prod_{i=M+1}^{N-1} \prod_{j=i+1}^N \left[\frac{w_{\epsilon_i \sigma_i} - qw_{\sigma_j}}{w_{\epsilon_i \sigma_i} - w_{\sigma_j}} \frac{1 - w_{\epsilon_i \sigma_i} w_{\sigma_j}}{1 - qw_{\epsilon_i \sigma_i} w_{\sigma_j}} \right] \times \right. \\ &\quad \left. \times \prod_{i=1}^M \left[\prod_{j=i+1}^N \left[\frac{w_{\epsilon_i \sigma_i} - qw_{\sigma_j}}{w_{\epsilon_i \sigma_i} - w_{\sigma_j}} \frac{1 - w_{\epsilon_i \sigma_i} w_{\sigma_j}}{1 - qw_{\epsilon_i \sigma_i} w_{\sigma_j}} \right] \varphi_{y_i}(w_{\epsilon_i \sigma_i}) \right] \right). \quad (6.58) \end{aligned}$$

The sum over the signs $\{\epsilon_i = \pm 1\}$ in (6.58) can now be performed independently for each i . Using $w_{-i} = 1/qw_i$ for $i = 1, \dots, N$, and absorbing the pre-factor $q^{\binom{N-M}{2}}$ we thus get

$$\begin{aligned} &\tilde{\mathcal{F}}_y(w_1, \dots, w_N) \\ &= \alpha^{N-M} \sum_{\sigma \in S_N} \sigma \left\{ \prod_{i=M+1}^{N-1} \left(q^{N-i} \prod_{j=i+1}^N \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] + \prod_{j=i+1}^N \left[\frac{qw_i - w_j}{w_i - w_j} \frac{1 - q^2 w_i w_j}{1 - qw_i w_j} \right] \right) \times \right. \\ &\quad \left. \times \prod_{i=1}^M \left(\prod_{j=i+1}^N \left[\frac{w_i - qw_j}{w_i - w_j} \frac{1 - w_i w_j}{1 - qw_i w_j} \right] \varphi_{y_i}(w_{\sigma_i}) + q^{y_i - N + i - 1} \prod_{j=i+1}^N \left[\frac{qw_i - w_j}{w_i - w_j} \frac{1 - q^2 w_i w_j}{1 - qw_i w_j} \right] \tilde{\varphi}_{y_i}(w_{\sigma_i}) \right) \right\}. \quad (6.59) \end{aligned}$$

where

$$\tilde{\varphi}_y(w) := q^{-y+1} \varphi_y \left(\frac{1}{qw} \right) = \frac{qw - a}{1 - qw^2} \frac{(1-q)w}{1 - qw} \left(\frac{1-w}{1 - qw} \right)^{y-1}.$$

The assumption $y_M > N - M + 1$ implies more generally that $y_i > N - i + 1$ for each $i = 1, \dots, N$ (because $y_1 > y_2 > \dots > y_M$), and under this condition the limit $q \rightarrow 0$ can be taken term-wise in

(6.59)⁹. By using the symmetrisation to simplify, we arrive at the following expression:

$$\lim_{q \rightarrow 0} \tilde{\mathcal{F}}_y(w_1, \dots, w_N) = \alpha^{N-M} \sum_{\sigma \in S_N} \sigma \left(\prod_{1 \leq i < j \leq N} \frac{1}{w_i - w_j} \prod_{i=1}^N w_i^{N-i} \prod_{i=1}^M \left[\frac{(1 - \alpha + \alpha w_i) w_i}{(1 - w_i)^{y_i}} \prod_{j=i+1}^N (1 - w_i w_j) \right] \right). \quad (6.60)$$

Overall factors can be removed from the symmetrisation over permutations (6.60), which yields:

$$\lim_{q \rightarrow 0} \tilde{\mathcal{F}}_y(w_1, \dots, w_N) = \alpha^{N-M} \prod_{1 \leq i < j \leq N} \frac{1 - w_i w_j}{w_i - w_j} \times \sum_{\sigma \in S_N} (-1)^{|\sigma|} \sigma \left(\prod_{M+1 \leq i < j \leq N} \frac{1}{1 - w_i w_j} \prod_{i=M+1}^N w_i^{N-i} \prod_{i=1}^M \frac{(1 - \alpha + \alpha w_i) w_i^{N-i+1}}{(1 - w_i)^{y_i}} \right). \quad (6.61)$$

From here, we may decompose the sum over the symmetric group using the following identity:

$$\begin{aligned} & \sum_{\sigma \in S_N} (-1)^{|\sigma|} f_1(w_{\sigma_1}, \dots, w_{\sigma_M}) f_2(w_{\sigma_{M+1}}, \dots, w_{\sigma_N}) \\ &= (-1)^{\binom{N+1}{2} + \binom{M+1}{2}} \sum_{\substack{T \subseteq \{1, \dots, N\} \\ |T| = N-M}} (-1)^{\sum_i T_i} \\ & \times \sum_{\rho \in S_M} (-1)^{|\rho|} f_1(w_{T_{\rho_1}^c}, \dots, w_{T_{\rho_M}^c}) \sum_{\sigma \in S_{N-M}} (-1)^{|\sigma|} f_2(w_{T_{\sigma_1}}, \dots, w_{T_{\sigma_{N-M}}}), \end{aligned} \quad (6.62)$$

for arbitrary appropriate functions f_1, f_2 , where T^c denotes the complement of $T \subseteq \{1, \dots, N\}$. And so, the symmetrisation (6.61) can be simplified using identity (6.62) to yield:

$$\begin{aligned} \lim_{q \rightarrow 0} \tilde{\mathcal{F}}_y(w_1, \dots, w_N) &= (-1)^{\binom{N+1}{2} + \binom{M+1}{2}} \alpha^{N-M} \prod_{1 \leq i < j \leq N} \frac{1 - w_i w_j}{w_i - w_j} \\ & \times \sum_{\substack{T \subseteq \{1, \dots, N\} \\ |T| = N-M}} (-1)^{\sum_i T_i} \left[\left(\sum_{\sigma \in S_{N-M}} (-1)^{|\sigma|} \prod_{1 \leq i < j \leq N-M} \frac{1}{1 - w_{T_{\sigma_i}} w_{T_{\sigma_j}}} \prod_{i=1}^{N-M} w_{T_{\sigma_i}}^{N-M-i} \right) \right. \\ & \left. \times \left(\sum_{\rho \in S_M} (-1)^{|\rho|} \prod_{i=1}^M \frac{(1 - \alpha + \alpha w_{T_{\rho_i}^c}) w_{T_{\rho_i}^c}^{N-i+1}}{(1 - w_{T_{\rho_i}^c})^{y_i}} \right) \right]. \end{aligned} \quad (6.63)$$

Now, for each fixed $T \subseteq \{1, \dots, N\}$, the sum over σ in (6.63) can be identified as

⁹The $q \rightarrow 0$ limit still exists when this restriction is not satisfied, however, the evaluation is more complicated.

$$\begin{aligned} & \sum_{\sigma \in S_{N-M}} (-1)^{|\sigma|} \prod_{1 \leq i < j \leq N-M} \frac{1}{1 - w_{T_{\sigma_i}} w_{T_{\sigma_j}}} \prod_{i=1}^{N-M} w_{T_{\sigma_i}}^{N-M-i} \\ &= \prod_{1 \leq i < j \leq N-M} \frac{w_{T_i} - w_{T_j}}{1 - w_{T_i} w_{T_j}} = \begin{cases} \text{Pf} [S(w_{T_i}, w_{T_j})]_{1 \leq i, j \leq N-M} & \text{if } N - M \text{ is even,} \\ \text{Pf} \begin{bmatrix} S(w_{T_i}, w_{T_j})_{1 \leq i, j \leq N-M} & \tilde{\mathbf{1}}_{N-M} \\ \tilde{\mathbf{1}}_{N-M}^T & 0 \end{bmatrix} & \text{if } N - M \text{ is odd,} \end{cases} \end{aligned}$$

where we have used the evaluation of the Vandermonde determinant and the Stembridge Pfaffian of Lemma A.2. The sum over ρ in (6.63) is then also identified with the following determinant

$$\sum_{\rho \in S_M} (-1)^{|\rho|} \prod_{i=1}^M \frac{(1 - \alpha + \alpha w_{T_{\rho_i}^c}) w_{T_{\rho_i}^c}^{N-i+1}}{(1 - w_{T_{\rho_i}^c})^{y_i}} = \det [g_{j,y_j}(w_{T_i^c})]_{1 \leq i, j \leq M}.$$

Using both of these identifications, both the odd and even cases of the right hand side of (6.63) may be simplified using Proposition A.1. The result for both even (6.55) and odd (6.56) $N + M$ follow once the following identity is applied

$$(-1)^{\binom{N+1}{2} + \binom{M+1}{2}} (-1)^{\binom{N}{2}} = (-1)^{\binom{M}{2} - \binom{N-M}{2}}. \quad \square$$

Proof of Theorem 6.17. We proceed by considering the $q = \gamma = 0$ case of the half-space ASEP transition probability which is obtained as the $q \rightarrow 0$ limit of the expression (6.13). The evaluation of the $q \rightarrow 0$ limit on $\mathcal{F}_y(w)$ follows from Lemma 6.20. For simplicity, we will only present the case where $N + M$ is even, where the odd case follows similarly. In the case where $N + M$ is even, the this limit is given by

$$\begin{aligned} \lim_{q \rightarrow 0} \mathbb{P}_t(y \rightarrow x) &= \alpha^N e^{-\alpha t} \oint_{\mathcal{D}_1} \frac{dw_1}{2\pi i} \dots \oint_{\mathcal{D}_N} \frac{dw_N}{2\pi i} \prod_{i=1}^N \left[\frac{(1 - w_i)^{x_i-1} \exp\left(\frac{w_i t}{1-w_i}\right)}{w_i^{N-i+1} \alpha - 1 - \alpha w_i} \right] \\ &\quad \times \text{Pf} \begin{bmatrix} S(w_i, w_j)_{1 \leq i, j \leq N} & g_{k,y_k}(w_i)_{\substack{1 \leq i \leq N \\ 1 \leq k \leq M}} \\ -g_{k,y_k}(w_j)_{\substack{1 \leq k \leq M \\ 1 \leq j \leq N}} & 0 \end{bmatrix}, \quad (6.64) \end{aligned}$$

where all contours \mathcal{D}_i satisfy the constraints of Definition 6.3. In particular, they all surround the point $w_i = 1$. It is important to note that, precisely when $q = 0$, we are able to deform all contours to be the same without crossing any singularities of the integrand. That is, for all i we deform $\mathcal{D}_i \mapsto \mathcal{D}$, for some appropriate contour \mathcal{D} which surrounds the point 1. Since all

integrations are over the same contour, we may symmetrise the integrand of (6.64) to obtain

$$\begin{aligned} \lim_{q \rightarrow 0} \mathbb{P}_t(y \rightarrow x) &= \frac{\alpha^{N-M} e^{-\alpha t}}{N!} (-1)^{\binom{M}{2}} \oint_{\mathcal{D}} \frac{dw_1}{2\pi i} \cdots \oint_{\mathcal{D}} \frac{dw_N}{2\pi i} \prod_{i=1}^N \frac{\exp\left(\frac{w_i t}{1-w_i}\right)}{\alpha - 1 - \alpha w_i} \\ &\quad \times \det \left[\frac{(1-w_i)^{x_j-1}}{w_i^{N-j+1}} \right]_{1 \leq i, j \leq N} \text{Pf} \begin{bmatrix} S(w_i, w_j)_{1 \leq i, j \leq N} & g_{k, y_k}(w_i)_{\substack{1 \leq i \leq N \\ 1 \leq k \leq M}} \\ -g_{k, y_k}(w_j)_{\substack{1 \leq k \leq M \\ 1 \leq j \leq N}} & 0 \end{bmatrix}, \end{aligned} \quad (6.65)$$

while the odd case follows similarly with an enlarged Pfaffian as in Lemma 6.20. We then make the change of integration variable $w_i \mapsto 1 - w_i^{-1}$ and suitably deform the contours.

We note here that whenever $y = \emptyset$ (or $M = 0$), for both odd and even N , the Pfaffian in (6.65) is reduced to $\prod_{1 \leq i < j \leq N} \frac{w_i - w_j}{1 - w_i w_j}$, which is equal to Stembridge's Pfaffian from Lemma A.2. Theorem 6.13 follows by appropriate manipulation after applying de Bruijn's integration formula from Lemma A.4.

Whenever $M > 0$, regardless of whether $N + M$ is even or odd, we apply generalised version of de Bruijn's integration formula of Lemma A.5 to (6.65). When $N + M$ is even, this yields the following single Pfaffian expression

$$\lim_{q \rightarrow 0} \mathbb{P}_t(y \rightarrow x) = e^{-\alpha t} \text{Pf} \begin{bmatrix} Q_{N-i+1, N-j+1}(x_i, x_j)_{1 \leq i, j \leq N} & U_{N-i+1-k}(x_i - y_{M-k+1})_{\substack{1 \leq i \leq N \\ 1 \leq k \leq M}} \\ -U_{N-j+1-k}(x_j - y_{M-k+1})_{\substack{1 \leq k \leq M \\ 1 \leq j \leq N}} & 0 \end{bmatrix}, \quad (6.66)$$

where the factor of α^{N-M} has been absorbed into the Pfaffian by simultaneously multiplying into the first N rows and columns and dividing the last M rows and columns by a factor of α . The Pfaffian expression (6.66) can be brought into the form (6.51) by simultaneously reversing the order of the first N rows and columns which generates an overall factor of $(-1)^{\binom{N}{2}}$. The case where $N + M$ is odd case follows analogously to yield (6.52). \square

6.5.2 Kernel recurrence relations

The Pfaffian kernel functions of Theorems 6.13 and 6.17 satisfy important recurrence relations. Analogous relations have been used to construct a determinantal point process from Schütz's transition probability for the full-space TASEP in [Sas05, BFPS07] and will be used to define a Pfaffian point process for the half-space TASEP in Section 7.1.

Lemma 6.21. The functions defined by (6.49), (6.50) and (6.53) satisfy the following recurrence

relations

$$\begin{aligned} Q_{i+1,j}(x_i, x_j) &= \sum_{y_i \geq x_i} Q_{i,j}(y_i, x_j), \\ Q_{i,j+1}(x_i, x_j) &= \sum_{y_j \geq x_j} Q_{i,j}(x_i, y_j), \\ U_{k+1}(z) &= \sum_{y \geq z} U_k(y), \\ p_{k+1}(z) &= \sum_{y \geq z} p_k(y). \end{aligned}$$

Proof. The argument follows a straightforward geometric series where uniform convergence is assured by deforming the contour radii of (6.49), (6.50) and (6.53) to be sufficiently large. \square

The recurrence relations also have a useful finite-summation version.

Lemma 6.22. The functions defined by (6.49), (6.50) and (6.53) satisfy the following recurrence relations

$$\begin{aligned} Q_{i+1,j}(s_i, x_j) - Q_{i+1,j}(x_i, x_j) &= \sum_{y_i=s_i}^{x_i-1} Q_{i,j}(y_i, x_j), \\ Q_{i,j+1}(x_i, s_j) - Q_{i,j+1}(x_i, x_j) &= \sum_{y_j=s_j}^{x_j-1} Q_{i,j}(x_i, y_j), \\ U_{k+1}(s) - U_{k+1}(z) &= \sum_{y=s}^{z-1} R_k(y), \\ p_{k+1}(s) - p_{k+1}(z) &= \sum_{y=s}^{z-1} p_k(y). \end{aligned}$$

6.5.3 Joint distribution

From Theorem 6.17, we can derive directly by summation a similar formula for the joint distribution of the particle currents in the half-space TASEP.

Proposition 6.23. Let $N \geq M \geq 0$ be fixed integers and consider integers $s_1 > s_2 > \dots > s_N \geq 1$ and $y_1 > y_2 > \dots > y_M$. Then the joint distribution of particle positions in the half-space TASEP, given initial condition $y = (y_1, \dots, y_M)$ satisfying $y_M > N - M + 1$, is given by the following Pfaffian formulas: when $N + M$ is even

$$\mathbb{P} \left(\bigcap_{k=1}^N \{X_t(k) \geq s_k\} \text{ and } |X_t| = N \mid X_0 = y \right) = (-1)^{\binom{N}{2}} e^{-\alpha t} \text{Pf} \begin{bmatrix} \tilde{Q} & \tilde{U} \\ -\tilde{U}^T & 0 \end{bmatrix}, \quad (6.67)$$

and when $N + M$ is odd

$$\mathbb{P}\left(\bigcap_{k=1}^N \{X_t(k) \geq s_k\} \text{ and } |X_t| = N \mid X_0 = y\right) = (-1)^{\binom{N}{2}} e^{-\alpha t} \text{Pf} \begin{bmatrix} \tilde{Q} & \tilde{p} & \tilde{U} \\ -\tilde{p}^T & 0 & 0 \\ -\tilde{U}^T & 0 & 0 \end{bmatrix}. \quad (6.68)$$

Here, \tilde{Q}, \tilde{U} are $N \times N$ and $N \times M$ -dimensional matrices respectively and \tilde{p} and a vector of length N whose entries are given by

$$\begin{aligned} [\tilde{Q}]_{i,j} &:= Q_{i+1,j+1}(s_{N-i+1}, s_{N-j+1}), \\ [\tilde{p}]_i &:= p_{i+1}(s_{N-i+1}), \\ [\tilde{U}]_{i,k} &:= U_{i-k+1}(s_{N-i+1} - y_{M-k+1}), \end{aligned} \quad (6.69)$$

for all $1 \leq i, j \leq N$ and $1 \leq k \leq M$ where the functions are given by (6.49), (6.50) and (6.53) respectively.

A formula of this type can be derived also for TASEP on the full-line, either by arguing similarly directly on (6.54), or by specialising the half-space formula to full-space as done in the proof of Corollary 6.19. The disadvantage of such a formula is that it appears to be ill-suited for asymptotic analysis (although for periodic TASEP the idea has been used fruitfully, see e.g. [BL18]).

In our setting, we are interested in the proposition mostly due to the following corollary, which computes the probability of observing exactly N particles in the half-space TASEP at time t (this is equivalent to observing the integrated particle current at the boundary). This yields the normalisation constant which will later appear in the conditional joint distribution in Section 7.2, and the explicit formula will be useful in the derivation there.

Corollary 6.24. Let $N \geq M \geq 0$ be a fixed integers and let $y = (y_1, \dots, y_M)$ be a fixed half-space configuration satisfying $y_M > N - M + 1$. Then the probability of the half-TASEP having exactly N particles, i.e.

$$\mathbb{P}(|X_t| = N \mid X_0 = y),$$

is given by the Pfaffian on the right hand sides of (6.67) when $N + M$ is even and (6.68) when $N + M$ is odd, where the matrix entries are given by (6.69) with $s_1 = \dots = s_N = 1$.

Proof of Theorem 6.23. The joint distribution may be calculated from the transition probability as

$$\mathbb{P}\left(\bigcap_{k=1}^N \{X_t(k) \geq s_k\} \text{ and } |X_t| = N \mid X_0 = y\right) = \sum_{x_1=s_1}^{\infty} \sum_{x_2=s_2}^{x_1-1} \cdots \sum_{x_N=s_N}^{x_{N-1}-1} \mathbb{P}_t(y \rightarrow x), \quad (6.70)$$

where the transition probabilities are given by the Pfaffian expressions of Theorem 6.13 when $y = \emptyset$ and Theorem 6.17 when y is non-empty. Note that all these sums are over non-empty ranges thanks to our assumption on the s_i 's. Let us consider the particular case of the Schütz-type Pfaffian transition probability where $N + M$ is odd (6.52) since the even case is simpler. We evaluate the sum over x_N in (6.70) first, where we note that the x_N -dependence only appears in the first row and column of the underlying matrix (6.52). The first row can be written as

$$\left[0 \quad Q_{1,2}(x_N, x_{N-1}) \quad \cdots \quad Q_{1,N}(x_N, x_1) \quad p_1(x_N) \quad U_0(x_N - y_M) \quad \cdots \quad U_{1-M}(x_N - y_1) \right],$$

where the sum over x_N can be evaluated on the individual rows and columns of the Pfaffian simultaneously using the multi-linearity of the Pfaffian. And so, the sum over x_N is evaluated readily using the recurrence relation of Lemma 6.22. This leads the the first row becoming

$$\begin{aligned} & \left[0 \quad Q_{2,2}(s_N, x_{N-1}) \quad \cdots \quad Q_{2,N}(s_N, x_1) \quad p_2(s_N) \quad U_1(s_N - y_M) \quad \cdots \quad U_{2-M}(s_N - y_1) \right] \\ - & \left[0 \quad Q_{2,2}(x_{N-1}, x_{N-1}) \quad \cdots \quad Q_{2,N}(x_{N-1}, x_1) \quad p_2(x_{N-1}) \quad U_1(x_{N-1} - y_M) \quad \cdots \quad U_{2-M}(x_{N-1} - y_1) \right], \end{aligned} \quad (6.71)$$

while the first column is modified simultaneously in the analogous way. Observe that, beyond the first entry, the second term in (6.71) is the equal to the second-row of the subsequent matrix after the x_N -summation has taken place. So, we apply the simultaneous row and column operations, $r_1 \mapsto r_1 + r_2$ and $c_1 \mapsto c_1 + c_2$, which do not affect the value of the Pfaffian. These operations leave the first row of the resulting matrix as the first term in (6.71).

We will now perform the subsequent summations over x_{N-1}, \dots, x_2 along the same lines. That is, for each $1 < k < N$ we evaluate the sum over x_{N-k+1} in (6.70) by applying the recurrences of Lemma 6.22 simultaneously to the k -th row and column of the matrix. This is then followed by the simultaneous row and column operations $r_k \mapsto r_k + r_{k+1}$ and $c_k \mapsto c_k + c_{k+1}$.

Finally, after performing this procedure for all in order for all k , we may perform the summation over x_1 in (6.70) using Lemma 6.21. This process results in the joint distribution given as the Pfaffian over the matrices (6.68) or (6.67) depending on whether $N + M$ is odd or even respectively. \square

Proof of Corollary 6.24. Setting $s_i = N + 1 - i$ for $1 \leq i \leq N$ in Theorem 6.23 we get immediately that $\mathbb{P}(|X_t| = N \mid X_0 = y)$ is given by the claimed formulas with that choice of s_i 's. Now from Lemma 6.22 we obtain the recursions $Q_{i,j}(i, j) = Q_{i,j}(i - 1, j) - Q_{i-1,j}(i - 1, j)$, $Q_{i,j}(i, j) = Q_{i,j}(i, j - 1) - Q_{i,j-1}(i, j - 1)$, $U_i(s) = U_i(s - 1) - U_{i-1}(s - 1)$ and $p_i(s) = p_i(s - 1) - p_{i-1}(s - 1)$, so applying repeatedly row and column operations to the Pfaffians in the formulas, in a similar

way as done in the previous proof, we can successively lower the values of the s_i 's to obtain the same formula evaluated at $s_1 = \cdots = s_N = 1$; we omit the details. \square

Chapter 7

Pfaffian Point Processes from TASEP in Half-space

7.1 Pfaffian point processes

7.1.1 Overview

Here we provide an overview of the essential definitions and properties of Pfaffian point processes.

Definition 7.1 (Pfaffian point process). Let \mathcal{Z} be a set of points and let $\mu : 2^{\mathcal{Z}} \rightarrow \mathbb{C}$ be a measure on the powerset $2^{\mathcal{Z}}$. This measure is called a *Pfaffian point process* if there exists a matrix valued kernel $K : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathbb{C}^{2 \times 2}$ which is skew-symmetric, i.e. $K_{ij}(x, y) = -K_{ji}(y, x)$, given by

$$K(x, y) = \begin{bmatrix} K_{11}(x, y) & K_{12}(x, y) \\ K_{21}(x, y) & K_{22}(x, y) \end{bmatrix},$$

such that correlation functions $\rho(z_1, \dots, z_m) := \mu(\{A \in 2^{\mathcal{Z}} : z_1, \dots, z_m \in A\})$ have the form

$$\rho(z_1, \dots, z_m) = \text{Pf} \begin{bmatrix} K_{11}(z_i, z_j) & K_{12}(z_i, z_j) \\ K_{21}(z_i, z_j) & K_{22}(z_i, z_j) \end{bmatrix}_{1 \leq i, j \leq m}.$$

for all finite subsets $\{z_1, \dots, z_m\} \subseteq \mathcal{Z}$. The kernel K is known as the *correlation kernel* of the process.

If μ is a Pfaffian point process on \mathcal{Z} and $\mathcal{B} \subseteq \mathcal{Z}$ then the gap probability corresponding to

This chapter is based on the preprint [P2]. Minor modifications have been made for overall coherence and so that the thesis references its other chapters where relevant.

seeing no points in \mathcal{B} is given as a Fredholm Pfaffian:

$$\sum_{X \in \mathcal{Z} \setminus \mathcal{B}} \mu(X) = \text{Pf}(J - K)_{\ell^2(\mathcal{B})}. \quad (7.1)$$

This follows e.g. from Theorem 8.2 in [Rai00]. The Fredholm Pfaffian of a 2×2 matrix kernel K acting on a space $L^2(\mathcal{X}, \lambda)$ can be defined through its series expansion

$$\text{Pf}(J + K)_{L^2(\mathcal{X})} = 1 + \sum_{n \geq 1} \frac{1}{n!} \int_{\mathcal{X}^n} d\lambda^{\otimes n}(x_1, \dots, x_n) \text{Pf}[K(x_i, x_j)]_{i,j=1}^n;$$

see Section 8 of [Rai00] or Appendix B of [OQR17] for more details on and properties of Fredholm Pfaffians. The kernel J here is defined as

$$J(x, y) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \delta_{x,y},$$

for $x, y \in \mathcal{Z}$.

Definition 7.2 (Pfaffian L -ensemble). Let \mathcal{Z} be a set of points, and let $L : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathbb{C}^{2 \times 2}$ be a skew-symmetric 2×2 -matrix valued function. A measure μ on $2^{\mathcal{Z}}$ is a *Pfaffian L -ensemble* on \mathcal{Z} if its correlation function ρ is given by

$$\rho(X) = \frac{\text{Pf } L_X}{\text{Pf}(J + L)}, \quad X \subset \mathcal{Z}. \quad (7.2)$$

Here L_X denotes the $(2|X|) \times (2|X|)$ matrix $[L(x, x')]_{x, x' \in X}$, while $\text{Pf}(J + L)$ denotes the Fredholm Pfaffian of L on $\ell^2(\mathcal{Z})$ (which coincides with the Pfaffian of the matrix $J + L$ when \mathcal{Z} is finite).

Definition 7.3 (Conditional Pfaffian L -ensemble). Let \mathcal{Z} be a set of points with subset $\mathcal{D} \subseteq \mathcal{Z}$ and complement $\mathcal{D}^c = \mathcal{Z} \setminus \mathcal{D}$, and let $L : \mathcal{Z} \times \mathcal{Z} \rightarrow \mathbb{C}^{2 \times 2}$ be a skew-symmetric 2×2 -matrix valued function. A measure μ on the powerset of the subset $2^{\mathcal{D}}$ is a *conditional Pfaffian L -ensemble* on \mathcal{D} within \mathcal{Z} if its correlation function ρ is given by

$$\rho(Y) = \frac{\text{Pf } L_{Y \cup \mathcal{D}^c}}{\text{Pf}(J_{\mathcal{D}} + L)}, \quad (7.3)$$

where $J_{\mathcal{D}}$ denotes the kernel J restricted to \mathcal{D} , i.e.

$$J_{\mathcal{D}}(x, y) = J(x, y) \mathbb{1}_{x, y \in \mathcal{D}}.$$

Note that the conditional Pfaffian L -ensemble (7.3) reduces to the regular Pfaffian L -ensemble (7.2) when the chosen subset is the whole set of points $\mathcal{D} = \mathcal{Z}$.

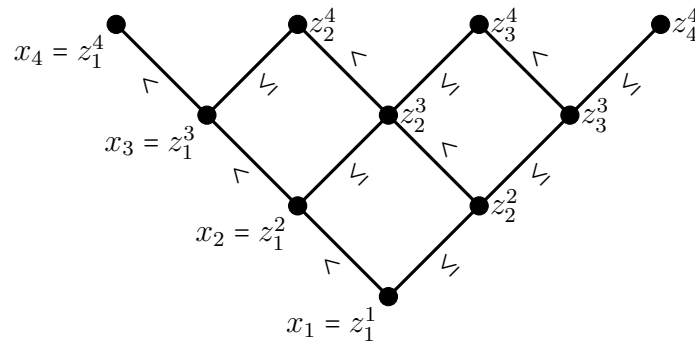


Figure 7.1: The Gelfand–Tsetlin pattern $\text{GT}_4(x)$ with left-edges fixed by $x = (x_1, x_2, x_3, x_4)$.

The following proposition is due to Borodin and Rains, Proposition 1.7 in [BR05].

Proposition 7.4. Let \mathcal{Z} be a finite set of points with subset $\mathcal{D} \subseteq \mathcal{Z}$ and let μ be a conditional Pfaffian L -ensemble on \mathcal{D} within \mathbb{Z} as in Definition 7.3. Then μ is a Pfaffian point process on \mathcal{D} whose correlation kernel is given by

$$K = J_{\mathcal{D}} + (J_{\mathcal{D}} + L)^{-1} \Big|_{\mathcal{D} \times \mathcal{D}},$$

where the second term denotes the inverse of the matrix $J_{\mathcal{D}} + L$ indexed by $\mathcal{Z} \times \mathcal{Z}$ restricted to $\mathcal{D} \times \mathcal{D}$.

7.1.2 Half-space TASEP measure

We will now proceed to develop the Schütz-type Pfaffian formulas for half-space TASEP from Section 6.5.1 into a Pfaffian point process. Theorems 6.13 and 6.17 give the transition probability for both odd and even values of $N + M$. However, for simplicity, we will restrict our analysis to the even case henceforth.

Let us denote the set of all *integer valued triangular arrays* of size N as

$$\Lambda_N := \{z = (z_i^k) \in \mathbb{Z}^{N(N+1)/2}\}, \tag{7.4}$$

where an individual entry on the triangular array is denoted by the coordinate $z_i^k \in \mathbb{Z}$ for $1 \leq i \leq k \leq N$. We must also define a more restricted subset of Λ_N : the *Gelfand–Tsetlin patterns* (sometimes spelled Gelfand–Zetlin or Gelfand–Cetlin) which we define as

$$\text{GT}_N := \{z = (z_i^k) \in \mathbb{Z}^{N(N+1)/2} : z_i^{k+1} < z_i^k \leq z_{i+1}^{k+1}, 1 \leq i \leq k < N\}.$$

We also let, for $x = (x_1, \dots, x_N)$ with $x_1 > x_2 > \dots > x_N$,

$$\Lambda_N(x) := \{z \in \Lambda_N : z_1^k = x_k\}, \quad \text{GT}_N(x) := \{z \in \text{GT}_N : z_1^k = x_k\},$$

which are the spaces of triangular arrays and Gelfand–Tsetlin patterns whose left-edges are fixed by the coordinates of x . An example of the Gelfand–Tsetlin pattern $\text{GT}_N(x)$ appears in Figure 7.1.

To simplify the exposition, and in particular the presentation of the proofs, we treat separately the cases of empty and general initial conditions throughout most of the remainder of the paper.

Proposition 7.5 (Triangular array marginal for empty initial conditions). Let N be an even integer and let $x = (x_1, \dots, x_N)$ be the ordered coordinates of TASEP particles. Then the half-line open TASEP transition probability from an empty state to x can be written as a sum over Pfaffian kernels indexed by Gelfand–Tsetlin patterns with endpoints fixed by $x = (x_1, \dots, x_N)$:

$$\mathbb{P}_t(\emptyset \rightarrow x) = (-1)^{\binom{N}{2}} e^{-\alpha t} \sum_{z \in \text{GT}_N(x)} \text{Pf} [\Psi(z_i^N, z_j^N)]_{1 \leq i, j \leq N}, \tag{7.5}$$

where we have defined

$$\Psi(x, y) := Q_{1,1}(x, y) = \alpha^2 \oint_{\beta} \frac{du}{2\pi i} \oint_{\beta} \frac{dw}{2\pi i} \frac{u-w}{1-u-w} \frac{w^{1-x} e^{t(w-1)}}{(w-\alpha)(w-1)} \frac{u^{1-y} e^{t(u-1)}}{(u-\alpha)(u-1)}, \tag{7.6}$$

where contour β , as in Definition 6.12, surrounds poles at $u, w = 1, 0, \alpha, 1 - \alpha$ and omits all other singularities of the integrand. The kernel $\Psi(x, y)$ is skew-symmetric under the interchange of x and y when $x, y \geq 1$.

The proof of this proposition will use the following result, which appears in [BFPS07].

Lemma 7.6. Let f be an anti-symmetric function in N variables and let $x = (x_1, \dots, x_N)$ where $x_1 > \dots > x_N$ are integers. We have the following identity of sums over triangular arrays:

$$\sum_{\substack{z \in \Lambda_N(x) \\ z_i^{k+1} \geq z_i^k}} f(z_1^N, \dots, z_N^N) = \sum_{z \in \text{GT}_N(x)} f(z_1^N, \dots, z_N^N).$$

Proof of Proposition 7.5. The result will follow from an expansion of the Pfaffian from (6.48) where we identify the coordinates $(x_1, \dots, x_N) = (z_1^1, \dots, z_1^N)$:

$$\text{Pf} [Q_{i,j}(z_1^{N-i+1}, z_1^{N-j+1})]$$

$$= \text{Pf} \begin{pmatrix} 0 & Q_{1,2}(z_1^N, z_1^{N-1}) & \cdots & Q_{1,N-1}(z_1^N, z_1^2) & Q_{1,N}(z_1^N, z_1^1) \\ Q_{2,1}(z_1^{N-1}, z_1^N) & 0 & \cdots & Q_{2,N-1}(z_1^{N-1}, z_1^2) & Q_{2,N}(z_1^{N-1}, z_1^1) \\ \vdots & & \ddots & & \vdots \\ Q_{N-1,1}(z_1^2, z_1^N) & Q_{N-1,2}(z_1^2, z_1^{N-1}) & \cdots & 0 & Q_{N-1,N}(z_1^2, z_1^1) \\ Q_{N,1}(z_1^1, z_1^N) & Q_{N,2}(z_1^1, z_1^{N-1}) & \cdots & Q_{N,N-1}(z_1^1, z_1^2) & 0 \end{pmatrix}. \quad (7.7)$$

Recall the property whereby Pfaffians are linear in expanding along rows and columns simultaneously. We may use the recurrence relations of Lemma 6.21 in the last row and column of (7.7). Expansion along this row and column yields

$$\begin{aligned} & \text{Pf}[Q_{i,j}(z_1^{N-i+1}, z_1^{N-j+1})] \\ &= \sum_{z_2^2 \geq z_1^1} \text{Pf} \begin{pmatrix} 0 & Q_{1,2}(z_1^N, z_1^{N-1}) & \cdots & Q_{1,N-1}(z_1^N, z_1^2) & Q_{1,N-1}(z_1^N, z_2^2) \\ Q_{2,1}(z_1^{N-1}, z_1^N) & 0 & \cdots & Q_{2,N-1}(z_1^{N-1}, z_1^2) & Q_{2,N-1}(z_1^{N-1}, z_2^2) \\ \vdots & & \ddots & & \vdots \\ Q_{N-1,1}(z_1^2, z_1^N) & Q_{N-1,2}(z_1^2, z_1^{N-1}) & \cdots & 0 & Q_{N-1,N-1}(z_1^2, z_2^2) \\ Q_{N-1,1}(z_2^2, z_1^N) & Q_{N-1,2}(z_2^2, z_1^{N-1}) & \cdots & Q_{N-1,N-1}(z_2^2, z_1^2) & 0 \end{pmatrix}. \end{aligned}$$

This step can be repeated so that it occurs $N - 1$ times in total:

$$\begin{aligned} & \text{Pf}[Q_{i,j}(z_1^{N-i+1}, z_1^{N-j+1})] \\ &= \sum_{z_N^N \geq \cdots \geq z_2^2 \geq z_1^1} \text{Pf} \begin{pmatrix} 0 & Q_{1,2}(z_1^N, z_1^{N-1}) & \cdots & Q_{1,N-1}(z_1^N, z_1^2) & Q_{1,1}(z_1^N, z_N^N) \\ Q_{2,1}(z_1^{N-1}, z_1^N) & 0 & \cdots & Q_{2,N-1}(z_1^{N-1}, z_1^2) & Q_{2,1}(z_1^{N-1}, z_N^N) \\ \vdots & & \ddots & & \vdots \\ Q_{N-1,1}(z_1^2, z_1^N) & Q_{N-1,2}(z_1^2, z_1^{N-1}) & \cdots & 0 & Q_{N-1,1}(z_1^2, z_N^N) \\ Q_{1,1}(z_N^N, z_1^N) & Q_{1,2}(z_N^N, z_1^{N-1}) & \cdots & Q_{1,N-1}(z_N^N, z_1^2) & 0 \end{pmatrix}. \end{aligned}$$

A similar procedure can then be applied to each row and column so that the j -th row has the recurrence relations of Lemma 6.21 applied $j - 1$ times. This yields

$$\begin{aligned} \text{Pf}[Q_{i,j}(z_1^{N-i+1}, z_1^{N-j+1})] &= \sum_{z_N^N \geq \cdots \geq z_2^2 \geq z_1^1} \sum_{z_{N-1}^N \geq \cdots \geq z_2^3 \geq z_1^2} \cdots \sum_{z_2^N \geq z_1^{N-1}} \\ & \times \text{Pf} \begin{pmatrix} 0 & Q_{1,1}(z_1^N, z_2^N) & \cdots & Q_{1,1}(z_1^N, z_{N-1}^N) & Q_{1,1}(z_1^N, z_N^N) \\ Q_{1,1}(z_2^N, z_1^N) & 0 & \cdots & Q_{1,1}(z_2^N, z_{N-1}^N) & Q_{1,1}(z_2^N, z_N^N) \\ \vdots & & \ddots & & \vdots \\ Q_{1,1}(z_{N-1}^N, z_1^N) & Q_{1,1}(z_{N-1}^N, z_2^N) & \cdots & 0 & Q_{1,1}(z_{N-1}^N, z_N^N) \\ Q_{1,1}(z_N^N, z_1^N) & Q_{1,1}(z_N^N, z_2^N) & \cdots & Q_{1,1}(z_N^N, z_{N-1}^N) & 0 \end{pmatrix}, \end{aligned}$$

which can be identified as a sum over integer valued triangular arrays with endpoints fixed by

the coordinates $x_1 > \cdots > x_N$:

$$\text{Pf} \left[Q_{i,j}(z_1^{N-i+1}, z_1^{N-j+1}) \right]_{1 \leq i, j \leq N} = \sum_{\substack{z \in \Lambda_N \\ z_i^{k+1} \geq z_i^k, z_1^j = x_j}} \text{Pf} \left[\Psi(z_i^N, z_j^N) \right]_{1 \leq i, j \leq N}. \quad (7.8)$$

Recall the property whereby a Pfaffian is anti-symmetric under the simultaneous interchange of two rows and columns. Therefore the function defined by

$$\text{Pf} \left[\Psi(z_i^N, z_j^N) \right]_{1 \leq i, j \leq N}$$

is an anti-symmetric function under the permutation of the alphabet z_1^N, \dots, z_N^N . Then, using Lemma 7.6, the sum on the right hand side of (7.8) can be restricted to be of Gelfand–Tsetlin patterns which completes the argument. \square

Proposition 7.7 (Triangular array marginal for general initial conditions). Let $0 \leq M \leq N$ be integers such that $N + M$ is even and let $x = (x_1, \dots, x_N)$ and $y = (y_1, \dots, y_M)$ be ordered coordinates of TASEP particles satisfying $y_M > N - M + 1$. Then the half-line open TASEP transition probability from y to x can be written as a sum over Pfaffian kernels indexed by Gelfand–Tsetlin patterns with endpoints fixed by $x = (x_1, \dots, x_N)$:

$$\mathbb{P}_t(y \rightarrow x) = (-1)^{\binom{N}{2}} e^{-\alpha t} \sum_{z \in \text{GT}_N(x)} \text{Pf} \begin{bmatrix} \Psi(z_i^N, z_j^N)_{1 \leq i, j \leq N} & \Xi_{N-k}(z_i^N)_{\substack{1 \leq i \leq N \\ 1 \leq k \leq M}} \\ -\Xi_{N-k}(z_j^N)_{\substack{1 \leq k \leq M \\ 1 \leq j \leq N}} & 0 \end{bmatrix}, \quad (7.9)$$

where we define for each $1 \leq k \leq M$ the function¹⁰:

$$\Xi_{N-k}(z) = (-1)^k U_{k-M}(z - y_k) = (-1)^k \oint_{\gamma} \frac{dw}{2\pi i} \frac{(w-1)^{N-k} e^{t(w-1)}}{w^{z-y_k+N-k+1}}, \quad (7.10)$$

where the contour γ encloses the origin.

Proof. The proof follows a similar argument to the proof of Proposition 7.5 while using the recurrence relations of Lemma 6.21 for the kernel U_k . This leads to a sum over Gelfand–Tsetlin patterns whose terms are given by

$$(-1)^{\binom{N}{2}} e^{-\alpha t} \begin{bmatrix} \Psi(z_i^N, z_j^N)_{1 \leq i, j \leq N} & U_{1-k}(z_i^N - y_{M-k+1})_{\substack{1 \leq i \leq N \\ 1 \leq k \leq M}} \\ -U_{1-k}(z_j^N - y_{M-k+1})_{\substack{1 \leq k \leq M \\ 1 \leq j \leq N}} & 0 \end{bmatrix}. \quad (7.11)$$

Now, we may reverse the direction of last M rows and columns so that an element of the

¹⁰The choice of indexing $(\Xi_{N-k})_{k=1, \dots, M}$ is made to better match notation in the literature, see Remark 7.9 below.

upper-right block of (7.11) has elements

$$U_{k-M}(z_i^N - y_k),$$

whose rows are represented by $1 \leq i \leq N$ and columns by $1 \leq k \leq M$. The process of performing these swaps contributes an overall factor of $(-1)^{\binom{M}{2}}$ to the Pfaffian expression. Following this, we multiply each of the last M rows and columns by $(-1)^k$ for each $1 \leq k \leq M$. This multiplication then exactly cancels the overall sign from which the result follows. \square

The goal of the rest of this section is to show that the Pfaffians appearing as summands in Propositions 7.5 and 7.7 constitute Pfaffian point processes supported on Gelfand–Tsetlin patterns, whose correlation kernels can be computed thanks to Proposition 7.4. We now proceed in deriving them, following the analogous derivation for TASEP on the full line [BFPS07], adjusting it to the Pfaffian case using the framework developed in [BR05] and [Rai00].

We are thinking of the summand $\text{Pf}[\Psi(z_i^N, z_j^N)]_{1 \leq i, j \leq N}$ in (7.5) and the analogous one in (7.9) as measures on Gelfand–Tsetlin patterns \mathbf{z} . It will actually be more convenient to extend them to measures over the whole set of integer valued triangular arrays (7.4), by simply assigning zero weight to configurations outside GT_N . To this end we employ an identity due to [BFPS07], for any triangular array $\mathbf{z} \in \Lambda_N$ satisfying $z_1^{k+1} < z_i^k$ for $k = 1, \dots, N-1$, one has

$$\mathbb{1}_{\mathbf{z} \in \text{GT}_N} = \prod_{k=2}^N \det \left[\mathbb{1}_{z_i^{k-1} > z_j^k} \right]_{1 \leq i, j \leq k}, \quad (7.12)$$

where z_2^1, \dots, z_N^{N-1} play the role of *virtual coordinates* (added due to technical considerations) which should be thought of as taking the value ∞ , so that $\mathbb{1}_{z_k^{k-1} > y} = 1$ for any $y \in \mathbb{Z}$. We will also add an additional virtual coordinate z_1^0 which does not appear in (7.12) so that there are N in total; doing that, the product on the right hand side can be extended up to $k = 1$ straightforwardly. To easily distinguish these virtual coordinates from the coordinates on the triangular array, we will henceforth denote them by $\dagger_1, \dots, \dagger_N$ (so that \dagger_k plays the role of z_k^{k-1}).

Note that the last row in each determinant in (7.12) has all 1's. Subtracting it from each of the other rows flips the inequalities in each indicator function and (after accounting for the minus signs which result from this) leads to the following: letting¹¹

$$\phi_k(x, y) = \begin{cases} \mathbb{1}_{x \leq y} & \text{if } x \in \mathbb{Z}, \\ 1 & \text{if } x = \dagger_k, \end{cases}$$

¹¹Note that the only difference between the different ϕ_k 's is that they are evaluated at different virtual variables. We could well have introduced a single function ϕ , but keeping the dependence on k makes some of the coming computations a bit more transparent.

for $k = 1, \dots, N$ and for $x \in \mathbb{Z} \cup \{\dagger_k\}$ and $y \in \mathbb{Z}$ (\dagger_k should now be thought of as taking the value 0), and assuming $z_1^{k+1} < z_1^k$ for $k = 1, \dots, N-1$, we have

$$\mathbb{1}_{\mathbf{z} \in \text{GT}_N} = (-1)^{\binom{N}{2}} \prod_{k=1}^N \det[\phi_k(z_i^{k-1}, z_j^k)]_{1 \leq i, j \leq k}, \quad (7.13)$$

with $z_k^{k-1} = \dagger_k$ for $k = 1, \dots, N$. This version of the identity will turn out to be more convenient for us than (7.12) (see Remark 7.14).

Definition 7.8. Given integers $N \geq 1$ and $M \geq 0$ such that $N + M$ is even, and given an ordered set of coordinates $y = (y_1, \dots, y_M)$ satisfying $y_M > N - M + 1$ ($y = \emptyset$ if $M = 0$), we define a measure on Λ_N corresponding to half-space TASEP with initial condition y as follows:

$$\mathcal{W}_{N,M}(\mathbf{z}|y) = \prod_{k=1}^N \det \begin{bmatrix} \phi_k(z_i^{k-1}, z_j^k)_{\substack{1 \leq i \leq k-1 \\ 1 \leq j \leq k}} \\ \phi_k(\dagger_k, z_j^k)_{1 \leq j \leq k} \end{bmatrix} \text{Pf} \begin{bmatrix} \Psi(z_i^N, z_j^N)_{\substack{1 \leq i, j \leq N}} & \Xi_{N-k}(z_i^N)_{\substack{1 \leq i \leq N \\ 1 \leq k \leq M}} \\ -\Xi_{N-k}(z_j^N)_{\substack{1 \leq k \leq M \\ 1 \leq j \leq N}} & 0 \end{bmatrix}. \quad (7.14)$$

Due to (7.13), the restriction of this measure to $\Lambda_N(x)$, with $x_1 > x_2 > \dots > x_N$, is supported on Gelfand–Tsetlin patterns.

Note that when $M = 0$, which corresponds to half-space TASEP with empty initial condition, only the block $\Psi(z_i^N, z_j^N)_{1 \leq i, j \leq N}$ remains in the Pfaffian on the right hand side of (7.14).

Remark 7.9. Just as in Corollary 6.19, the case $N = M$ recovers the known results for TASEP on the full line. In fact, the Pfaffian on the right hand side of (7.14) becomes $\det[\Xi_{N-k}(z_i^N; t)]_{\substack{1 \leq i \leq N \\ 1 \leq k \leq M}}$ thanks to (A.2), while the Ξ_{N-k} 's in this case become, modulo a sign change, the functions Ψ_{N-k}^N appearing in [BFPS07]. In that setting, the family $(\Psi_k^N)_{k=0, \dots, N-1}$ can be thought of follows: Ψ_k^N is the Charlier orthogonal polynomial of degree k , multiplied by the Charlier (i.e. Poisson) weight, and shifted by y_{N-k} . In our case (for general N), the family $(\Xi_k)_{k=N-M, \dots, N-1}$ can be interpreted similarly: Ξ_k is the Charlier orthogonal polynomial of degree $N - M + k$, multiplied by the Charlier weight, and shifted by y_{M-k} .

Using Propositions 7.5 and 7.7, the transition probability of the half-space TASEP can be recovered as a marginal of the measure from Definition 7.8 as:

$$\mathbb{P}_t(y \rightarrow x) = e^{-\alpha t} \sum_{\mathbf{z} \in \Lambda_N(x)} \mathcal{W}_{N,M}(\mathbf{z}|y), \quad (7.15)$$

for x of size $N \geq 1$ (with $x_1 > x_2 > \dots > x_N \geq 1$) and y of size $M \geq 0$, and with $N + M$ being even.

7.1.3 Half-space TASEP as a Pfaffian point process

In this section, the measures defined in Definition 7.8 are shown to be conditional Pfaffian L -ensembles (see Definition 7.3). From this, using Proposition 7.4, we define a Pfaffian point process. We will first present the case of empty initial conditions using (7.14) with $y = \emptyset$ (or $M = 0$) before presenting the case of more general initial conditions.

L -ensemble for empty initial conditions

For fixed N , consider the spaces

$$\mathcal{X} = \{1, \dots, N\} \times \mathbb{N}, \quad \mathcal{V} = \{\dagger_1, \dots, \dagger_N\}, \quad \text{and} \quad \tilde{\mathcal{X}} = \mathcal{V} \cup \mathcal{X}. \quad (7.16)$$

We think of $\tilde{\mathcal{X}}$ as the state space of a point process; the \dagger_i 's correspond to the virtual coordinates, while (i, x) correspond to physical variables, representing a point with label i at x . We will refer to a point $(i, x) \in \mathcal{X}$ as belonging to the i -th fiber of \mathcal{X} .

We may identify a collection of points $A \subset \mathcal{X}$ as a triangular array, $A \in \Lambda_N$, if A has exactly i elements in its i -th fiber for all $1 \leq i \leq N$. In this way, we may (and will) identify Λ_N as a subset of \mathcal{X} . For example, the collection $A = \{(1, 2), (2, 2), (2, 3)\}$ is in Λ_2 .

Theorem 7.10. Let N be a fixed even integer. Let $L : \tilde{\mathcal{X}} \times \tilde{\mathcal{X}} \rightarrow \mathbb{C}^{2 \times 2}$ be defined as follows:

$$\begin{aligned} L(\dagger_i, \dagger_j) &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbb{1}_{i+1=j} + \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \mathbb{1}_{j+1=i}, \\ L(\dagger_i, (j, x_2)) &= \begin{bmatrix} 0 & 0 \\ \phi_i(\dagger_i, x_2) & 0 \end{bmatrix} \mathbb{1}_{i=j}, \\ L((i, x_1), \dagger_j) &= -L(\dagger_j, (i, x_1))^T, \\ L((i, x_1), (j, x_2)) &= \begin{bmatrix} 0 & 0 \\ \phi_i(x_1, x_2) & 0 \end{bmatrix} \mathbb{1}_{i+1=j < N} + \begin{bmatrix} 0 & \phi_j(x_2, x_1) \\ 0 & 0 \end{bmatrix} \mathbb{1}_{j+1=i < N} \\ &\quad + \begin{bmatrix} 0 & 0 \\ 0 & \Psi(x_1, x_2) \end{bmatrix} \mathbb{1}_{i=j=N}, \end{aligned} \quad (7.17)$$

for $1 \leq i, j \leq N$ and $x_1, x_2 \in \mathbb{N}$. Then for $A \subset \mathcal{X}$ we have

$$\text{Pf}[L|_{\mathcal{V} \cup A}] = \begin{cases} \mathcal{W}_{N,0}(A|\emptyset) & \text{if } A \in \Lambda_N, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first part of the proof will consist of showing that for $A \in \Lambda_N$, the measure

$\mathcal{W}_{N,0}(A|\emptyset)$ defined in (7.14) can be written as the Pfaffian of the matrix $L|_{\mathcal{V} \cup A}$.

The measure $\mathcal{W}_{N,0}(\cdot|\emptyset)$ is defined over triangular arrays $\mathbf{z} = (z_i^k)_{1 \leq i \leq k} \in \Lambda_N$. For $1 \leq k \leq N$ let $\mathcal{Z}^{(k)}$ denote the set of points corresponding to the k -th row of the triangular array, i.e. $\{z_1^k, \dots, z_k^k\}$, and set $\mathcal{Z}^{(0)} = \emptyset$. Let also $(\mathcal{Z}^{(k)})'$ and $(\mathcal{Z}^{(k)})''$ denote identical copies of $\mathcal{Z}^{(k)}$ (this is just a notational device to help keep track of the two rows and columns associated to the 2×2 matrix entries of L). For $k = 0, \dots, N-1$ we introduce kernels V_k on $((\mathcal{Z}^{(k)})'' \cup \{\dagger_{k+1}\}) \times (\mathcal{Z}^{(k+1)})'$ defined by

$$V_k((z_i^k)'', (z_j^{k+1})') = \phi_{k+1}(z_i^k, z_j^{k+1}), \quad V_k(\dagger_{k+1}, (z_j^{k+1})') = \phi_{k+1}(\dagger_{k+1}, z_j^{k+1}).$$

We also regard Ψ as being defined on $(\mathcal{Z}^{(N)})'' \times (\mathcal{Z}^{(N)})''$.

Using the Pfaffian identity (A.2), we may write the measure (7.14) as

$$\mathcal{W}_{N,0}(\mathbf{z}|\emptyset) = \text{Pf} \begin{bmatrix} 0 & V_0 & \cdots & 0 & 0 & 0 \\ -V_0^T & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & V_{N-1} & 0 \\ 0 & 0 & \cdots & -V_{N-1}^T & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \Psi \end{bmatrix} (\tilde{\mathbf{z}}, \tilde{\mathbf{z}}), \quad (7.18)$$

where the rows and the columns of the matrix are indexed by

$$\{\dagger_1\} \cup (\mathcal{Z}^{(1)})' \cup ((\mathcal{Z}^{(1)})'' \cup \{\dagger_2\}) \cup (\mathcal{Z}^{(2)})' \cup \dots \cup ((\mathcal{Z}^{(N-1)})'' \cup \{\dagger_N\}) \cup (\mathcal{Z}^{(N)})' \cup (\mathcal{Z}^{(N)})''$$

and where $\tilde{\mathbf{z}}$ denotes a version of \mathbf{z} with duplicated $\mathcal{Z}^{(k)}$ variables for $k = 1, \dots, N$ and with the virtual variables added, all in the order specified by the above space. The kernel V_k can be split into its physical and virtual coordinate dependence as

$$V_k = \begin{bmatrix} W_k \\ e_k \end{bmatrix},$$

with W_k a kernel on $(\mathcal{Z}^{(k)})'' \times (\mathcal{Z}^{(k+1)})'$ and e_k a kernel on $\{\dagger_{k+1}\} \times (\mathcal{Z}^{(k+1)})'$, given by

$$W_k((z_i^k)'', (z_j^{k+1})') = \phi_{k+1}(z_i^k, z_j^{k+1}), \quad e_k(\dagger_{k+1}, (z_j^{k+1})') = \phi_{k+1}(\dagger_{k+1}, z_j^{k+1}).$$

We emphasise here that W_k depends only on the physical coordinates of the triangular array. Note also that W_0 is an empty array.

We aim to define a measure on the space of duplicated physical coordinates

$$\mathcal{Z} = (\mathcal{Z}^{(1)})' \cup (\mathcal{Z}^{(1)})'' \cup \dots \cup (\mathcal{Z}^{(N)})' \cup (\mathcal{Z}^{(N)})''.$$

It will be convenient first to duplicate the alphabet of virtual coordinates so that there are two copies \dagger'_k, \dagger''_k of each \dagger_k . For each $0 \leq k \leq N-1$, henceforth we will regard e_k as a kernel on $\{\dagger''_{k+1}\} \times (\mathcal{Z}^{(k+1)})'$. The first copy of the virtual coordinates will then be identified through the kernel on $\{\dagger'_1, \dots, \dagger'_N\} \times \{\dagger'_1, \dots, \dagger'_N\}$ defined by the matrix¹²

$$\Lambda_N(\dagger'_i, \dagger'_j) = \delta_{i,j+1} - \delta_{i+1,j}. \quad (7.19)$$

When N is even, the Pfaffian of the matrix (7.19) satisfies $\text{Pf}[\Lambda_N(\dagger'_i, \dagger'_j)]_{1 \leq i,j \leq N} = 1$, so that (7.18) can be expressed as

$$\mathcal{W}_{N,0}(\mathbf{z}|\emptyset) = \text{Pf} \begin{bmatrix} \Lambda_N & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & V_0 & \cdots & 0 & 0 & 0 \\ 0 & -V_0^T & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & V_{N-1} & 0 \\ 0 & 0 & 0 & \cdots & -V_{N-1}^T & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \Psi \end{bmatrix} (\tilde{\mathbf{z}}, \tilde{\mathbf{z}}),$$

where $\tilde{\mathbf{z}}$ is now a version of \mathbf{z} with duplicated physical and virtual variables. Now, recall the property by which a Pfaffian changes sign under the simultaneous interchange of two rows and columns. Performing row and column swaps shows that

$$\mathcal{W}_{N,0}(\mathbf{z}|\emptyset) = \text{Pf} \left[\begin{array}{cc|cccccc} \Lambda_N & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & E_0 & 0 & E_1 & \cdots & 0 & E_{N-1} & 0 \\ \hline 0 & -E_0^T & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & W_1 & \cdots & 0 & 0 & 0 \\ 0 & -E_1^T & 0 & -W_1^T & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & W_{N-1} & 0 \\ 0 & -E_{N-1}^T & 0 & 0 & 0 & \cdots & -W_{N-1}^T & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \Psi \end{array} \right] (\tilde{\mathbf{z}}, \tilde{\mathbf{z}}), \quad (7.20)$$

¹²In principle, we could have used any skew-symmetric $N \times N$ matrix whose Pfaffian is equal to 1.

where, for each $0 \leq k \leq N-1$, the kernel E_k over $\{\dagger_1'', \dots, \dagger_N''\} \times (\mathcal{Z}^{(k)})'$ is defined by

$$E_k(\dagger_i'', (z_j^{k+1})') = \begin{cases} e_k(\dagger_{k+1}, z_j^{k+1}) & \text{if } i = k+1 \\ 0 & \text{otherwise} \end{cases},$$

for each $1 \leq i \leq N, 1 \leq j \leq k+1$. Note here that, since each \dagger_k'' has to be swapped with all of the duplicated virtual variables of lower index, the number of simultaneous interchanges of rows and columns is even.

The rows and columns of the matrix in (7.20) are both indexed by the space

$$(\mathcal{V}' \cup \mathcal{V}'') \cup \left((\mathcal{Z}^{(1)})' \cup (\mathcal{Z}^{(1)})'' \cup \dots \cup (\mathcal{Z}^{(N)})' \cup (\mathcal{Z}^{(N)})'' \right)$$

(and the duplicated variables in $\tilde{\mathcal{Z}}$ are expressed now according to this ordering). Now we may permute rows and columns simultaneously so that pairs of duplicate variables appear next to each other. This corresponds to applying the permutation appearing in (A.1) separately to the blocks $\mathcal{V}' \cup \mathcal{V}''$ and $(\mathcal{Z}^{(k)})' \cup (\mathcal{Z}^{(k)})''$, $k = 1, \dots, N$, so using that identity we get, after reordering, an extra factor of $(-1)^{\binom{N}{2} + \sum_{k=1}^N \binom{k}{2}} = (-1)^{\binom{N}{3}}$, which equals 1 since N is even. If we now identify \mathcal{z} with a set $A \in \mathcal{X}$, we may regard the resulting identity as the Pfaffian of the minor of a suitable 2×2 matrix kernel L over the variables $\mathcal{V} \cup A$. The kernel L as given in (7.17) can now be identified by reading from the entries of (7.20). This shows that

$$\text{Pf} \left[L \Big|_{\mathcal{V} \cup A} \right] = \mathcal{W}_{N,0}(\mathcal{z} | \emptyset)$$

whenever $A \in \Lambda_N$.

It remains to extend the identity to $A \in 2^{\mathcal{X}} \setminus \Lambda_N$, in which case we need to show that the left hand side vanishes. Fix such an A and suppose first that there is no point in A in the i -th fiber of \mathcal{X} . Then the $(2i)$ -th row of $L \Big|_{\mathcal{V} \cup A}$ (corresponding to the second row coming from the 2×2 blocks indexed by row \dagger_i in (7.17)) is clearly just zero, which means that the Pfaffian of the matrix has to vanish. So in order for the Pfaffian not to vanish there necessarily have to be points in all of the N fibers of \mathcal{X} .

Now suppose that A has at least two points in the first fiber of \mathcal{X} , say $(1, a)$ and $(1, b)$, and consider rows $2N+1$ and $2N+3$ of $L \Big|_{\mathcal{V} \cup A}$. They correspond to the first rows coming from the blocks indexed respectively by rows $(1, a)$ and $(1, b)$. Each of these two blocks vanishes except at the second coordinate coming from the \dagger_1 column, so the two rows are linearly dependent and thus the Pfaffian again vanishes. So there has to be exactly one point in the first fiber of \mathcal{X} . The same argument allows one to show inductively that there have to be k points in the k -th fiber. \square

Now, we define the following measure on $\mathcal{X} = \{1, \dots, N\} \times \mathbb{N}$:

$$\mu_N(A) = \frac{\text{Pf}[L_{\mathcal{V} \cup A}]}{\text{Pf}[J_{\mathcal{X}} + L]}. \quad (7.21)$$

The following result follows from the last theorem and Proposition 7.4:

Corollary 7.11 (Pfaffian point process for empty initial configuration). The measure μ_N given in (7.21) defines a Pfaffian point process with correlation kernel given by

$$K = J_{\mathcal{X}} + (J_{\mathcal{X}} + L)^{-1} \Big|_{\mathcal{X} \times \mathcal{X}}.$$

Moreover, μ_N is supported on configurations in Λ_N , and is proportional to $\mathcal{W}_{N,0}(\cdot|\emptyset)$ (defined in (7.14) with $M = 0$) there.

Proof. Fix $m \in \mathbb{N}$ and let μ_N^m denote the restriction of the measure μ_N to $\mathcal{X}_m = \{1, \dots, N\} \times \{1, \dots, m\}$. Proposition 7.4 implies that μ_N^m defines a Pfaffian point process with correlation kernel given by

$$K_m = J_{\mathcal{X}_m} + (J_{\mathcal{X}_m} + L)^{-1} \Big|_{\mathcal{X}_m \times \mathcal{X}_m}.$$

This means that given any collection of points $x_1, \dots, x_n \in \mathcal{X}_m$,

$$\mu_N^m(\{A \subset \mathcal{X}_m : x_1, \dots, x_n \in A\}) = \text{Pf}[K_m(x_i, x_j)]_{i,j=1}^n. \quad (7.22)$$

Now we take $m \rightarrow \infty$ on both sides of the identity to get

$$\mu_N(\{A \subset \mathcal{X} : x_1, \dots, x_n \in A\}) = \text{Pf}[K(x_i, x_j)]_{i,j=1}^n$$

with K as in the statement of the result, which means precisely that the point process μ_N is Pfaffian, with correlation kernel K . That the left hand side of (7.22) converges as claimed follows directly, while for the right hand side it is enough to use that $(J_{\mathcal{X}_m} + L)^{-1} \Big|_{\mathcal{X}_m \times \mathcal{X}_m} (x_i, x_j) = (J_{\mathcal{X}_m} + L)^{-1} (x_i, x_j)$ for each i, j as long as $m > \max\{x_1, \dots, x_n\}$ while $J_{\mathcal{X}_m}$ converges to $J_{\mathcal{X}}$ in operator norm and L defines a bounded operator (which can be checked from its definition; we omit the details).

The remaining claims follow from Theorem 7.10. □

***L*-ensemble for general initial conditions**

Our goal now is to extend the above construction to the case of general initial configuration. In this case, in addition to the spaces \mathcal{X} and \mathcal{V} from (7.16), and for fixed integers M, N and

half-space TASEP ordered coordinates (y_1, \dots, y_M) , we need to introduce the space

$$\mathcal{U} = \{\dagger_{N+1}, \dots, \dagger_{N+M}\}$$

and redefine $\tilde{\mathcal{X}}$ as

$$\tilde{\mathcal{X}} = \mathcal{V} \cup \mathcal{U} \cup \mathcal{X}.$$

We state the result corresponding to Theorem 7.10 and Corollary 7.11 as a single result:

Theorem 7.12 (Pfaffian point process for general initial conditions). Let $N \geq M \geq 0$ be integers such that $N+M$ is even and fix $y = (y_1, \dots, y_M)$ satisfying $y_M > N - M + 1$. Let $L : \tilde{\mathcal{X}} \times \tilde{\mathcal{X}} \rightarrow \mathbb{C}^{2 \times 2}$ be defined by (7.17) where the domain of $L(\dagger_i, \dagger_j)$ is extended to $1 \leq i, j \leq N+M$. Additionally we require

$$\begin{aligned} L((i, x), \dagger_{N+\ell}) &= \begin{bmatrix} 0 & 0 \\ 0 & \Xi_{N-\ell}(x) \end{bmatrix} \mathbb{1}_{i=N}, \\ L(\dagger_{N+k}, (j, x)) &= -L((j, x), \dagger_{N+k})^T, \end{aligned} \tag{7.23}$$

for $1 \leq i, j \leq N$, $1 \leq k, \ell \leq M$ and $x \in \mathbb{N}$. Then

$$\mu_{N,M}(A|y) = \frac{\text{Pf}[L_{\mathcal{V} \cup \mathcal{U} \cup A}]}{\text{Pf}[J_{\mathcal{X}} + L]}.$$

defines a Pfaffian point process on \mathcal{X} with correlation kernel

$$K = J + (J_{\mathcal{X}} + L)^{-1} \Big|_{\mathcal{X} \times \mathcal{X}} \tag{7.24}$$

(with $(J_{\mathcal{X}} + L)^{-1}$ now computed on the whole space $\tilde{\mathcal{X}} = \mathcal{V} \cup \mathcal{U} \cup \mathcal{X}$) which is supported on configurations in Λ_N , and is proportional to $\mathcal{W}_{N,M}(\cdot|y)$ (defined in (7.14)) there.

Proof. We will keep the notation employed in the proof of Theorem 7.10. We also let \mathcal{U}' and \mathcal{U}'' be two copies of the new set of virtual coordinates \mathcal{U} and introduce a kernel $\tilde{\Xi}$ acting on $(\mathcal{Z}^{(N)})'' \times \mathcal{U}'$ as

$$\tilde{\Xi}((z_i^N)'', \dagger'_{N+\ell}) := \Xi_{N-\ell}(z_i^N),$$

for each $1 \leq i \leq N$ and $1 \leq \ell \leq M$, where the function $\Xi_{N-\ell}$ is defined by (7.10) (with explicit dependence on the initial coordinate y_ℓ).

Now, beginning with the measure (7.14), we follow the same procedure as the proof of Theorem

7.10. This leads us to the Pfaffian expression (analogous to (7.20))

$$\mathcal{W}_{N,M}(z|y) = \text{Pf} \left[\begin{array}{cc|cccccc|cc|c} \Lambda_{N+M} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & E_0 & 0 & E_1 & \cdots & 0 & E_{N-1} & 0 & 0 \\ \hline 0 & -E_0^T & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & W_1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -E_1^T & 0 & -W_1^T & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & W_{N-1} & 0 & 0 \\ 0 & -E_{N-1}^T & 0 & 0 & 0 & \cdots & -W_{N-1}^T & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \Psi & \tilde{\Xi} \\ \hline 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\tilde{\Xi}^T & 0 \end{array} \right] (\tilde{z}, \tilde{z}),$$

whose rows and columns are indexed by the space

$$(\mathcal{V}' \cup \mathcal{U}' \cup \mathcal{V}'') \cup \left((\mathcal{Z}^{(1)})' \cup (\mathcal{Z}^{(1)})'' \cup \cdots \cup (\mathcal{Z}^{(N)})' \cup (\mathcal{Z}^{(N)})'' \right) \cup \mathcal{U}''.$$

Here we are taking Λ_{N+M} to be indexed by $\mathcal{V}' \cup \mathcal{U}'$ in the obvious way. Now, we move the \mathcal{U}'' rows and columns to end up with a matrix indexed by

$$(\mathcal{V}' \cup \mathcal{U}' \cup \mathcal{V}'' \cup \mathcal{U}'') \cup \left((\mathcal{Z}^{(1)})' \cup (\mathcal{Z}^{(1)})'' \cup \cdots \cup (\mathcal{Z}^{(N)})' \cup (\mathcal{Z}^{(N)})'' \right), \quad (7.25)$$

leading to

$$\mathcal{W}_{N,M}(z|y) = \text{Pf} \left[\begin{array}{cc|cccccc|cc|c} \Lambda_{N+M} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & E_0 & 0 & E_1 & \cdots & 0 & E_{N-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\tilde{\Xi}^T \\ \hline 0 & -E_0^T & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & W_1 & \cdots & 0 & 0 & 0 \\ 0 & -E_1^T & 0 & 0 & -W_1^T & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & W_{N-1} & 0 \\ 0 & -E_{N-1}^T & 0 & 0 & 0 & 0 & \cdots & -W_{N-1}^T & 0 & 0 \\ 0 & 0 & \tilde{\Xi} & 0 & 0 & 0 & \cdots & 0 & 0 & \Psi \end{array} \right] (\tilde{z}, \tilde{z}). \quad (7.26)$$

The arguments and evaluations of the L -ensemble (7.17) and (7.23) can be identified from (7.26) in the same way as in the proof of Theorem 7.10. A careful counting of the number of interchanges of rows and columns gives the same prefactor of $(-1)^{\binom{N}{3}}$, which may now not

equal 1 since N may be odd. This shows that, for $A \in \Lambda_N$,

$$\text{Pf} [L|_{\mathcal{V} \cup \mathcal{U} \cup A}] = (-1)^{\binom{N}{3}} \mathcal{W}_{N,M}(A|y).$$

All that is left to prove is that if $A \in 2^{\mathcal{X}} \setminus \Lambda_N$ then $\text{Pf} [L|_{\mathcal{V} \cup \mathcal{U} \cup A}] = 0$. But this follows from exactly the same argument as the one used in Theorem 7.10 (noting that the relevant rows in the argument have been augmented with 0's on the $\mathcal{U}' \cup \mathcal{U}''$ coordinates). \square

7.2 Conditional joint distributions

We begin this section by explaining how (7.1) can be used to extract information about the finite-dimensional distributions of half-space TASEP. Consider the case of general initial conditions, and fix $1 \leq p_1 < \dots < p_m \leq N$ and $a_1, \dots, a_m \geq 0$. The identity (7.15) implies that

$$\begin{aligned} \mathbb{P} \left[\bigcap_{k=1}^m \{X_t(p_k) > a_k\}, |X_t| = N \right] &= e^{-\alpha t} \sum_{\substack{x_1 > \dots > x_N \geq 1: \\ x_{p_k} > a_k, k=1, \dots, m}} \sum_{z \in \Lambda_N(x)} \mathcal{W}_{N,M}(z|y) \\ &= e^{-\alpha t} \sum_{x_1 > \dots > x_N \geq 1} \sum_{z \in \Lambda_N(x)} \prod_{k=1}^m \mathbb{1}_{z_1^{p_k} > a_k} \mathcal{W}_{N,M}(z|y). \end{aligned}$$

Using Theorem 7.12, this can be rewritten as

$$\mathbb{P} \left[\bigcap_{k=1}^m \{X_t(p_k) > a_k\}, |X_t| = N \right] = c_N \sum_{A \in 2^{\mathcal{X}} \setminus \mathcal{B}} \frac{\text{Pf} [L|_{\mathcal{V} \cup \mathcal{U} \cup A}]}{\text{Pf} [J_{\mathcal{X}} + L]} \Big|_{v_1 = \dots = v_N = 1}$$

for some $c_N \neq 0$ and for $\mathcal{B} = \{B \in 2^{\mathcal{X}} : (p_k, x) \in B \text{ for some } k = 1, \dots, m \text{ and some } x \leq a_k\}$. The summand on the right hand side corresponds to the measure $\mu_{N,M}(A|y)$, which defines a Pfaffian point process thanks to the same theorem, with the correlation kernel K given in (7.24). So (7.1) implies that

$$\mathbb{P} \left[\bigcap_{k=1}^m \{X_t(p_k) > a_k\}, |X_t| = N \right] = c_N \text{Pf}(J - K)_{\ell^2(\mathcal{B})} = c_N \text{Pf}(J - \bar{\chi}_a K \bar{\chi}_a)_{\ell^2(\{p_1, \dots, p_m\} \times \mathbb{N})}, \quad (7.27)$$

where

$$\bar{\chi}_a(p, x) = \mathbb{1}_{x \leq a_p}. \quad (7.28)$$

Setting $a_k = 0$ for all k , the identity becomes $\mathbb{P} [|X_t| = N] = c_N \text{Pf}(J)_{\ell^2(\{p_1, \dots, p_m\} \times \mathbb{N})} = c_N^{13}$. This implies that the Fredholm Pfaffian on the right hand side of (7.27), which comes from representing half-space TASEP as a conditional Pfaffian L -ensemble, computes the finite-

¹³We note that the inverse of the normalisation coefficient c_N coincides with result of Corollary 6.24

dimensional distributions of the system conditioned on the number of particles at time t :

$$\mathbb{P} \left[\bigcap_{k=1}^m \{X_t(p_k) > a_k\} \mid |X_t| = N \right] = \text{Pf}(J - \bar{\chi}_a K \bar{\chi}_a)_{\ell^2(\{p_1, \dots, p_k\} \times \mathbb{N})}. \quad (7.29)$$

The goal of the rest of this section is to obtain an expression for the kernel K appearing on the right hand side of (7.29).

7.2.1 Preliminaries

In order to explicitly evaluate the kernel, we must first introduce some notation and some preliminary computations. Given two kernels A, B we will denote their *half-space convolution* as

$$(A \star B)(x, y) = \sum_{\nu \geq 1} A(x, \nu) B(\nu, y).$$

Similarly, if A is a kernel and f a function of a single variable, we write

$$(A \star f)(x) = \sum_{\nu \geq 1} A(x, \nu) f(\nu).$$

We will also need to employ a convolution over the whole integers: we will denote the *full-space convolution* as

$$(A \diamond B)(x, y) = \sum_{\nu \in \mathbb{Z}} A(x, \nu) B(\nu, y).$$

We use the half-space convolution to define the following family of kernels on $\mathbb{N} \times \mathbb{N}$: for $1 \leq k, \ell, \leq N$,

$$\phi_{(k, \ell]}(x, y) = \begin{cases} (\phi_{k+1} \star \dots \star \phi_\ell)(x, y) & \text{for } k < \ell, \\ \text{id}(x, y) & \text{for } k = \ell, \\ 0 & \text{for } k > \ell. \end{cases} \quad (7.30)$$

These kernels play an important role in the arguments that follow and we will now demonstrate some of their properties. For $k < \ell$ the kernel (7.30) can be evaluated explicitly as

$$\phi_{(k, \ell]}(x, y) = \oint_{\gamma_0} \frac{dw}{2\pi i} \frac{w^{x-y-1}}{(1-w)^{\ell-k}} = \binom{y-x+\ell-k-1}{\ell-k-1} \mathbb{1}_{x \leq y}, \quad (7.31)$$

where γ_0 encloses only the singularity at the origin $w = 0$. This follows from a simple computation which is in fact more general, and which we state as a lemma (see also [MR23], Lemma 5.6, for a full space version):

Lemma 7.13. Let S_1 and S_2 be two kernels defined on $\mathbb{N} \times \mathbb{N}$, which are given by

$$S_i(x, y) = \frac{1}{2\pi i} \oint_{\gamma} dw w^{x-y-1} g_i(w),$$

where g_1, g_2 are complex functions which are both analytic on the disk $\{z \in \mathbb{C}: |z| < r\}$ for some $r > 0$ and γ is any simple closed contour contained in that disk. Then the sum defining the convolution $S_1 \star S_2$ is absolutely convergent and

$$S_1 \star S_2(x, y) = \frac{1}{2\pi i} \oint_{\gamma} dw w^{x-y-1} g_1(w) g_2(w) \quad (7.32)$$

for all $x, y \in \mathbb{N}$.

Proof. We have

$$\begin{aligned} S_1 \star S_2(x, y) &= \sum_{z \geq 1} \frac{1}{(2\pi i)^2} \oint_{\gamma_1} dw \oint_{\gamma_2} dv w^{x-z-1} v^{z-y-1} g_1(w) g_2(v) \\ &= \frac{1}{(2\pi i)^2} \oint_{\gamma_1} dw \oint_{\gamma_2} dv \frac{w^{x-1} v^{-y}}{w-v} g_1(w) g_2(v) = \frac{1}{2\pi i} \oint_{\gamma} dv v^{x-y-1} g_1(v) g_2(v), \end{aligned}$$

where in the second equality the contours are chosen so that $|w| > |v|$ and in the third one we have computed the residue at $w = v$ (note that there is no residue at $w = 0$ because $x \geq 1$). \square

The formula (7.31) follows from this and the simple fact that $\phi_i(x, y) = \oint_{\gamma_0} \frac{dw}{2\pi i} \frac{w^{x-y-1}}{1-w}$ (recall that $\phi_i(x, y)$ actually does not depend on i , see the footnote in page 146).

We introduce another family of kernels on $\mathbb{N} \times \mathbb{N}$, defined for $1 \leq j \leq m \leq N$ as

$$\phi_{-(j,m]}(x, y) = \oint_{\gamma_0} \frac{dw}{2\pi i} w^{x-y-1} (1-w)^{m-j} = (-1)^{y-x} \binom{m-j}{y-x} \mathbb{1}_{x \leq y}, \quad (7.33)$$

where γ_0 again encloses only the singularity at the origin $w = 0$. Thanks to (7.32), the kernels (7.33) satisfy the following annihilation relation with those defined by (7.30):

$$\phi_{(\ell, N]} \star \phi_{-(j, N]} = \begin{cases} \phi_{(\ell, j]} & \text{if } \ell \leq j, \\ \phi_{-(j, \ell]} & \text{if } \ell > j. \end{cases}$$

We extend the notation $\phi_{(k, \ell]}$ to $\phi_{[k, \ell]} = \phi_{(k-1, \ell]}$ and $\phi_{[k, \ell]} = \phi_{[k, \ell-1]}$, and we also extend $\phi_{-(k, \ell]}$ analogously.

The kernel $\phi_{[k, \ell]}$, which is originally defined on $\mathbb{N} \times \mathbb{N}$, has a natural extension to virtual

coordinates: we define it on $\{\dagger_k\} \times \mathbb{N}$ by

$$\phi_{[k,\ell]}(\dagger_k, x) = \phi_k \star \phi_{(k,\ell]}(\dagger_k, x) = \sum_{\nu \geq 1} \phi_{(k,\ell]}(\nu, x) = \frac{1}{2\pi i} \oint_{\gamma} dw \frac{w^{-x}}{(1-w)^{\ell-k+1}} \mathbb{1}_{k \leq \ell}, \quad (7.34)$$

where we computed again a geometric sum, with the contour γ chosen so that $|w| < 1$.

Remark 7.14. The identity (7.34) is the reason why it is useful to flip the indicator functions in (7.12). In fact, it is straightforward to check that if the $\phi_k(x, y)$'s were defined as $\mathbb{1}_{x > y}$ instead of $\mathbb{1}_{x \leq y}$, the convolution after the second equal sign above would be divergent. An alternative, though more cumbersome solution in our setting, would be to introduce additional parameters to our measure which, under the right ordering, make the convolutions convergent, and then to extend the final answer analytically to the original situation; see e.g. [BF08, MR23], where this is done for full-space models, and where the additional parameters play the role of particle speeds.

For $x \geq 1$ and $k \leq \ell$ the integral in (7.34) can be computed explicitly as $\binom{x+\ell-k-1}{\ell-k}$, which is a polynomial in x on the positive integers of degree $\ell - k$. It will be convenient for us to extend the definition of $\phi_{[k,\ell]}(\dagger_k, x)$ to a polynomial in all $x \in \mathbb{Z}$, which may be done as follows:

$$\phi_{[k,\ell]}(\dagger_k, x) = \frac{1}{2\pi i} \oint_{\gamma} dw \frac{(1-w)^{-x}}{w^{\ell-k+1}} = \frac{(x)_{\ell-k}}{\Gamma(\ell-k+1)}, \quad (7.35)$$

where $(x)_n = x(x+1)\cdots(x+n-1)$ is the Pochhammer symbol (with $(x)_0 = 1$ and $(x)_n = 0$ for $n < 0$). Note that this extension coincides with the right hand side of (7.34) for all $x \geq 1$ and all k, ℓ , that it vanishes for $k > \ell$, and that it defines a polynomial of degree $\ell - k$ for $k \leq \ell$.

One reason why the extension (7.35) will be useful is the following full-space convolution identity¹⁴

$$\phi_{[\ell,N]} \diamond \phi_{-(j,N]}(\dagger_{\ell}, x) = \phi_{[\ell,j]}(\dagger_{\ell}, x) \quad (7.36)$$

for every $j, \ell = 1, \dots, N$ and any $x \in \mathbb{Z}$, and where $\phi_{-(j,N]}(x, y)$ is being extended to a kernel in $\mathbb{Z} \times \mathbb{Z}$ straightforwardly using (7.33). To prove (7.36), compute the convolution over \mathbb{N} and over $\mathbb{Z} \setminus \mathbb{N}$ separately through geometric sums to get $\frac{1}{(2\pi i)^2} \oint_{|u| < |1-w|} dw du \frac{u^{-x}(u-1)^{N-j}}{w^{N-j+1}(w+u-1)} - \frac{1}{(2\pi i)^2} \oint_{|u| > |1-w|} dw du \frac{u^{-x}(u-1)^{N-j}}{w^{N-j+1}(w+u-1)}$, and then shrink the u contour in the second term to cancel the first, picking up a residue at $u = 1 - w$ which gives $\phi_{[\ell,j]}(\dagger_{\ell}, x)$.

¹⁴This identity fails in general if the full-space convolution \diamond is replaced by its half-space version \star ; for example, one has $\phi_{[N-1,N]} \star \phi_{-(N-1,N]}(\dagger_{N-1}, x) = \mathbb{1}_{x=1}$, while $\phi_{[N,N-1]}(\dagger_N, x) = 0$ by definition. It is not strictly necessary to convolve over all integers to get this identity in our setting, where we are only interested in evaluating at $x \geq 1$: convolving over $\mathbb{Z}_{>-N}$ would do. But since it makes no difference and it makes the presentation simpler, we stick to the full-space convolution in (7.33).

The functions (7.30) also allow for a natural extension of the kernel (7.6):

$$\Psi_{(k,\ell)}^N = \left(\phi_{(k,N]} \star \Psi \star \left(\phi_{(\ell,N]} \right)^T \right), \quad (7.37)$$

where $\left(\phi_{(\ell,N]} \right)^T(x, y) = \phi_{(\ell,N]}(y, x)$. Noting the form of the functions (7.31), the convolutions in (7.37) may be explicitly calculated as $\sum_{\nu \geq x} \sum_{\nu' \geq y} \phi_{(k,N]}(x, \nu) \Psi(\nu, \nu') \phi_{(\ell,N]}(\nu', y)$, which can be expressed explicitly as

$$\begin{aligned} \Psi_{(k,\ell)}^N(x, y) &= \frac{\alpha^2}{(2\pi i)^4} \oint \oint \oint \oint du d\tilde{u} dw d\tilde{w} \frac{1}{(1-\tilde{w})^{N-k}} \frac{1}{(1-\tilde{u})^{N-\ell}} \\ &\quad \times \frac{u-w}{1-u-w} \frac{w^{2-x}}{1-w\tilde{w}} \frac{u^{2-y}}{1-u\tilde{u}} \frac{e^{t(w-1)}}{(w-\alpha)(w-1)} \frac{e^{t(u-1)}}{(u-\alpha)(u-1)}, \end{aligned}$$

whose contours satisfy $|u\tilde{u}| > 1$ and $|w\tilde{w}| > 1$ so that the \tilde{u} and \tilde{w} contours include the singularities at $\tilde{w} = 1/w$ and $\tilde{u} = 1/u$ while the u and w contours need to be large enough and include $0, 1, \alpha$ and $1-\alpha$. The simple poles at $\tilde{w} = 1/w$ and $\tilde{u} = 1/u$ can be evaluated to give

$$\Psi_{(k,\ell)}^N(x, y) = \frac{\alpha^2}{(2\pi i)^2} \oint \oint du dw \frac{u-w}{1-u-w} \frac{w^{N-k+1-x}}{(w-1)^{N-k+1}} \frac{u^{N-\ell+1-y}}{(u-1)^{N-\ell+1}} \frac{e^{t(w-1)}}{(w-\alpha)(w-1)} \frac{e^{t(u-1)}}{(u-\alpha)(u-1)}.$$

The kernel $\Psi_{(k,\ell)}^N$ is defined on $\mathbb{N} \times \mathbb{N}$, but we can extend the definition naturally to virtual variables through

$$\begin{aligned} \Psi_{(k,\ell]}^N(x, \dagger_\ell) &= \left(\phi_{(k,N]} \star \Psi \star \left(\phi_{[\ell,N]} \right)^T \right)(x, \dagger_\ell), & \Psi_{[k,\ell]}^N(\dagger_k, y) &= \left(\phi_{[k,N]} \star \Psi \star \left(\phi_{(\ell,N]} \right)^T \right)(\dagger_k, y), \\ \Psi_{[k,\ell]}^N(\dagger_k, \dagger_\ell) &= \left(\phi_{[k,N]} \star \Psi \star \left(\phi_{[\ell,N]} \right)^T \right)(\dagger_k, \dagger_\ell). \end{aligned} \quad (7.38)$$

7.2.2 Empty initial conditions

We are ready to state our main result on the conditional multipoint distribution of half-space TASEP. We do it first for empty initial data.

Theorem 7.15 (Conditional multipoint distribution for empty initial condition). Let $N > 0$ be a fixed even integer and let $\Phi_0(x), \dots, \Phi_{N-1}(x)$ be a family of polynomials where Φ_k is of

degree k . Let these polynomials satisfy the following skew-biorthogonality relations:

$$\begin{aligned} \sum_{x,y=1}^{\infty} \Phi_{2i}(x)\Psi(x,y)\Phi_{2j}(y) &= 0, \\ \sum_{x,y=1}^{\infty} \Phi_{2i+1}(x)\Psi(x,y)\Phi_{2j+1}(y) &= 0, \\ \sum_{x,y=1}^{\infty} \Phi_{2i}(x)\Psi(x,y)\Phi_{2j+1}(y) &= -\delta_{i,j}, \end{aligned} \tag{7.39}$$

for all $0 \leq i, j < N/2$. These polynomials are specified uniquely up to N free parameters which play no role in the sequel (see Remark 7.16). Then for a fixed integer $m \geq 0$, let $\{p_1, \dots, p_m\} \subseteq \{1, \dots, N\}$ be indices denoting particle labels such that $p_1 < p_2 < \dots < p_m$. The half-line TASEP with empty initial conditions has conditional joint distribution given by a Fredholm Pfaffian of the form

$$\mathbb{P} \left[\bigcap_{k=1}^m \{X_t(p_k) > a_k\} \mid |X_t| = N \right] = \text{Pf} \left((J - \bar{\chi}_a K \bar{\chi}_a)_{\ell^2(\{p_1, \dots, p_m\} \times \mathbb{N})} \right), \tag{7.40}$$

where $\bar{\chi}_a(p, x) = \mathbb{1}_{x \leq a_p}$ as in (7.28), where $a_1, \dots, a_m \geq 0$. The kernel K can be expressed as the sum of two terms

$$K = K^0 - K^A \tag{7.41}$$

which have 2×2 block matrix entries given as follows:

$$K^0(i, x_1; j, x_2) = \begin{bmatrix} \Psi_{(i,j)} & -\mathbb{1}_{i < j} \phi_{(i,j]} \\ \mathbb{1}_{j < i} \phi_{(j,i]}^T & 0 \end{bmatrix} (x_1, x_2) \tag{7.42}$$

and

$$K^A(i, x_1; j, x_2) = \begin{bmatrix} \Psi_{(i,N)} \star \mathcal{T}_N \star \Psi_{(N,j)} & -\Psi_{(i,N)} \star \mathcal{T}_N \diamond \phi_{-(j,N]} \\ \phi_{-(i,N]}^T \diamond \mathcal{T}_N \star \Psi_{(N,j)} & -\phi_{-(i,N]}^T \diamond \mathcal{T}_N \diamond \phi_{-(j,N]} \end{bmatrix} (x_1, x_2), \tag{7.43}$$

with

$$\mathcal{T}_N(\nu_1, \nu_2) = \sum_{k=0}^{N/2-1} \Phi_{2k}(\nu_1)\Phi_{2k+1}(\nu_2) - \sum_{k=0}^{N/2-1} \Phi_{2k+1}(\nu_1)\Phi_{2k}(\nu_2).$$

Remark 7.16. Choosing the family of polynomials $\{\Phi_0, \dots, \Phi_{N-1}\}$ corresponds to specifying $\frac{1}{2}N(N+1)$ parameters, while (7.39) corresponds to solving $\frac{1}{2}(N-1)N$ equations, which leaves us with N free parameters. On the other hand, a skew-biorthogonalisation problem of the form (7.39) always has $N/2$ free parameters, because the solution is preserved under the replacements

$$\Phi_{2k+1} \longmapsto \Phi_{2k+1} + \beta_k \Phi_{2k}$$

for $k = 0, \dots, N/2 - 1$ and any choices of $\beta_0, \dots, \beta_{N/2-1} \in \mathbb{R}$. But the final answer (7.41) for the

kernel K is clearly also preserved under these replacements, due to the structure of K^A , so we are indeed free to choose those $N/2$ parameters freely. This leaves us with $N/2$ additional free parameters, but they can be accounted for similarly from the structure of K^A , which only depends on the products $\Phi_{2k}(x)\Phi_{2k+1}(y)$ for $k = 0, \dots, N/2 - 1$, and not on each function separately¹⁵.

Remark 7.17. While the Φ_k 's are defined implicitly in the statement of the theorem as the solution of the skew-biorthogonalisation problem (7.39), in the proof they will be constructed somewhat more explicitly in terms of the functions $\phi_{[j,N]}(\dagger_j, \cdot)$ and the skew-Borel decomposition of the matrix $[\Psi_{[k,\ell]}^N(\dagger_k, \dagger_\ell)]_{k,\ell=1}^N$, see (7.53). In Appendix B we provide formulas which can be used in principle to derive explicit formulas for the Φ_k 's (and, hence, for the solution of the skew-biorthogonalisation problem) based on that representation.

Before presenting the proof of the theorem, we need to state the following simple lemma.

Lemma 7.18. The following skew-symmetric matrix written in block form has an inverse given by

$$\begin{bmatrix} A & B \\ -B^T & D \end{bmatrix}^{-1} = \begin{bmatrix} \mathcal{H}^{-1} & -\mathcal{H}^{-1}BD^{-1} \\ D^{-1}B^T\mathcal{H}^{-1} & D^{-1} - D^{-1}B^T\mathcal{H}^{-1}BD^{-1} \end{bmatrix}, \quad \mathcal{H} = A + BD^{-1}B^T,$$

assuming that all relevant inverses exist.

Proof of Theorem 7.15. Recall that the kernel K is given as

$$K = J_{\mathcal{X}} + (J_{\mathcal{X}} + L)^{-1} \Big|_{\mathcal{X} \times \mathcal{X}} \quad (7.44)$$

with L as in Theorem 7.10. Consider the matrix form of the kernel L indexed by the space $\tilde{\mathcal{X}} \times \tilde{\mathcal{X}}$. It involves duplicate sets of variables as in the proof of Theorem 7.10, which we may order as in that proof¹⁶. With this ordering convention, L can be expressed as the matrix on

¹⁵So, for instance, we could specify the N free parameters by insisting that $\Phi_{2k+1}(x)$ is monic and its x^{2k} coefficient vanishes.

¹⁶As seen in that proof, this reordering does not change the sign in front of the Pfaffian (and, in any case, we will undo it once we get to the final formula for K in this proof).

the right hand side of (7.20), and then $J_{\mathcal{X}} + L$ is represented by the matrix

$$(J_{\mathcal{X}} + L) \Big|_{\tilde{\mathcal{X}} \times \tilde{\mathcal{X}}} = \left[\begin{array}{cc|ccccccccc} \Lambda_N & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & E_0 & 0 & E_1 & \cdots & 0 & E_{N-1} & 0 & 0 \\ \hline 0 & -E_0^T & 0 & \text{id} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -\text{id} & 0 & W_1 & \cdots & 0 & 0 & 0 & 0 \\ 0 & -E_1^T & 0 & -W_1^T & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & W_{N-1} & 0 & 0 \\ 0 & -E_{N-1}^T & 0 & 0 & 0 & \cdots & -W_{N-1}^T & 0 & \text{id} & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -\text{id} & \Psi & 0 \end{array} \right]. \quad (7.45)$$

Let us define the following kernels using the blocks of (7.45)¹⁷:

$$\begin{aligned} A(\dagger_i, \dagger_j) &= L(\dagger_i, \dagger_j), \\ B(\dagger_i, (j, x_2)) &= L(\dagger_i, (j, x_2)), \\ D((i, x_1), (j, x_2)) &= \begin{bmatrix} 0 & \text{id}(x_1, x_2) \\ -\text{id}(x_1, x_2) & 0 \end{bmatrix} \delta_{i,j} + L((i, x_1), (j, x_2)). \end{aligned}$$

In a similar way to [BR05], the matrix inverse of the function D is calculated explicitly as

$$D^{-1}((i, x_1), (j, x_2)) = \begin{bmatrix} \Psi_{(i,j)}(x_1, x_2) & -\phi_{(i,j]}(x_1, x_2) \\ (\phi_{(j,i]})^T(x_1, x_2) & 0 \end{bmatrix}. \quad (7.46)$$

In order to invert the matrix (7.45) using Lemma 7.18 we are required to take products like BD^{-1} . These are evaluated as matrix products indexed by \mathcal{X} , as

$$B \star D^{-1}(\dagger_i, (j, x)) = \sum_{(k,\nu) \in \mathcal{X}} B(\dagger_i, (k, \nu)) D^{-1}((k, \nu), (j, x)) = B(\dagger_i, (i, \cdot)) \star D^{-1}((i, \cdot), (j, x)),$$

for $1 \leq i, j \leq N$ and $x \in \mathbb{N}$. Here, due to the form of the L -ensemble (7.17), the matrix product reduces to a half-space convolution. Through a slight abuse of notation, we will denote this and similar products as $(B \star D^{-1})(\dagger_i, (j, x))$. So, the matrix may be inverted using Lemma 7.18 as

$$[(J_{\mathcal{X}} + L)^{-1}]((i, x_1), (j, x_2)) = [D^{-1} - D^{-1} \star B^T \mathcal{H}^{-1} B \star D^{-1}]((i, x_1), (j, x_2)), \quad (7.47)$$

where \mathcal{H} is the $N \times N$ -dimensional skew-symmetric matrix where entries are given by 2×2

¹⁷The last kernel is defined explicitly on $\mathcal{X} \times \mathcal{X}$ as $D = J + L|_{\mathcal{X} \times \mathcal{X}}$.

matrices defined by

$$[\mathcal{H}]_{k,\ell} = (A + B \star D^{-1} \star B^T)(\dagger_k, \dagger_\ell),$$

and as long as \mathcal{H} is invertible. The matrix entries of \mathcal{H} are explicitly evaluated as

$$[\mathcal{H}]_{k,\ell} = \begin{bmatrix} [\Lambda_N]_{k,\ell} & 0 \\ 0 & [\mathcal{N}]_{k,\ell} \end{bmatrix}$$

where Λ_N and \mathcal{N} are the $N \times N$ -dimensional skew-symmetric matrices defined by (7.19) and

$$[\mathcal{N}]_{k,\ell} = \Psi_{[k,\ell]}^N(\dagger_k, \dagger_\ell) \quad (7.48)$$

respectively. Note that this means $[\mathcal{N}]_{k,\ell} = \phi_{(k,N)} \star \Psi^N \star (\phi_{(\ell,N)})^T(\dagger_k, \dagger_\ell)$, so (7.35) and Lemma B.3 imply that

$$[\mathcal{N}]_{k,\ell} = Q_{N-k+2, N-\ell+2}(1, 1).$$

In particular, $\text{Pf}[\mathcal{N}] = (-1)^{N/2} \text{Pf}[Q_{i+1, j+1}(1, 1)]_{i, j=1}^N$ which is non-zero by Corollary 6.24 (it being proportional to the probability of there being N particles within the system at time t). Hence \mathcal{N} is invertible.

We now turn our attention to the calculation of the second term from (7.47). This is

$$\begin{aligned} & [D^{-1} \star B^T \mathcal{H}^{-1} B \star D^{-1}]((i, x_1), (j, x_2)) \\ &= \sum_{k,\ell=1}^N \begin{bmatrix} 0 & \Psi_{(i,k)}(x_1, \dagger_k) \\ 0 & (\phi_{[k,i]})^T(x_1, \dagger_k) \end{bmatrix} [\mathcal{H}^{-1}]_{k,\ell} \begin{bmatrix} 0 & 0 \\ \Psi_{[\ell,j]}(\dagger_\ell, x_2) & -\phi_{[\ell,j]}(\dagger_\ell, x_2) \end{bmatrix}. \end{aligned}$$

The matrix multiplication is calculated as

$$\begin{aligned} & [D^{-1} \star B^T \mathcal{H}^{-1} B \star D^{-1}]((i, x_1), (j, x_2)) \\ &= \sum_{k,\ell=1}^N \begin{bmatrix} \Psi_{(i,k)}(x_1, \dagger_k) [\mathcal{N}^{-1}]_{k,\ell} \Psi_{[\ell,j]}(\dagger_\ell, x_2) & -\Psi_{(i,k)}(x_1, \dagger_k) [\mathcal{N}^{-1}]_{k,\ell} \phi_{[\ell,j]}(\dagger_\ell, x_2) \\ (\phi_{[k,i]})^T(x_1, \dagger_k) [\mathcal{N}^{-1}]_{k,\ell} \Psi_{[\ell,j]}(\dagger_\ell, x_2) & -(\phi_{[k,i]})^T(x_1, \dagger_k) [\mathcal{N}^{-1}]_{k,\ell} \phi_{[\ell,j]}(\dagger_\ell, x_2) \end{bmatrix}. \end{aligned} \quad (7.49)$$

Now define the following skew-symmetric kernel \mathcal{T}_N on $\mathbb{Z} \times \mathbb{Z}$:

$$\mathcal{T}_N(\nu_1, \nu_2) = \sum_{k,\ell=1}^N (\phi_{[k,N]})^T(\nu_1, \dagger_k) [\mathcal{N}^{-1}]_{k,\ell} \phi_{[\ell,N]}(\dagger_\ell, \nu_2), \quad (7.50)$$

Then using (7.36), we may rewrite (7.49) as

$$[D^{-1} \star B^T \mathcal{H}^{-1} B \star D^{-1}]((i, x_1), (j, x_2))$$

$$= \begin{bmatrix} (\Psi_{(i,N)} \star \mathcal{T}_N \star \Psi_{(N,j)})(x_1, x_2) & -(\Psi_{(i,N)} \star \mathcal{T}_N \diamond \phi_{-(j,N)})(x_1, x_2) \\ ((\phi_{-(i,N)})^T \diamond \mathcal{T}_N \star \Psi_{(N,j)})(x_1, x_2) & -((\phi_{-(i,N)})^T \diamond \mathcal{T}_N \diamond \phi_{-(j,N)})(x_1, x_2) \end{bmatrix}. \quad (7.51)$$

Recall now the skew-Borel decomposition of a skew-symmetric matrix (Proposition B.1) whereby the matrix \mathcal{N} can be uniquely factorised as

$$\mathcal{N} = \mathcal{R}_N J_N \mathcal{R}_N^T, \quad (7.52)$$

where \mathcal{R}_N is a uniquely determined upper-triangular $N \times N$ matrix and J_N is the $N \times N$ block-diagonal matrix defined by

$$J_N = \text{diag} \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

(so that there are $N/2$ blocks in total). The inverse of \mathcal{N} can be expressed as

$$\mathcal{N}^{-1} = -\mathcal{R}_N^{-T} J_N \mathcal{R}_N^{-1},$$

where \mathcal{R}_N^{-T} is the inverse transpose of \mathcal{R}_N . With this in mind, let us define a family of polynomials $\Phi_0(x), \dots, \Phi_{N-1}(x)$ by

$$\Phi_{N-k}(x) = \sum_{j=1}^N [\mathcal{R}_N^{-1}]_{k,j} \phi_{[j,N]}(\dagger_j, x). \quad (7.53)$$

Recall from (7.35) that each $\phi_{[j,N]}(\dagger_j, x)$ is a polynomial of degree $N - j$ so, since \mathcal{R}_N^{-1} is upper-triangular, $\Phi_{N-k}(x)$ is a polynomial of degree $N - k$, as desired.

The kernel (7.50) can then be written in terms of these functions as

$$\mathcal{T}_N(\nu_1, \nu_2) = - \sum_{a,b=1}^N \Phi_{N-a}(\nu_1) [J_N]_{a,b} \Phi_{N-b}(\nu_2) = \sum_{k=0}^{N/2-1} \Phi_{2k}(\nu_1) \Phi_{2k+1}(\nu_2) - \sum_{k=0}^{N/2-1} \Phi_{2k+1}(\nu_1) \Phi_{2k}(\nu_2).$$

Substituting into (7.51) we obtain the kernel of the Fredholm Pfaffian (7.40) and using the result together with (7.46) and (7.47) in (7.44) leads to the desired formula (7.41) for K , with $K^0 = J + D^{-1}$ and $K^A = D^{-1} \star B^T \mathcal{H}^{-1} B \star D^{-1}$.

To derive the skew-biorthogonality, recall the definition of \mathcal{N} in (7.48) and the skew-Borel factorisation (7.52). They yield

$$[J_N]_{k,\ell} = [\mathcal{R}_N^{-1} \mathcal{N} \mathcal{R}_N^{-T}]_{k,\ell} = \sum_{i,j=1}^N [\mathcal{R}_N^{-1}]_{k,i} \Psi_{[i,j]}(\dagger_i, \dagger_j) [\mathcal{R}_N^{-T}]_{j,\ell}.$$

Using (7.38) and the definition (7.53) of the functions Φ_k , this identity can now be expressed

using the half-space convolution as

$$[J_N]_{k,\ell} = \Phi_{N-k} \star \Psi \star \Phi_{N-\ell}.$$

so the Φ_k 's solve the skew-biorthogonalisation problem (7.39). \square

7.2.3 General initial conditions

In this section we will present an extension of Theorem 7.15. That is, a Fredholm Pfaffian expression for the conditional joint distribution of the half-space TASEP in the case of a restricted subset of general deterministic initial conditions. In order to state the result we need to introduce a new family of functions: for fixed $1 \leq i \leq N$ and $1 \leq k \leq M$, we let

$$\Xi_{N-k}^{(i)}(x) = \phi_{(i,N]} \star \Xi_{N-k}(x),$$

where Ξ_{N-k} is given by (7.10) (note in particular that when $i = N$ we have $\Xi_{N-k}^{(N)}(x) = \Xi_{N-k}(x)$). We can compute the convolution explicitly as

$$\phi_{(i,N]} \star \Xi_{N-k}(x) = \frac{(-1)^k}{(2\pi i)^2} \oint \oint dw du \frac{w^{x-1}(u-1)^{N-k}}{(1-w)^{N-i}u^{N-k+1-y_k}} \frac{e^{t(u-1)}}{uw-1}$$

with contours enclosing $w = 0$ and $u = 0$ but not $w = 1$, and such that $|w||u| > 1$; note that, assuming the contours are taken to be circles centered at the origin, this implies that $|u| > 1$. Since $x \geq 1$, the w integral has no singularity at the origin, and computing the remaining residue at $w = 1/u$ we get

$$\Xi_{N-k}^{(i)}(x) = (-1)^i \oint_{\gamma_{0,1}} \frac{du}{2\pi i} \frac{(1-u)^{i-k}}{u^{x-y_k+i-k+1}} e^{t(u-1)}, \quad (7.54)$$

where the contour $\gamma_{0,1}$ now encircles the singularities at both $u = 0$ and $u = 1$. It is important to emphasise here that this function has explicit dependence on the initial particle coordinate $y_k \in \mathbb{N}$ due to the definition of the function Ξ_{N-k} (to be more precise, Ξ_ℓ depends explicitly on $y_{N-\ell}$). It is useful to think of this as a dependence on the virtual coordinate \dagger_{N+k} , which we may identify with y_k , but in any case we will omit this dependence from the notation.

In the proof of the theorem that follows we will also need to use an extension of $\Xi_{N-k}^{(i)}$ to virtual coordinates:

$$\Xi_{N-k}^{[i]}(\dagger_i) = \phi_i \star \Xi_{N-k}^{(i)}(\dagger_i) = (-1)^{i+1} \oint_{\gamma_{0,1}} \frac{du}{2\pi i} \frac{(1-u)^{i-k-1}}{u^{-y_k+i-k+1}} e^{t(u-1)}. \quad (7.55)$$

From this it follows that, for fixed $k \in \{1, \dots, M\}$,

$$\Xi_{N-k}^{[i]}(\dagger_i) = 0 \quad \text{for all } i > k \quad \text{if } y_k > i - k. \quad (7.56)$$

Theorem 7.19. Let $0 < N \geq M \geq 0$ be fixed integers such that $N + M$ is even. Consider two families of polynomials $\Phi_0(x), \dots, \Phi_{N-M-1}(x)$ and $\Upsilon_{N-M}(x), \dots, \Upsilon_{N-1}(x)$ which are such that Φ_k is of degree k and Υ_ℓ is of degree ℓ . Additionally, for all $1 \leq j \leq N, 1 \leq k \leq M$, define

$$\Upsilon_{N-k}^{(j)}(x) = \Upsilon_{N-k} \diamond \phi_{-(j,N]}(x), \quad (7.57)$$

so that in particular $\Upsilon_{N-k}^{(N)} = \Upsilon_{N-k}$. Suppose that the first family of polynomials satisfies the following skew-biorthogonality relations:

$$\begin{aligned} \sum_{x,y=1}^{\infty} \Phi_{2i}(x) \Psi(x,y) \Phi_{2j}(y) &= 0, \\ \sum_{x,y=1}^{\infty} \Phi_{2i+1}(x) \Psi(x,y) \Phi_{2j+1}(y) &= 0, \\ \sum_{x,y=1}^{\infty} \Phi_{2i}(x) \Psi(x,y) \Phi_{2j+1}(y) &= -\delta_{i,j}, \end{aligned} \quad (7.58)$$

for all $0 \leq i, j < (N - M)/2$. Next, suppose that the second family of polynomials satisfy the following biorthogonality relation:

$$\sum_{x=1}^{\infty} \Upsilon_{N-k}(x) \Xi_{N-\ell}(x) = \delta_{k,\ell}, \quad (7.59)$$

for all $1 \leq k, \ell \leq M$. Additionally, suppose that both families satisfy the following orthogonality relation:

$$\sum_{x,y=1}^{\infty} \Upsilon_{N-k}(x) \Psi(x,y) \Phi_i(y) = 0 \quad (7.60)$$

for all $1 \leq k \leq M$ and $0 \leq i < N - M$. Then for a fixed integer $m \geq 0$, let $\{p_1, \dots, p_m\} \subseteq \{1, \dots, N\}$ be indices denoting particles such that $p_1 < p_2 < \dots < p_m$. The half-space TASEP with initial conditions given by $y = (y_1, \dots, y_M)$ where $y_M > N - M + 1$ has conditional joint distribution given by the Fredholm Pfaffian

$$\mathbb{P} \left[\bigcap_{k=1}^m \{X_t(p_k) > a_k\} \mid |X_t| = N \right] = \text{Pf} (J - \bar{\chi}_a K \bar{\chi}_a)_{\ell^2(\{p_1, \dots, p_m\} \times \mathbb{N})}, \quad (7.61)$$

where $\bar{\chi}_a(p, x) = \mathbb{1}_{x \leq a_p}$ and K is the kernel given by

$$K = K^0 - (K^A + K^B + K^C)$$

with the blocks on the right hand side specified as follows. Firstly, K^0 is given by (7.42), namely

$$K^0(i, x_1; j, x_2) = \begin{bmatrix} \Psi_{(i,j)} & -\mathbb{1}_{i < j} \phi_{(i,j]} \\ \mathbb{1}_{j < i} \phi_{(j,i]}^T & 0 \end{bmatrix} (x_1, x_2), \quad (7.62)$$

while K^A is given as in (7.43) but with \mathcal{T}_N replaced by \mathcal{T}_{N-M} , namely

$$K^A(i, x_1; j, x_2) = \begin{bmatrix} \Psi_{(i,N)} \star \mathcal{T}_{N-M} \star \Psi_{(N,j)} & -\Psi_{(i,N)} \star \mathcal{T}_{N-M} \diamond \phi_{-(j,N]} \\ \phi_{-(i,N]}^T \diamond \mathcal{T}_{N-M} \star \Psi_{(N,j)} & -\phi_{-(i,N]}^T \diamond \mathcal{T}_{N-M} \diamond \phi_{-(j,N]} \end{bmatrix} (x_1, x_2), \quad (7.63)$$

Next, K^B is given as

$$K^B(i, x_1; j, x_2) = \begin{bmatrix} K_{11}^B(i, x_1; j, x_2) & K_{12}^B(i, x_1; j, x_2) \\ K_{21}^B(i, x_1; j, x_2) & K_{22}^B(i, x_1; j, x_2) \end{bmatrix} \quad (7.64)$$

with

$$K_{11}^B(i, x_1; j, x_2) = \sum_{k=1}^M \left(\Xi_{N-k}^{(i)}(x_1) (\Upsilon_{N-k} \star \Psi_{(N,j)})(x_2) + (\Psi_{(i,N)} \star \Upsilon_{N-k})(x_1) \Xi_{N-k}^{(j)}(x_2) \right),$$

$$K_{12}^B(i, x_1; j, x_2) = - \sum_{k=1}^M \Xi_{N-k}^{(i)}(x_1) \Upsilon_{N-k}^{(j)}(x_2),$$

$$K_{21}^B(i, x_1; j, x_2) = \sum_{k=1}^M \Upsilon_{N-k}^{(i)}(x_1) \Xi_{N-k}^{(j)}(x_2)$$

$$K_{22}^B(i, x_1; j, x_2) = 0.$$

Finally, K^C is given by

$$K^C(i, x_1; j, x_2) = \begin{bmatrix} \sum_{k,\ell=1}^M \Xi_{N-k}^{(i)}(x_1) (\Upsilon_{N-k} \star \Psi \star \Upsilon_{N-\ell}) \Xi_{N-\ell}^{(j)}(x_2) & 0 \\ 0 & 0 \end{bmatrix}. \quad (7.65)$$

Remark 7.20. Determining the combined family of polynomials

$$\{\Phi_0, \dots, \Phi_{N-M-1}, \Upsilon_{N-M}, \dots, \Upsilon_{N-1}\}$$

corresponds to specifying $\frac{1}{2}N(N+1)$ parameters. Just as in Theorem 7.15 (see Remark 7.16), the family of polynomials $\{\Phi_0, \dots, \Phi_{N-M-1}\}$ is specified uniquely up to $N-M$ parameters which do not affect the Pfaffian since they only appear explicitly in the kernel through the form of K^A (7.63). So it remains to determine the family of polynomials $\{\Upsilon_{N-M}, \dots, \Upsilon_{N-1}\}$, a task which ultimately requires specifying $\frac{1}{2}M(M+1) + (N-M)M$ parameters. They are determined partially from the biorthogonality relations (7.59) for $1 \leq k \leq \ell \leq M$, which correspond to $\frac{1}{2}M(M+1)$ equations.

The $(N - M)M$ remaining parameters are determined by the orthogonality relations (7.60), which yield precisely that number of equations. However, we need to make sure that the outstanding $N - M$ free parameters involved in the choice of the Φ_i polynomial family do not interfere with the uniqueness of the Υ_{N-k} family (and hence do not affect the K^B and K^C parts of the kernel). To see this, recall that these $N - M$ free parameters arise from the freedom we have in making the replacements

$$\Phi_{2k+1} \mapsto \Phi_{2k+1} + \beta_k \Phi_{2k}$$

for $k = 0, \dots, (N - M)/2 - 1$ and any choices of $\beta_0, \dots, \beta_{(N-M)/2-1} \in \mathbb{R}$ and the simultaneous replacements

$$\Phi_{2k+1} \mapsto \gamma_k \Phi_{2k+1}, \quad \Phi_{2k} \mapsto \Phi_{2k}/\gamma_k$$

for $k = 0, \dots, (N - M)/2 - 1$ and any choices of $\gamma_0, \dots, \gamma_{(N-M)/2-1} \in \mathbb{R} \setminus \{0\}$. It is not hard to see that the solution of the Υ_{N-k} polynomials in the orthogonality relations (7.60) are independent of the values of the free parameters β_i, γ_j for all $0 \leq i, j < (N - M)/2$.

Proof of Theorem 7.19. The argument will follow the proof of Theorem 7.15 very closely. We will assume first for simplicity that N and M are even, and explain later how to extend the proof to the case when $N + M$ remains even but both N and M are odd.

The kernel K is given as

$$K = J + (J_{\mathcal{X}} + L)^{-1} \Big|_{\mathcal{X} \times \mathcal{X}}.$$

Using (7.26), we may represent $J_{\mathcal{X}} + L$ as the following matrix

$$(J_{\mathcal{X}} + L) \Big|_{\tilde{\mathcal{X}} \times \tilde{\mathcal{X}}} = \left[\begin{array}{ccc|cccccc|ccc} \Lambda_{N+M} & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & E_0 & 0 & E_1 & \cdots & 0 & E_{N-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & -\tilde{\Xi}^T \\ \hline 0 & -E_0^T & 0 & 0 & \text{id} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & -\text{id} & 0 & W_1 & \cdots & 0 & 0 & 0 \\ 0 & -E_1^T & 0 & 0 & -W_1^T & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & W_{N-1} & 0 \\ 0 & -E_{N-1}^T & 0 & 0 & 0 & 0 & \cdots & -W_{N-1}^T & 0 & \text{id} \\ 0 & 0 & \tilde{\Xi} & 0 & 0 & 0 & \cdots & 0 & -\text{id} & \tilde{\Psi} \end{array} \right]. \quad (7.66)$$

Note that we are presenting the right hand side with rows and columns in the order prescribed by (7.25); this incurs in a sign change in front of the Pfaffian, which will be undone once we get to the expressions for K in the theorem, so for simplicity we omit it.

We define the following kernels as extensions of those that appear in the proof of Theorem 7.15:

$$\begin{aligned}\tilde{A}(\dagger_m, \dagger_n) &= L(\dagger_m, \dagger_n), \\ \tilde{B}(\dagger_m, (j, x)) &= L(\dagger_m, (j, x)),\end{aligned}$$

for each $1 \leq m, n \leq N + M$ and $1 \leq j \leq N, x \in \mathbb{N}$ and where the L -ensemble is defined by (7.17) and (7.23). We emphasise here that the definitions of \tilde{A}, \tilde{B} agree with the definitions of A, B from the proof of Theorem 7.15 when $1 \leq m, n \leq N$, but are otherwise extended to incorporate the extended virtual alphabet. The kernel denoted by D remains the same as in Theorem 7.15.

The $\mathcal{X} \times \mathcal{X}$ block of the inverse of the matrix (7.66) can be written, using Lemma 7.18, as

$$(J_{\mathcal{X}} + L)^{-1} \Big|_{\mathcal{X} \times \mathcal{X}} = D^{-1} - D^{-1} \star \tilde{B}^T \tilde{\mathcal{H}}^{-1} \tilde{B} \star D^{-1} \quad (7.67)$$

with $\tilde{\mathcal{H}} = \tilde{A} + \tilde{B} \star D^{-1} \star \tilde{B}^T$. This matrix is represented as a matrix of 2×2 blocks which is explicitly calculated to yield

$$[\tilde{\mathcal{H}}]_{a,b} = \begin{cases} \begin{bmatrix} [\Lambda_{N+M}]_{a,b} & 0 \\ 0 & [\mathcal{N}]_{a,b} \end{bmatrix} & \text{if } a, b \leq N, \\ \begin{bmatrix} [\Lambda_{N+M}]_{a,b} & 0 \\ 0 & -[\mathcal{P}]_{a,b-N} \end{bmatrix} & \text{if } a \leq N, b > N, \\ \begin{bmatrix} [\Lambda_{N+M}]_{a,b} & 0 \\ 0 & [\mathcal{P}^T]_{a-N,b} \end{bmatrix} & \text{if } a > N, b \leq N, \\ \begin{bmatrix} [\Lambda_{N+M}]_{a,b} & 0 \\ 0 & 0 \end{bmatrix} & \text{if } a, b > N, \end{cases}$$

where \mathcal{N} is the $N \times N$ -dimensional skew-symmetric matrix defined by (7.48) while the $N \times M$ -dimensional matrix \mathcal{P} is defined by

$$[\mathcal{P}]_{a,b} = \Xi_{N-b}^{[a]}(\dagger_a) \quad (7.68)$$

for $1 \leq a \leq N, 1 \leq b \leq M$, where the function $\Xi_{N-k}^{[a]}(\dagger_a)$ is defined by (7.55). The inverse of $\tilde{\mathcal{H}}$

can be calculated using Lemma 7.18 again, after suitably reordering rows and columns, giving

$$[\tilde{\mathcal{H}}^{-1}]_{a,b} = \begin{cases} \begin{bmatrix} [\Lambda_{N+M}^{-1}]_{a,b} & 0 \\ 0 & [\mathcal{N}^{-1} - \mathcal{N}^{-1}\mathcal{P}\mathcal{M}^{-1}\mathcal{P}^T\mathcal{N}^{-1}]_{a,b} \end{bmatrix} & \text{if } a, b \leq N \\ \begin{bmatrix} [\Lambda_{N+M}^{-1}]_{a,b} & 0 \\ 0 & [\mathcal{N}^{-1}\mathcal{P}\mathcal{M}^{-1}]_{a,b-N} \end{bmatrix} & \text{if } a \leq N, b > N \\ \begin{bmatrix} [\Lambda_{N+M}^{-1}]_{a,b} & 0 \\ 0 & -[\mathcal{M}^{-1}\mathcal{P}^T\mathcal{N}^{-1}]_{a-N,b} \end{bmatrix} & \text{if } a > N, b \leq N \\ \begin{bmatrix} [\Lambda_{N+M}^{-1}]_{a,b} & 0 \\ 0 & [\mathcal{M}^{-1}]_{a-N,b-N} \end{bmatrix} & \text{if } a, b > N \end{cases},$$

where $\mathcal{M} = \mathcal{P}^T\mathcal{N}^{-1}\mathcal{P}$. \mathcal{M} is $M \times M$ -dimensional and skew-symmetric, and we will see shortly that it is in fact invertible (recall also that \mathcal{N} is invertible under our assumptions, as was checked in the proof of Theorem 7.15).

And so, the Pfaffian kernel can be identified, using (7.67), as

$$K = K^0 - (K^A + K^B + K^C) = J + D^{-1} - D^{-1} \star \tilde{B}^T \tilde{\mathcal{H}}^{-1} \tilde{B} \star D^{-1}.$$

The first term in the kernel is identified as $K^0 = J + D^{-1}$. This is explicitly calculated as (7.62) using the explicit formula for D^{-1} given by (7.46) (and is the same as K^0 in the case of empty initial condition).

Ultimately, in order to calculate the remaining terms in the kernel we will identify $K^A + K^B + K^C = D^{-1} \star \tilde{B}^T \tilde{\mathcal{H}}^{-1} \tilde{B} \star D^{-1}$. This may be written explicitly as

$$\begin{aligned} & [D^{-1} \star \tilde{B}^T \tilde{\mathcal{H}}^{-1} \tilde{B} \star D^{-1}]((i, x_1), (j, x_2)) \\ &= \sum_{a,b=1}^N \begin{bmatrix} 0 & \Psi_{(i,a]}(x_1, \dagger_a) \\ 0 & (\phi_{[a,i]}^T(x_1, \dagger_a)) \end{bmatrix} [\tilde{\mathcal{H}}^{-1}]_{a,b} \begin{bmatrix} 0 & 0 \\ \Psi_{[b,j]}(\dagger_b, x_2) & -\phi_{[b,j]}(\dagger_b, x_2) \end{bmatrix} \\ &+ \sum_{a=1}^N \sum_{d=1}^M \begin{bmatrix} 0 & \Psi_{(i,a]}(x_1, \dagger_a) \\ 0 & (\phi_{[a,i]}^T(x_1, \dagger_a)) \end{bmatrix} [\tilde{\mathcal{H}}^{-1}]_{a,N+d} \begin{bmatrix} 0 & 0 \\ -\Xi_{N-d}^{(j)}(x_2) & 0 \end{bmatrix} \\ &+ \sum_{c=1}^M \sum_{b=1}^N \begin{bmatrix} 0 & -\Xi_{N-c}^{(i)}(x_1) \\ 0 & 0 \end{bmatrix} [\tilde{\mathcal{H}}^{-1}]_{N+c,b} \begin{bmatrix} 0 & 0 \\ \Psi_{[b,j]}(\dagger_b, x_2) & -\phi_{[b,j]}(\dagger_b, x_2) \end{bmatrix} \\ &+ \sum_{c,d=1}^M \begin{bmatrix} 0 & -\Xi_{N-c}^{(i)}(x_1) \\ 0 & 0 \end{bmatrix} [\tilde{\mathcal{H}}^{-1}]_{N+c,N+d} \begin{bmatrix} 0 & 0 \\ -\Xi_{N-d}^{(j)}(x_2) & 0 \end{bmatrix}. \end{aligned} \quad (7.69)$$

Recall from the proof of Theorem 7.15 that the matrix \mathcal{N} has the skew-Borel decomposition given by (7.52) whereby the upper-triangular matrix \mathcal{R}_N defines the family of skew-orthogonal

polynomials $\Phi_0(x), \dots, \Phi_{N-1}(x)$ by

$$\Phi_{N-k}(x) = \sum_{j=1}^N [\mathcal{R}_N^{-1}]_{k,j} \phi_{[j,N]}(\dagger_j, x). \quad (7.70)$$

We emphasise here that, while this defines a family of N functions Φ_k , only the first $N - M$ will explicitly appear in final expression for the kernel.

Now consider the matrix \mathcal{P} defined by (7.68). Using (7.56), and due to the restriction on the initial coordinates $y_M > N - M + 1$, which implies that $y_k > N - k + 1$ for each $1 \leq k \leq M$, the matrix \mathcal{P} may be decomposed as

$$\mathcal{P} = \begin{bmatrix} \mathcal{S}_M \\ 0 \end{bmatrix}, \quad (7.71)$$

where \mathcal{S}_M is an upper-triangular $M \times M$ -dimensional matrix. Its diagonal entries are calculated using (7.55) as $[\mathcal{S}_M]_{k,k} = \Xi_{N-k}^{[k]}(\dagger_k) = (-1)^{k+1} \oint_{\gamma_{0,1}} \frac{du}{2\pi i} \frac{1}{(1-u)u^{-y_k+i-k+1}} e^{t(u-1)} = (-1)^k$ for $1 \leq k \leq M$ (again due to our condition on y_M), so it follows that this matrix is invertible.

From this and the skew-Borel decomposition of \mathcal{N} , the skew-symmetric matrix \mathcal{M} is given by

$$\mathcal{M} = - \begin{bmatrix} \mathcal{S}_M^T & 0 \end{bmatrix} \mathcal{R}_N^{-T} J_N \mathcal{R}_N^{-1} \begin{bmatrix} \mathcal{S}_M \\ 0 \end{bmatrix}.$$

Since the matrix \mathcal{R}_N in the skew-Borel decomposition of \mathcal{N} is upper-triangular, one checks directly that if $\widehat{\mathcal{R}}_M^N$ is the $M \times M$ -dimensional minor of \mathcal{R}_N consisting of only the first M rows and columns¹⁸, then the top left $M \times M$ block of \mathcal{R}_N^{-1} is $(\widehat{\mathcal{R}}_M^N)^{-1}$, and thus the above can be written as

$$\mathcal{M} = -\mathcal{S}_M^T (\widehat{\mathcal{R}}_M^N)^{-T} J_M (\widehat{\mathcal{R}}_M^N)^{-1} \mathcal{S}_M.$$

Thus the inverse of \mathcal{M} can be calculated as

$$\mathcal{M}^{-1} = \mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N J_M (\widehat{\mathcal{R}}_M^N)^T \mathcal{S}_M^{-T}. \quad (7.72)$$

Let us now define the family of functions $\Upsilon_{N-M}(x), \dots, \Upsilon_{N-1}(x)$ by

$$\Upsilon_{N-k}(x) = \sum_{j=1}^N \left[\mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N \begin{bmatrix} I_M & 0 \end{bmatrix} \mathcal{R}_N^{-1} \right]_{k,j} \phi_{[j,N]}(\dagger_j, x) \quad (7.73)$$

for each $1 \leq k \leq M$.

We now proceed to evaluate each of the sums in (7.69) individually in order to calculate the

¹⁸In general, the minor $\widehat{\mathcal{R}}_M^N$ is not equal to the matrix \mathcal{R}_M which emerges from the skew-Borel decomposition of the analogous matrix \mathcal{N} for a system of size M . Rather, \mathcal{R}_M is the minor consisting of the *last* M rows and columns of \mathcal{R}_N .

terms K^A, K^B, K^C in the Pfaffian kernel explicitly.

- (A) The function $K^A(i, x_1; j, x_2)$ is identified with the first sum of (7.69) which is over $1 \leq a, b \leq N$. The calculation of this sum follows in a similar way to that of the proof of Theorem 7.15. We note that, using (7.72), we may express the product $\mathcal{N}^{-1}\mathcal{P}\mathcal{M}^{-1}\mathcal{P}^T\mathcal{N}^{-1}$ as

$$\begin{aligned} & \mathcal{R}_N^{-T} J_N \mathcal{R}_N^{-1} \begin{bmatrix} \mathcal{S}_M \\ 0 \end{bmatrix} \mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N J_M (\widehat{\mathcal{R}}_M^N)^T \mathcal{S}_M^{-T} \begin{bmatrix} \mathcal{S}_M^T & 0 \end{bmatrix} \mathcal{R}_N^{-T} J_N \mathcal{R}_N^{-1} \\ &= \mathcal{R}_N^{-T} J_N \mathcal{R}_N^{-1} \begin{bmatrix} \widehat{\mathcal{R}}_M^N \\ 0 \end{bmatrix} J_M \begin{bmatrix} (\widehat{\mathcal{R}}_M^N)^T & 0 \end{bmatrix} \mathcal{R}_N^{-T} J_N \mathcal{R}_N^{-1} = -\mathcal{R}_N^{-T} \begin{bmatrix} J_M & 0 \\ 0 & 0 \end{bmatrix} \mathcal{R}_N^{-1}. \end{aligned}$$

In this calculation we have used the fact that $\mathcal{R}_N^{-1} \begin{bmatrix} \widehat{\mathcal{R}}_M^N \\ 0 \end{bmatrix} = \begin{bmatrix} I_M \\ 0 \end{bmatrix}$, which follows from the upper-triangularity of \mathcal{R}_N . And so, we may express

$$\mathcal{N}^{-1} - \mathcal{N}^{-1}\mathcal{P}\mathcal{M}^{-1}\mathcal{P}^T\mathcal{N}^{-1} = -\mathcal{R}_N^{-T} \begin{bmatrix} 0 & 0 \\ 0 & J_{N-M} \end{bmatrix} \mathcal{R}_N^{-1}. \quad (7.74)$$

Now let us define a kernel $\mathcal{T}_A : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{C}$ by

$$\mathcal{T}_A(\nu_1, \nu_2) = \sum_{k, \ell=1}^N (\phi_{[k, N]})^T(\nu_1, \dagger_k) [\mathcal{N}^{-1} - \mathcal{N}^{-1}\mathcal{P}\mathcal{M}^{-1}\mathcal{P}^T\mathcal{N}^{-1}]_{k, \ell} \phi_{[\ell, N]}(\dagger_\ell, \nu_2). \quad (7.75)$$

Using this kernel, we may write K^A given by the first sum in (7.69) as

$$\begin{aligned} & K^A((i, x_1), (j, x_2)) \\ &= \begin{bmatrix} (\Psi_{(i, N]} \star \mathcal{T}_A \star \Psi_{[N, j]}) (x_1, x_2) & -(\Psi_{(i, N]} \star \mathcal{T}_A \star \phi_{-(j, N]}) (x_1, x_2) \\ \left((\phi_{-(i, N]} \right)^T \star \mathcal{T}_A \star \Psi_{[N, j]} \right) (x_1, x_2) & -\left((\phi_{-(i, N]} \right)^T \star \mathcal{T}_A \star \phi_{-(j, N]} \right) (x_1, x_2) \end{bmatrix}, \end{aligned} \quad (7.76)$$

where, using the matrix decomposition (7.74) and (7.70), the kernel (7.75) can be expressed as

$$\begin{aligned} \mathcal{T}_A(\nu_1, \nu_2) &= - \sum_{a, b=M+1}^N \Phi_{N-a}(\nu_1) [J_N]_{a, b} \Phi_{N-b}(x) \\ &= \sum_{k=0}^{(N-M)/2-1} \Phi_{2k}(\nu_1) \Phi_{2k+1}(\nu_2) - \sum_{k=0}^{(N-M)/2-1} \Phi_{2k+1}(\nu_1) \Phi_{2k}(\nu_2), \end{aligned}$$

so that \mathcal{T}_A coincides with the definition of \mathcal{T}_{N-M} . After substituting this into (7.76) we may identify the form of the kernel (7.63).

- (B) The function $K^B(i, x_1; j, x_2)$ is identified with the sum of the second and third sums of (7.69) which are over $1 \leq a \leq N, 1 \leq d \leq M$ and $1 \leq c \leq M, 1 \leq b \leq N$ respectively. We will

explicitly calculate the third sum since the second sum follows in a similar fashion via skew-symmetry.

We have

$$\mathcal{M}^{-1} \mathcal{P}^T \mathcal{N}^{-1} = -\mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N J_M (\widehat{\mathcal{R}}_M^N)^T \mathcal{S}_M^{-T} \begin{bmatrix} \mathcal{S}_M^T & 0 \end{bmatrix} \mathcal{R}_N^{-T} J_N \mathcal{R}_N^{-1} = \mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N \begin{bmatrix} I_M & 0 \end{bmatrix} \mathcal{R}_N^{-1}.$$

We may then calculate the third sum of (7.69) as

$$\sum_{c=1}^M \sum_{b=1}^N \begin{bmatrix} 0 & -\Xi_{N-c}^{(i)}(x_1) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -[\mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N \begin{bmatrix} I_M & 0 \end{bmatrix} \mathcal{R}_N^{-1}]_{c,b} \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ \Psi_{[b,j]}(\dagger_b, x_2) & -\phi_{[b,j]}(\dagger_b, x_2) \end{bmatrix}.$$

Using the definition of the family of functions Υ_{N-k} (7.73), the previous expression may be written as

$$\sum_{c=1}^M \sum_{k=1}^M \begin{bmatrix} 0 & -\Xi_{N-c}^{(i)}(x_1) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & [I_M]_{c,k} \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -(\Upsilon_{N-k} \star \Psi_{(N,j)})(x_2) & (\Upsilon_{N-k} \diamond \phi_{-(j,N)})(x_2) \end{bmatrix} \\ = \sum_{k=1}^M \begin{bmatrix} \Xi_{N-k}^{(i)}(x_1) (\Upsilon_{N-k} \star \Psi_{(N,j)})(x_2) & -\Xi_{N-k}^{(i)}(x_1) (\Upsilon_{N-k} \diamond \phi_{-(j,N)})(x_2) \\ 0 & 0 \end{bmatrix},$$

in which the function $\Upsilon_{N-k}^{(j)}$ defined by (7.57) may be identified. Now accounting for both the second and third sums of (7.69), we may express the kernel term as the expression (7.64) for K^B .

- (C) The function $K^C(i, x_1; j, x_2)$ is identified with the fourth sum of (7.69) which is over $1 \leq c, d \leq M$. This is explicitly calculated as

$$K^C(i, x_1; j, x_2) = \sum_{c,d=1}^M \begin{bmatrix} \Xi_{N-c}^{(i)}(x_1) [\mathcal{M}^{-1}]_{c,d} \Xi_{N-d}^{(j)}(x_2) & 0 \\ 0 & 0 \end{bmatrix}.$$

To see that this coincides with the expression for K^C given by (7.65) we need to show that

$$[\mathcal{M}^{-1}]_{c,d} = \Upsilon_{N-c} \star \Psi \star \Upsilon_{N-d}.$$

In order to check this we use the definition of the functions Υ_k to write the right hand side

as

$$\begin{aligned}
& \sum_{i,j=1}^N \left[\mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N \begin{bmatrix} I_M & 0 \end{bmatrix} \mathcal{R}_N^{-1} \right]_{c,i} \phi_{[i,N]}(\dagger_i, \cdot) \star \Psi \star (\phi_{[j,N]})^T(\cdot, \dagger_j) \left[\mathcal{R}_N^{-T} \begin{bmatrix} I_M \\ 0 \end{bmatrix} (\widehat{\mathcal{R}}_M^N)^T \mathcal{S}_M^{-T} \right]_{j,d} \\
&= \sum_{i,j=1}^N \left[\mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N \begin{bmatrix} I_M & 0 \end{bmatrix} \mathcal{R}_N^{-1} \right]_{c,i} [\mathcal{N}]_{i,j} \left[\mathcal{R}_N^{-T} \begin{bmatrix} I_M \\ 0 \end{bmatrix} (\widehat{\mathcal{R}}_M^N)^T \mathcal{S}_M^{-T} \right]_{j,d} \\
&= \left[\mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N J_M (\widehat{\mathcal{R}}_M^N)^T \mathcal{S}_M^{-T} \right]_{c,d},
\end{aligned}$$

where we used again the skew-Borel decomposition of \mathcal{N} . In view of (7.72), the right hand side is exactly $[\mathcal{M}^{-1}]_{c,d}$.

With these calculations, we have explicitly determined the Pfaffian kernel in terms of known functions and the new functions $\Phi_0, \dots, \Phi_{N-M-1}$ and $\Upsilon_{N-M}, \dots, \Upsilon_{N-1}$. In order to finish proving that the kernel coincides with the one in the statement of the theorem, we need to prove that these functions are polynomials of the correct degrees and that they satisfy their defining relations (7.58), (7.59) and (7.60).

The functions $\Phi_0, \dots, \Phi_{N-M-1}$ were defined by (7.70) in the exact same way as the extended family $\Phi_0, \dots, \Phi_{N-1}$ was in the proof of Theorem 7.15, so, the first $N - M$ of these functions are polynomials of the correct degrees and they inherit their skew-orthogonality relations in exactly the same manner. Note that the relations (7.58) for the first $N - M$ functions are self-contained, so these functions are determined by these relations (up to $N - M$ free parameters which are not consequential) as in the previous proof.

That Υ_ℓ is a polynomial of degree ℓ follows from (7.73) because $[\mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N [I_M \ 0] \mathcal{R}_N^{-1}]_{k,j} = 0$ for $j > k$ and each $\phi_{[j,N]}(\dagger_j, x)$ is a polynomial of degree $N - j$. In order to determine the relation (7.59) we use the decomposition of the matrix \mathcal{P} from (7.71) to write $(\phi_{[j,N]} \star \Xi_{N-\ell})(\dagger_j) = \begin{bmatrix} \mathcal{S}_M \\ 0 \end{bmatrix}_{j,\ell}$, so that

$$\begin{aligned}
\sum_{x=1}^{\infty} \Upsilon_{N-k}(x) \Xi_{N-\ell}(x) &= \sum_{j=1}^N \left[\mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N \begin{bmatrix} I_M & 0 \end{bmatrix} \mathcal{R}_N^{-1} \right]_{k,j} (\phi_{[j,N]} \star \Xi_{N-\ell})(\dagger_j) \\
&= \left[\mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N \begin{bmatrix} I_M & 0 \end{bmatrix} \mathcal{R}_N^{-1} \begin{bmatrix} \mathcal{S}_M \\ 0 \end{bmatrix} \right]_{k,\ell} = \left[\begin{bmatrix} \mathcal{S}_M^{-1} & \Delta \end{bmatrix} \begin{bmatrix} \mathcal{S}_M \\ 0 \end{bmatrix} \right]_{k,\ell} = \delta_{k,\ell},
\end{aligned}$$

where in the third equality we used once again the fact the top left $M \times M$ block of \mathcal{R}_N^{-1} is $(\widehat{\mathcal{R}}_M^N)^{-1}$ and where the value of the Δ block is irrelevant.

To prove the skew-orthogonality relation between the two families of functions, (7.60), we write, for $1 \leq k \leq M$ and $1 \leq i \leq N - M$,

$$\begin{aligned} \sum_{x_1, x_2=1}^{\infty} \Upsilon_{N-k}(x_1) \Psi(x_1, x_2) \Phi_i(x_2) &= \sum_{j=1}^N \left[\mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N \begin{bmatrix} I_M & 0 \end{bmatrix} \mathcal{R}_N^{-1} \right]_{k,j} \phi_{[j,N]} \star \Psi \star \Phi_i(\dagger_j) \\ &= \sum_{\ell=1}^N \left[\mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N \begin{bmatrix} I_M & 0 \end{bmatrix} \right]_{k,\ell} \Phi_{N-\ell} \star \Psi \star \Phi_i = \left[\mathcal{S}_M^{-1} \widehat{\mathcal{R}}_M^N \begin{bmatrix} J_M & 0 \end{bmatrix} \right]_{k, N-i} \end{aligned}$$

where we have used both the definition and the skew-biorthogonality relation of the entire family of functions $\{\Phi_0, \dots, \Phi_{N-1}\}$. The relation (7.60) follows by noting that the matrix entries of the final expression above are equal to zero whenever $i \leq N - M$.

To finish the proof we need to extend the result to the case of odd N and M . We will do this by comparing with a system with $\bar{M} = M + 1$ initial particles and $\bar{N} = N + 1$ final ones, placing an additional initial particle very far to the right. To this end, given the initial particle positions $y = (y_1, \dots, y_M)$, we introduce a new collection of particle positions $\bar{y} = (\bar{y}_1, \dots, \bar{y}_{\bar{M}})$ by

$$\bar{y}_1 = L, \quad \bar{y}_i = y_{i-1}, \quad i = 2, \dots, M + 1,$$

where L is an auxiliary parameter which we will ultimately take to infinity. Note that, for large L , \bar{y} satisfies $\bar{y}_k > \bar{N} - k + 1$ thanks to the analogous conditions for y . Then from the case with N and M even,

$$\mathbb{P}_{\bar{y}} \left[\bigcap_{k=1}^m \{X_t(\bar{p}_k) > a_k\} \mid |X_t| = \bar{N} \right] = \text{Pf} \left(J - \bar{\chi}_a \bar{K} \bar{\chi}_a \right)_{\ell^2(\{\bar{p}_1, \dots, \bar{p}_m\} \times \mathbb{N})} = \text{Pf} \left(J - \bar{\chi}_a \hat{K} \bar{\chi}_a \right)_{\ell^2(\{p_1, \dots, p_m\} \times \mathbb{N})}, \quad (7.77)$$

where we have included the initial condition \bar{y} explicitly in the subscript, where $\bar{p}_i = p_i + 1$ with $1 \leq p_1 < \dots < p_m \leq N$ and where \bar{K} denotes the kernel K computed above, with N and M replaced by \bar{N} and \bar{M} and $\hat{K}(i, \cdot; j, \cdot) = \bar{K}(i + 1, \cdot; j + 1, \cdot)$. Now for very large L , the first particle (which starts at L) is very likely to not interact with the rest of the system by time t : in fact, the probability that the second particle is blocked by the first one at any moment before time t is bounded by the probability that a Poisson random variable with parameter t is greater than $L - y_1 - 2$. Using this, and after suitably shifting the particle labels, it is not hard to see that the left hand side above converges as $L \rightarrow \infty$ to

$$\mathbb{P}_y \left[\bigcap_{k=1}^m \{X_t(p_k) > a_k\} \mid |X_t| = N \right].$$

So what remains is to compute the limit of the Fredholm Pfaffian on the right hand side of (7.77). The argument is relatively simple, so we will only sketch it.

Consider each of the terms making up \hat{K} . The first one is \hat{K}^0 , which does not depend on N ,

M or L , and which clearly satisfies $\hat{K}^0 = K^0$. The second one is given by (7.63), and now reads

$$\hat{K}^A(i, x_1; j, x_2) = \begin{bmatrix} \Psi_{(i+1, \bar{N})} \star \mathcal{T}_{N-M} \star \Psi_{(\bar{N}, j+1)} & -\Psi_{(i+1, \bar{N})} \star \mathcal{T}_{N-M} \diamond \phi_{-(j+1, \bar{N})} \\ \phi_{-(i+1, \bar{N})}^T \diamond \mathcal{T}_{N-M} \star \Psi_{(\bar{N}, j+1)} & -\phi_{-(i+1, \bar{N})}^T \diamond \mathcal{T}_{N-M} \diamond \phi_{-(j+1, \bar{N})} \end{bmatrix} (x_1, x_2).$$

Note that it also does not depend on L and that, in fact, $\hat{K}^A = K^A$.

For the remaining two terms we need to compute limits in L . But this is simple once one notes that $\Xi_{\bar{N}-1}(x) \xrightarrow{L \rightarrow \infty} 0$, as follows easily from (7.10) since $y_1 = L$. So, for instance,

$$\lim_{L \rightarrow \infty} \hat{K}_{12}^B(i, x_1; j, x_2) = - \sum_{k=2}^{\bar{M}} \Xi_{\bar{N}-k}^{(i+1)}(x_1) \Upsilon_{\bar{N}-k}^{(j+1)}(x_2) = K_{12}^B(i, x_1; j, x_2).$$

Similar expressions hold for the other limits, and we deduce that $\lim_{L \rightarrow \infty} \hat{K}^B = K^B$. In a similar way we get $\lim_{L \rightarrow \infty} \hat{K}^C = K^C$. A bit more work shows that the limit holds in a strong enough sense (e.g. trace class) so that it can be taken inside the Fredholm Pfaffian. Using this on the right hand side of (7.77) yields the claimed formula. \square

7.2.4 Reduction to full-space TASEP

One important consequence of Theorem 7.19 is that the conditional joint distribution of the half-space TASEP reduces the well-known joint distribution of the full-space TASEP. In this way, Theorem 7.19 may be regarded as a generalisation of the Fredholm determinant expressions obtained in the literature.

Corollary 7.21 (of Theorem 7.19). Let $N > 0$ be a fixed integer and let $y = (y_1, \dots, y_N)$ be an ordered set of coordinates, and assume that $y_N > 1$. Moreover, let $\Upsilon_0(x), \dots, \Upsilon_{N-1}(x)$ be a family of polynomials where Υ_k is of degree k . Let these polynomials satisfy the biorthogonality relation (7.59). The full-space TASEP with initial conditions given by y has joint distribution given by the Fredholm determinant expression

$$\mathbb{P} \left[\bigcap_{k=1}^m \{X_t(p_k) > a_k\} \right] = \det(I - \bar{\chi}_a K \bar{\chi}_a)_{\ell^2(\{p_1, \dots, p_m\} \times \mathbb{N})}, \quad (7.78)$$

whose kernel is given by

$$K(i, x_1; j, x_2) = -\mathbb{1}_{i < j} \phi_{[i, j]}(x_1, x_2) + \sum_{k=1}^j \Xi_{N-k}^{(i)}(x_1) \Upsilon_{N-k}^{(j)}(x_2). \quad (7.79)$$

More generally, if $y = (y_1, \dots, y_N)$ is any ordered set of coordinates in \mathbb{Z} , the full-space TASEP joint distribution with initial conditions given by y can be written as (note that the Fredholm

determinant is now over $\ell^2(\{p_1, \dots, p_m\} \times \mathbb{Z})$

$$\mathbb{P} \left[\bigcap_{k=1}^m \{X_t(p_k) > a_k\} \right] = \det(I - \bar{\chi}_a K \bar{\chi}_a)_{\ell^2(\{p_1, \dots, p_m\} \times \mathbb{Z})}, \quad (7.80)$$

and the kernel K can be expressed as

$$K(i, x_1; j, x_2) = -\mathbb{1}_{i < j} \phi_{[i,j]}(x_1, x_2) + \sum_{k=1}^j f_{i-k}^i(x_1) g_{j-k}^j(x_2), \quad (7.81)$$

with

$$f_{i-k}^i(x) = (-1)^k \Xi_{N-k}^{(i)}(x),$$

which we regard as being defined on \mathbb{Z} directly through (7.54), and where the family $\{g_0^j, \dots, g_{j-1}^j\}$ is defined as follows: g_ℓ^j is a polynomial of degree ℓ , and these j polynomials satisfy the (full-space) biorthogonality relations

$$\sum_{x \in \mathbb{Z}} f_k^j(x) g_\ell^j(x) = \delta_{k,\ell}. \quad (7.82)$$

Moreover, the formula (7.80) recovers the one derived for this model in Proposition 3.1 of [BF08].

The formula derived in [BF08] is for a more general model in full-space, namely PushASEP with jump rates which depend on time and on the particle; to compare their formula to (7.80) one needs to set all their v_i 's to 1 and take $a(t) = 1$ and $b(t) = 0$. Similar formulas were derived earlier for full-space TASEP in [Sas05, BFPS07], under a formalism where the ϕ_i 's are flipped with respect to our setting.

Proof of Corollary 7.21. If $y_N > 1$, the probability on the left hand side of (7.78) can be computed from Theorem 7.19 with $\alpha = 0$ and $M = N$, and is given by the Fredholm Pfaffian on the right hand side of (7.61) satisfying $|X_t| = N$. Now, since $M = N$, we note that the only (bi)orthogonality relation of the Υ polynomials is (7.59), while K^A vanishes. In particular, the K_{22} of the kernel completely vanishes. Now recall the following property, found in Section 8 of [Rai00], whereby a Fredholm Pfaffian reduces to a Fredholm determinant:

$$\text{Pf}(J - K)_{L^2(\mathcal{X})} = \det(I - K_{12})_{L^2(\mathcal{X})},$$

whenever $K_{22}(x_1, x_2) = 0$ for all $x_1, x_2 \in \mathcal{X}$. This property implies that the right hand side of (7.78) reduces as

$$\det(I - \bar{\chi}_a K \bar{\chi}_a)_{\ell^2(\{p_1, \dots, p_m\} \times \mathbb{N})}, \quad \text{with} \quad K = K_{12}^0 - K_{12}^B.$$

K can be expressed explicitly as

$$K(i, x_1; j, x_2) = -\mathbb{1}_{i < j} \phi_{[i, j]}(x_1, x_2) + \sum_{k=1}^N \Xi_{N-k}^{(i)}(x_1) \Upsilon_{N-k}^{(j)}(x_2). \quad (7.83)$$

Now note that since $g \diamond \phi_{-(j, j+1]}(x) = g(x) - g(x-1)$ and since Υ_{N-k} is a polynomial of degree $N-k$, then $\Upsilon_{N-k} \diamond \phi_{-(j, j+1]}$ is a polynomial of degree $N-k-1$, and thus $\Upsilon_{N-k}^{(j)}$ is a polynomial of degree $j-k$ if $k \leq j$, while

$$\Upsilon_{N-k}^{(j)} = 0 \quad \text{if } k > j.$$

In particular, the formula (7.83) becomes (7.79).

Now, assuming still that $y_N > 1$, define f_{i-k}^i as in the statement of the corollary and let

$$g_{j-k}^j(x) = (-1)^k \Upsilon_{N-k}^{(j)}(x),$$

so that (7.79) can be expressed as (7.81). We know already that g_ℓ^j is a polynomial of degree ℓ for $\ell = 0, \dots, j-1$, and we claim that the two families of functions satisfy (7.82). In fact,

$$\begin{aligned} \sum_{x \in \mathbb{Z}} f_{j-k}^j(x) g_{j-\ell}^j(x) &= \sum_{x, y \in \mathbb{Z}} \Xi_{N-k}^{(j)}(x) \Upsilon_{N-\ell}(y) \phi_{-(j, N]}(y, x) = \sum_{x \in \mathbb{Z}} \phi_{-(j, N]} \diamond \Xi_{N-k}^{(j)}(x) \Upsilon_{N-\ell}(x) \\ &= \sum_{x \in \mathbb{Z}} \Xi_{N-k}(x) \Upsilon_{N-\ell}(x), \end{aligned}$$

and the last sum can be restricted to $x \geq 1$ because $\Xi_{N-k}(x) = 0$ for $x \leq 0$ thanks to the condition $y_k \geq y_N + N - k > N - k$, so (7.59) ensures that it equals $\delta_{k, \ell}$.

Consider now a general choice of initial condition $y = (y_1, \dots, y_N)$. By translation invariance, if we fix $L > 0$ and shift all the y_i 's and all the a_i 's by L , the probability on the left hand side of (7.80) does not change. Then letting $\bar{y}_i = y_i + L$ and $\bar{a}_i = a_i + L$, if L is large enough so that $\bar{y}_N > 1$ then previous case yields

$$\mathbb{P} \left[\bigcap_{k=1}^m \{X_t(p_k) > a_k\} \right] = \det(I - \bar{\chi}_{\bar{a}} \bar{K} \bar{\chi}_{\bar{a}})_{\ell^2(\{p_1, \dots, p_m\} \times \mathbb{N})}, \quad (7.84)$$

where \bar{K} is defined as in (7.81) but using \bar{y} instead of y . But it is easy to see, from the definition of $\phi_{[i, j]}$ and f_{i-k}^i , and using the fact that the g_{i-k}^i 's are defined through the biorthogonalisation problem (7.82), that $\bar{K}(i, x_1 + L; j, x_2 + L) = K(i, x_1; j, x_2)$ (i.e., shifting the variables in the kernel is equivalent to shifting back the y_i 's). So performing this change of variables in the kernel inside the Fredholm determinant in (7.84) we get

$$\mathbb{P} \left[\bigcap_{k=1}^m \{X_t(p_k) > a_k\} \right] = \det(I - \bar{\chi}_a K \bar{\chi}_a)_{\ell^2(\{p_1, \dots, p_m\} \times \mathbb{Z}_{2-M})},$$

Taking $L \rightarrow \infty$ yields (7.80). Checking that this formula coincides with the one given in [BF08] in the case of full-space TASEP is straightforward. \square

Appendix A

Properties of Pfaffians

A.1 Definition and identities

A *Pfaffian* is defined for a skew-symmetric $2n \times 2n$ -dimensional matrix A with entries $a_{i,j}$ as

$$\text{Pf } A = \frac{1}{2^n n!} \sum_{\sigma \in S_n} (-1)^{|\sigma|} \prod_{i=1}^n a_{\sigma(2i-1), \sigma(2i)}.$$

The determinant of the same even dimensional skew-symmetric matrix A is the square of a polynomial in its entries. It can be shown that the determinant is exactly the square of the Pfaffian:

$$\det A = (\text{Pf } A)^2.$$

The Pfaffian of a skew-symmetric $2n \times 2n$ matrix A changes sign under simultaneous interchange of rows and columns of the same index. A simple consequence is the following: if A is given and \hat{A} is the reordered matrix

$$[\hat{A}]_{i,j} = A_{\sigma(i), \sigma(j)}$$

with $\sigma \in S_{2n}$ specified as

$$\sigma(2i-1) = i, \quad \sigma(2i) = n+i \quad \text{for each } i = 1, \dots, n,$$

then

$$\text{Pf}(\hat{A}) = (-1)^{\binom{n}{2}} \text{Pf}(A). \tag{A.1}$$

This Appendix appears in the preprint [\[P2\]](#), where Lemma [A.3](#) has been added.

Pfaffians may also be thought of as generalisations of determinants: for an arbitrary $m \times m$ -dimensional matrix B , the following identity also holds

$$\det(B) = (-1)^{\binom{m}{2}} \text{Pf} \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix}. \quad (\text{A.2})$$

The following Pfaffian identity, found in [Oka19], is a generalisation of (A.2).

Proposition A.1. Let A be a skew-symmetric $n \times n$ -dimensional matrix and let B be a $n \times m$ -dimensional matrix. The following Pfaffian evaluation holds: whenever $n + m$ is even

$$\text{Pf} \begin{bmatrix} A & B \\ -B^T & 0 \end{bmatrix} = (-1)^{\binom{n}{2}} \times \begin{cases} \sum_S (-1)^{\sum_i S_i} \text{Pf } A_S \det B_{(S^c; [m])} & \text{if } n > m \\ \det B & \text{if } n = m, \\ 0 & \text{if } n < m \end{cases}$$

and whenever $n + m$ is odd

$$\text{Pf} \begin{bmatrix} A & \vec{\mathbf{1}}_n & B \\ -\vec{\mathbf{1}}_n^T & 0 & 0 \\ -B^T & 0 & 0 \end{bmatrix} = (-1)^{\binom{n}{2} + n - m} \times \begin{cases} \sum_S (-1)^{\sum_i S_i} \text{Pf} \begin{bmatrix} A_S & \vec{\mathbf{1}}_{n-m} \\ -\vec{\mathbf{1}}_{n-m}^T & 0 \end{bmatrix} \det B_{(S^c; [m])} & \text{if } n > m, \\ 0 & \text{if } n < m \end{cases},$$

where the sum is over subsets $S \subseteq \{1, \dots, n\}$ of cardinality $n - m$ and S^c denotes its complement. We have used the shorthand notation $[m] = \{1, \dots, m\}$.

Proof. The version of the result where $n + m$ is even is given explicitly as Corollary 2.4 of [Oka19]. Meanwhile, the odd version can be obtained as a special case of the even version where the last row and column of an extended skew-symmetric matrix A are set to be ± 1 appropriately and the last row of an extended matrix B is set to zero. \square

The following identity, due to Stembridge [Ste90] is crucial to our work.

Lemma A.2 (Stembridge's Pfaffian). Let x_1, \dots, x_m be an alphabet of fixed length m . The following Pfaffian identities hold: when $m = 2n$ is even

$$\prod_{1 \leq i < j \leq 2n} \frac{x_i - x_j}{1 - x_i x_j} = \text{Pf} [S(x_i, x_j)]_{1 \leq i, j \leq 2n},$$

and when $m = 2n - 1$ is odd:

$$\prod_{1 \leq i < j \leq 2n-1} \frac{x_i - x_j}{1 - x_i x_j} = \text{Pf} \begin{bmatrix} S(x_i, x_j)_{1 \leq i, j \leq 2n-1} & \vec{\mathbf{1}}_{2n-1} \\ -\vec{\mathbf{1}}_{2n-1}^T & 0 \end{bmatrix},$$

where

$$S(x_i, x_j) = \frac{x_i - x_j}{1 - x_i x_j}.$$

The following identity also appears in [Ste90].

Lemma A.3. Let A and B be skew-symmetric $m \times m$ matrices. Then we have

$$\text{Pf}(A + B) = \sum_{r=0}^m (-1)^{r/2} \sum_{\substack{S \subseteq [m] \\ |S|=r}} (-1)^{\sum_i S_i} \text{Pf}(A_S) \text{Pf}(B_{S^c}), \quad (\text{A.3})$$

where S^c denotes the set which is the complement to the set S w.r.t. $[m]$.

A.2 Pfaffian integration formulas

The following result, due to de Bruijn [dB55], is the Pfaffian analogue of the Andréief identity.

Lemma A.4 (de Bruijn's integration formula). Let $m > 0$ be a fixed integer and let μ be an appropriate measure on a region X . Let g_1, \dots, g_m be functions and let h be an skew-symmetric function of two variables which are all integrable over X with respect to μ . Then the following identities hold: when $m = 2n$ is even

$$\frac{1}{(2n)!} \int_X d\mu(w_1) \cdots \int_X d\mu(w_{2n}) \det[g_j(w_i)]_{1 \leq i, j \leq 2n} \text{Pf}[h(w_i, w_j)]_{1 \leq i, j \leq 2n} = \text{Pf}(A),$$

and when $m = 2n - 1$ is odd

$$\begin{aligned} \frac{1}{(2n-1)!} \int_X d\mu(w_1) \cdots \int_X d\mu(w_{2n-1}) \det[g_j(w_i)]_{1 \leq i, j \leq 2n-1} \text{Pf} \begin{bmatrix} h(w_i, w_j)_{1 \leq i, j \leq 2n-1} & \vec{\mathbf{1}}_{2n-1} \\ -\vec{\mathbf{1}}_{2n-1}^T & 0 \end{bmatrix} \\ = \text{Pf} \begin{bmatrix} A & \vec{b} \\ -\vec{b}^T & 0 \end{bmatrix}, \end{aligned}$$

where A is a skew-symmetric $m \times m$ -dimensional matrix and \vec{b} is a vector of length m . The matrix elements are given as:

$$\begin{aligned} [A]_{i,j} &= \int_X d\mu(w) \int_X d\mu(u) g_i(w) g_j(u) h(w, u), \\ [\vec{b}]_i &= \int_X d\mu(w) g_i(w), \end{aligned}$$

for each $1 \leq i, j \leq m$.

The following Lemma is generalisation of Lemma A.4 which appeared in the work [KG10].

Lemma A.5 (Generalised de Bruijn's integration formula). Let $\ell \geq m > 0$ be fixed integers and let μ be an appropriate measure on a region X . Let $f_1, \dots, f_{\ell-m}, g_1, \dots, g_m$ be functions and let h be a skew-symmetric function of two variables which are all integrable over X with respect to μ . Additionally let S be a constant skew-symmetric matrix of dimension $(\ell - m) \times (\ell - m)$. Then the following identities hold: when ℓ is even

$$\begin{aligned} \frac{(-1)^{\binom{\ell-m}{2}}}{m!} \int_X d\mu(w_1) \cdots \int_X d\mu(w_m) \det [g_j(w_i)]_{1 \leq i, j \leq m} \text{Pf} \begin{bmatrix} h(w_i, w_j)_{1 \leq i, j \leq m} & f_k(w_i)_{\substack{1 \leq i \leq m \\ 1 \leq k \leq \ell-m}} \\ -f_k(w_j)_{\substack{1 \leq k \leq \ell-m \\ 1 \leq j \leq m}} & S \end{bmatrix} \\ = \text{Pf} \begin{bmatrix} A & C \\ -C^T & S \end{bmatrix} \end{aligned} \quad (\text{A.4})$$

and when ℓ is odd

$$\begin{aligned} \frac{(-1)^{\binom{\ell-m}{2}}}{m!} \int_X d\mu(w_1) \cdots \int_X d\mu(w_m) \det [g_j(w_i)]_{1 \leq i, j \leq m} \text{Pf} \begin{bmatrix} h(w_i, w_j)_{1 \leq i, j \leq m} & \vec{\mathbf{1}}_m & f_k(w_i)_{\substack{1 \leq i \leq m \\ 1 \leq k \leq \ell-m}} \\ -\vec{\mathbf{1}}_m^T & 0 & 0 \\ -f_k(w_j)_{\substack{1 \leq k \leq \ell-m \\ 1 \leq j \leq m}} & 0 & S \end{bmatrix} \\ = \text{Pf} \begin{bmatrix} A & \vec{b} & C \\ -\vec{b}^T & 0 & 0 \\ -C^T & 0 & S \end{bmatrix}. \end{aligned}$$

where A is a skew-symmetric $m \times m$ -dimensional matrix, C is an $m \times (\ell - m)$ -dimensional matrix and \vec{b} is a vector of length m . The matrix elements are given as:

$$\begin{aligned} [A]_{i,j} &= \int_X d\mu(u) \int_X d\mu(w) g_i(w) g_j(u) h(w, u), \\ [C]_{i,k} &= \int_X d\mu(w) g_i(w) f_k(w), \\ [\vec{b}]_i &= \int_X d\mu(w) g_i(w), \end{aligned}$$

for each $1 \leq i, j \leq m$ and $1 \leq k \leq \ell - m$.

In fact, a more general version of Lemma A.5 appeared in [KG10] involving a additional constant matrix augmented to the rows of the matrix in the determinant of (A.4). However, this more general result is not needed for our purposes here and we will not present it.

The well known Andréief integration identity can be recovered from (A.4) by taking $\ell = 2m$ with $h(w, u) = 0$ and $[S]_{i,j} = 0$ for all $1 \leq i, j \leq m$.

Appendix B

Skew-biorthogonal Polynomials

In this appendix we provide explicit formulas for the skew-biorthogonal polynomials which are characterised by the relations (7.39). We start with a general result for the Cholesky type skew-Borel factorisation of a skew-symmetric matrix.

Proposition B.1. Let m be a fixed integer and let M be a fixed $2m \times 2m$ -dimensional skew-symmetric matrix such that the minors

$$[M]_{k,\ell}, \quad \text{for } 2j-1 \leq k, \ell \leq 2m$$

are invertible for all $1 \leq j \leq m$. Then M has a *skew-Borel factorisation* of the form

$$M = R_{2m} J_{2m} R_{2m}^T,$$

where R_{2m} is an upper-triangular $2m \times 2m$ -dimensional matrix which is uniquely determined up to setting $[R_{2m}]_{2j-1,2j} = 0$ and $[R_{2m}]_{2j,2j} = 1$ for each $1 \leq j \leq m$. Also where

$$J_{2m} = \text{diag} \left\{ \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}.$$

A proof of this result can be found in [BBF⁺00] with an alternative formulation for the matrix J_{2m} . Instead of presenting a proof of Proposition B.1, we demonstrate that the matrix R_{2m} can be explicitly given in terms of 2×2 blocks of Pfaffian minors. For a $2m \times 2m$ matrix, let us denote its construction in terms of 2×2 blocks by:

$$\langle A \rangle_{i,j} = \begin{bmatrix} [A]_{2i-1,2j-1} & [A]_{2i-1,2j} \\ [A]_{2i,2j-1} & [A]_{2i,2j} \end{bmatrix},$$

This Appendix appears in the preprint [P2].

for $1 \leq i, j \leq m$.

Proposition B.2. Let $I \subseteq \{1, \dots, 2m\}$ be an ordered list and let

$$\text{Pf } M_I := \text{Pf}[M_{k,\ell}]_{k,\ell \in I}$$

denote the Pfaffian of the sub-matrix of M consisting of columns and rows labeled by I with the convention that $M_\emptyset = 1$. The matrix R_{2m} of the skew-Borel factorisation of M has the explicit form given in terms of 2×2 blocks as:

$$\langle R_{2m} \rangle_{i,j} = \begin{cases} \begin{pmatrix} \frac{\text{Pf } M_{\{2i-1,2j\} \cup [2j+1,2m]}}{\text{Pf } M_{[2j+1,2m]}} & -\frac{\text{Pf } M_{\{2i-1,2j-1\} \cup [2j+1,2m]}}{\text{Pf } M_{[2j-1,2m]}} \\ \frac{\text{Pf } M_{\{2i,2j\} \cup [2j+1,2m]}}{\text{Pf } M_{[2j+1,2m]}} & -\frac{\text{Pf } M_{\{2i,2j-1\} \cup [2j+1,2m]}}{\text{Pf } M_{[2j-1,2m]}} \end{pmatrix}, & 1 \leq i < j \leq m \\ \begin{pmatrix} \frac{\text{Pf } M_{[2i-1,2m]}}{\text{Pf } M_{[2i+1,2m]}} & 0 \\ 0 & 1 \end{pmatrix}, & i = j \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & 1 \leq j < i \leq m \end{cases},$$

for each block $1 \leq i, j \leq m$. The inverse R_{2m}^{-1} similarly may also be written in terms of 2×2 blocks and is given explicitly by

$$\langle R_{2m}^{-1} \rangle_{i,j} = \begin{cases} \begin{pmatrix} -\frac{\text{Pf } M_{[2i-1,2m] \setminus \{2i,2j-1\}}}{\text{Pf } M_{[2i-1,2m]}} & \frac{\text{Pf } M_{[2i-1,2m] \setminus \{2i,2j\}}}{\text{Pf } M_{[2i-1,2m]}} \\ -\frac{\text{Pf } M_{[2i,2m] \setminus \{2j-1\}}}{\text{Pf } M_{[2i+1,2m]}} & \frac{\text{Pf } M_{[2i,2m] \setminus \{2j\}}}{\text{Pf } M_{[2i+1,2m]}} \end{pmatrix}, & 1 \leq i < j \leq m \\ \begin{pmatrix} \frac{\text{Pf } M_{[2i+1,2m]}}{\text{Pf } M_{[2i-1,2m]}} & 0 \\ 0 & 1 \end{pmatrix}, & i = j \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & 1 \leq j < i \leq m \end{cases}.$$

Recall that $\Psi(x, y) = Q_{1,1}(x, y)$, where $Q_{k,\ell}(x, y)$ is defined in (6.49) as

$$Q_{k,\ell}(x, y) = \alpha^2 \oint_{\beta} \frac{dv}{2\pi i} \oint_{\beta} \frac{dw}{2\pi i} \frac{v-w}{1-v-w} \frac{w^{k-x} e^{t(w-1)}}{(w-\alpha)(w-1)^k} \frac{v^{\ell-y} e^{t(v-1)}}{(v-\alpha)(v-1)^{\ell}},$$

The following is a simple application of the generalised binomial identity on the integrand of $Q_{k,\ell}(x, y)$

Lemma B.3. Let us denote, by Θ_i , the polynomials from (7.34) so that for a fixed integer N

$$\Theta_i(x) := \phi_{[N-i+1, N]}(\dagger_i, x) = \oint_{\gamma} \frac{dz}{2\pi i} \frac{1}{z^i} \left(\frac{1}{1-z} \right)^x = \binom{x+i-2}{x-1},$$

where the contour surrounds $z = 0$ while omitting all other singularities of the integrand. Then the following holds for all $1 \leq i, j \leq N$:

$$\sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \Theta_i(x) Q_{1,1}(x, y) \Theta_j(y) = Q_{i+1, j+1}(1, 1).$$

Recall the definition of the skew-symmetric matrix \mathcal{N} from (7.48). Lemma implies that the entries of this matrix are given by

$$[\mathcal{N}]_{k, \ell} = \phi_{[k, N]} \star \Psi \star (\phi_{[\ell, N]})^T(\dagger_k, \dagger_{\ell}) = Q_{N-k+2, N-\ell+2}(1, 1),$$

for each $1 \leq k, \ell \leq N$. The following result follows by the application of the explicit skew-Borel factorisation of $\mathcal{N} = \mathcal{R}_N J_N \mathcal{R}_N^T$ from (7.52), so that \mathcal{R}_N is an upper-triangular $N \times N$ matrix and determined by Proposition B.2. Let us also denote, for $I \subseteq [1, N]$ which has an even cardinality, the following sub-Pfaffian of the matrix which appears in Corollary 6.24:

$$\text{Pf } \tilde{Q}_I := \text{Pf} [Q_{i+1, j+1}(1, 1)]_{i, j \in I}.$$

Proposition B.4. Let us define the family of polynomials $\tilde{\Phi}_0, \dots, \tilde{\Phi}_{N-1}$ which satisfy the skew-biorthogonalisation relations from (7.39), so that $\tilde{\Phi}_k(x)$ is a polynomial in x of degree k :

$$\sum_{x=1}^{\infty} \sum_{y=1}^{\infty} \tilde{\Phi}_{N-k}(x) \Psi(x, y) \tilde{\Phi}_{N-\ell}(y) = [J_N]_{k, \ell}, \quad 1 \leq k, \ell \leq N. \quad (\text{B.1})$$

These polynomials are constructed explicitly as

$$\tilde{\Phi}_{N-k}(x) = \sum_{j=1}^N [\mathcal{R}_N^{-1}]_{k, j} \Theta_{N-j+1}(x),$$

where \mathcal{R}_N^{-1} is the inverse of the upper-triangular skew-Borel matrix of \mathcal{N} . Moreover, it is

constructed explicitly out of 2×2 -blocks, where¹⁹

$$\langle \mathcal{R}_N^{-1} \rangle_{i,j} = \begin{cases} \begin{pmatrix} \frac{\text{Pf } \tilde{Q}_{[1,N-2i+2]\setminus\{N-2i+1,N-2j+2\}}}{\text{Pf } \tilde{Q}_{[1,N-2i+2]}} & -\frac{\text{Pf } \tilde{Q}_{[1,N-2i+2]\setminus\{N-2i+1,N-2j+1\}}}{\text{Pf } \tilde{Q}_{[1,N-2i+2]}} \\ -\frac{\text{Pf } \tilde{Q}_{[1,N-2i+1]\setminus\{N-2j+2\}}}{\text{Pf } \tilde{Q}_{[1,N-2i]}} & \frac{\text{Pf } \tilde{Q}_{[1,N-2i+1]\setminus\{N-2j+1\}}}{\text{Pf } \tilde{Q}_{[1,N-2i]}} \end{pmatrix}, & 1 \leq i < j \leq \frac{N}{2} \\ \begin{pmatrix} -\frac{\text{Pf } \tilde{Q}_{[1,N-2i]}}{\text{Pf } \tilde{Q}_{[1,N-2i+2]}} & 0 \\ 0 & 1 \end{pmatrix}, & i = j \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, & 1 \leq j < i \leq \frac{N}{2} \end{cases},$$

for each $1 \leq i, j \leq N/2$.

Recall Remark 7.16, whereby the skew-biorthogonal polynomials which satisfy (B.1) are defined up to N free parameters. Proposition B.4 constructs an explicit family of these polynomials, $\tilde{\Phi}_k$, which corresponds to the choices $[\mathcal{R}_N]_{2j-1,2j} = 0$ and $[\mathcal{R}_N]_{2j,2j} = 1$ for each $1 \leq j \leq N/2$; as is made in Proposition B.1.

¹⁹The signs have been modified from the conventions of Proposition B.2 since the underlying matrices satisfy $\text{Pf } \mathcal{N} = (-1)^{N/2} \text{Pf}[Q_{k+1,\ell+1}(1,1)]_{1 \leq k,\ell \leq N}$.

Appendix C

Matching with Tracy–Widom Transition Probability

Let us consider the following expression from the proof of the initial condition of Theorem 2.1 of [TW08]. In the aforementioned work, the authors consider ASEP on \mathbb{Z} is considered with a left-hopping rate of q and a right-hopping rate of p , subject to the normalisation $p + q = 1$. We will consider a modification of their integral expression where the normalisation is relaxed and we set $p = 1$. This is given by:

$$\mathcal{I}_{x,y}^{\text{TW}}(\sigma) := \oint_{\delta} \frac{d\xi_1}{2\pi i} \cdots \oint_{\delta} \frac{d\xi_N}{2\pi i} \prod_{\substack{1 \leq i < j \leq N \\ \sigma_i > \sigma_j}} f(\xi_{\sigma(i)}, \xi_{\sigma(j)}) \prod_{i=1}^N \xi_{\sigma(i)}^{x_{N-i+1}} \xi_i^{-y_{N-i+1}-1}, \quad (\text{C.1})$$

where

$$f(\xi_i, \xi_j) = -\frac{1 + q\xi_i\xi_j - (1+q)\xi_i}{1 + q\xi_i\xi_j - (1+q)\xi_j},$$

and where δ is a contour surrounding the origin, $\xi_i = 0$, with a sufficiently small radius to avoid all other singularities. Let us also emphasise that the convention of labelling individual particle coordinates in (C.1) are consistent with the main body of this text. We note that the particle coordinates of [TW08] are the reverse to our conventions. That is, we consider $x_1 > \cdots > x_N$ as opposed to $x_1 < \cdots < x_N$, which is the reason for x_{N-i+1} appearing in (C.1) rather than x_i .

For any configurations x, y , it is proven in [TW08] that the following sum vanishes:

$$\sum_{\sigma \in S_N \setminus \{\text{id}\}} \mathcal{I}_{x,y}^{\text{TW}}(\sigma) = 0. \quad (\text{C.2})$$

The proof of this statement is involved. It is made necessarily more complicated by the fact

This Appendix appears in the preprint [P2].

that individual terms in the sum (C.2) are in general non-zero after the integrals are evaluated.

Now recall the integral expression from the proof of Lemma 6.6 given by (6.46). We will show here that this integral expression obeys the vanishing summation property (6.47) by explicitly matching our expression with (C.1). Let us consider the integral expression (6.46) under the change of integration coordinates $w_i \mapsto \frac{1-\xi_i}{1-q\xi_i}$. After straightforward algebraic manipulation, for any $\tau \in S_N$, (6.46) is given by:

$$\mathcal{I}_{x,y}(\tau) := \oint_{\delta} \frac{d\xi_1}{2\pi i} \cdots \oint_{\delta} \frac{d\xi_N}{2\pi i} \prod_{\substack{1 \leq i < j \leq N \\ \tau(i) > \tau(j)}} f(\xi_{\tau(i)}, \xi_{\tau(j)}) \prod_{i=1}^N \xi_i^{x_i} \xi_{\tau(i)}^{-y_i-1}. \quad (\text{C.3})$$

From here, for each individual fixed τ , we may simultaneously relabel the integration coordinates of (C.3) by $\xi_i \mapsto \xi_{\rho(i)}$ for the permutation defined by $\rho = (r(\tau))^{-1}$, where $r(\tau) = (\tau(N), \dots, \tau(1))$ is permutation which is the reverse ordering of τ . We emphasise here that we are allowed to freely change the order of integration after relabelling since all contours may be deformed to be the same. Under this procedure, the quantity (C.3) associated with τ is

$$\mathcal{I}_{x,y}(\tau) = \oint_{\delta} \frac{d\xi_1}{2\pi i} \cdots \oint_{\delta} \frac{d\xi_N}{2\pi i} \prod_{\substack{1 \leq i < j \leq N \\ \tau(i) > \tau(j)}} f(\xi_{N-i+1}, \xi_{N-j+1}) \prod_{i=1}^N \xi_{r(\rho)(i)}^{x_{N-i+1}} \xi_i^{-y_{N-i+1}-1}. \quad (\text{C.4})$$

We now claim that the integrands of (C.1) and (C.4) agree whenever their respective permutations are related by the following

$$\tau = r((r(\sigma))^{-1}).$$

This relationship is precisely equivalent to saying that τ is the *mirror image* of σ . That is, if, as expressed a reduced word in simple transpositions s_i which generate S_N , we have $\tau = s_{k_1} \cdots s_{k_n}$ for some sequence $k_1, \dots, k_n \in \{1, \dots, N-1\}$. Then we have the corresponding expression $\sigma = s_{N-k_1+1} \cdots s_{N-k_n+1}$. Additionally, we note that this correspondence preserves the identity, i.e. $\tau = \text{id}$ corresponds with $\sigma = \text{id}$. Using this correspondence, we may conclude that

$$\sum_{\tau \in S_N \setminus \{\text{id}\}} \mathcal{I}_{x,y}(\tau) = \sum_{\sigma \in S_N \setminus \{\text{id}\}} \mathcal{I}_{x,y}^{\text{TW}}(\sigma) = 0,$$

which is the desired result.

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